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Primitive idempotents of the Grothendieck ring of Mackey functors

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Abstract

We study the Grothendieck ring of the category of Mackey functors for a finite group and determine the primitive idempotents of the ring.

1 PRELIMINARIES

Let G be a finite group and an R a commutative ring. We denote by $S(G)$ the set of all subgroups of G and let $C(G)$ be the set of representatives of conjugacy classes of $S(G)$. For $H \in S(G)$ and $g \in G$ let ${}^gH = gHg^{-1}$, $H^g = g^{-1}Hg$. If $H \in S(G)$ and $L, K \in S(G)$ let $[L \setminus H/K]$ be a set of representatives of cosets LhK with $h \in H$. If $L \leq H \leq G$ let H/L be a set of representatives of cosets hL with $h \in H$.

A **Mackey functor** for G over R is a mapping

$$M : S(G) \longrightarrow R\text{-mod}$$

with morphisms

$$\begin{array}{rcll} I_K^H & : & M(K) \longrightarrow M(H) & \text{(induction)} \\ R_K^H & : & M(H) \longrightarrow M(K) & \text{(restriction)} \\ c_g^H & : & M(H) \longrightarrow M({}^gH) & \text{(conjugation)} \end{array}$$

whenever $K \leq H$ are subgroups of G and $g \in G$, such that

(M0) $I_H^H, R_H^H, c_h^H : M(H) \to M(H)$ are the identity morphisms for all subgroups H and $h \in H$,

(M1) $R_L^K R_K^H = R_L^H$, $I_K^H I_L^K = I_L^H$ for all subgroups $L \leq K \leq H$,

(M2) $c_g^{{}^hH} c_h^H = c_{gh}^H$ for all subgroups $H \leq G$ and $g, h \in G$,

(M3) $R_{{}^gK}^{{}^gH} c_g^H = c_g^K R_K^H$, $I_{{}^gK}^{{}^gH} c_g^K = c_g^H I_K^H$ for all subgroups $K \leq H$ and $g \in G$,

(M4) $R_L^H I_K^H = \sum_{x \in [L \setminus H/K]} I_{L \cap {}^xK}^L c_x^{L^x \cap K} R_{L^x \cap K}^K$ for all subgroups $L, K \leq H$.

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This definition was given by Green [Gr71]. Moreover, this definition is equivalent to the categorical definition given by Dress [Dr73]. The most important axiom is (M4), which is the **Mackey decomposition formula.** Note that the axioms (M0) and (M2) imply that $WH := W_G(H) := N_G(H)/H$ acts on a left R-module $M(H)$, so that $M(H)$ is a left $R[WH]$-module for each subgroup H of G.

A **morphism** of Mackey functors $f : M \to N$ is a family of R-homomorphisms $f(H) : M(H) \to N(H)$, for $H \in S(G)$, which commute with restriction, induction, and conjugation. In particular, since f commutes with conjugation, $f(H)$ is an $R[WH]$-homomorphism. We denote by $\mathrm{Mack}_R(G)$ the category of Mackey functors for G over R. It is easy to see that $\mathrm{Mack}_R(G)$ is an abelian category.

We summarise some of the basic constructions of Mackey functors. We denote by

$$\uparrow_H^G : \mathrm{Mack}_R(H) \to \mathrm{Mack}_R(G),\ \downarrow_H^G : \mathrm{Mack}_R(G) \to \mathrm{Mack}_R(H),\ \sigma_g^H : \mathrm{Mack}_R(H) \to \mathrm{Mack}_R({}^gH),$$

the **induction, restriction,** and **conjugation** of Mackey functors [Sa82]. Whenever we have a normal subgroup N of G and a Mackey functor L for $Q = G/N$ we can form the **infration** $\mathrm{Inf}_Q^G L$ which is a Mackye functor for G defined by

$$\mathrm{Inf}_Q^G L = \begin{cases} L(K/N) & \text{if } K \subseteq N \\ 0 & \text{otherwise} \end{cases}$$

with zero restriction and induction morphisms R_K^H, I_K^H unless $N \le K \le H$, in which case they are the mappings $R_{K/N}^{H/N}$, $I_{K/N}^{H/N}$ for L, and similarly with conjugations. If M is a Mackey functor for G over Rwe will write

$$\overline{M}(H) = M(H)/\sum_{J<H} I_J^H M(J).$$

Note that $\overline{M}(H)$ is an $R[WH]$-module. We recall the **simple Mackey functor** which constructed by Thévenaz and Webb [TW89]. For an $R[G]$-module we describe a Mackey functor $S_{1,V}^G$ for G as follows;

$$S_{1,V}^G(H) = (\sum_{h\in H} h)V, \quad H \in S(G).$$

Mreover, if H is any subgroup of G and V is a simple $R[WH]$-module we define $S_{H,V}^G = (\mathrm{Inf}_{WH}^{NH} S_{1,V}^{WH}) \uparrow_{NH}^G$, and this is in fact a simple Mackey functor. The $S_{H,V}^G$ so constructed constitute a complete set of representatives for the isomorphism classes of simple Mackey functors [TW89] 8.3.

Lemma 1.1 ([TW95] 6.4) *Let $S_{H,V}^G$ be a simple Mackey functor. Then*

$$\overline{S_{H,V}^G}(L) = \begin{cases} V & \textit{if } H \textit{ and } K \textit{ are conjugate} \\ 0 & \textit{otherwise.} \end{cases}$$

A Mackey functor for G over R is identified as a certain finite dimensional algebra $\mu_R(G)$ which called a **Mackey algebra** [TW95].

Lemma 1.2 ([TW95] 3.6) *Let K be a field. Then K is a splitting field for $\mu_K(G)$ if and only if K is a splitting field for the representations of WH for every subgroup $H \in S(G)$.*

The set of G-isomorphism classes of finite G-sets becomes a commutative ring $\Omega(G)$ whose name is **Buernside ring**, with addition defined by disjoint union and multiplication defined by cartesian product with diagonal action. The Burnside ring over R of G is the free R-module with basis the G-sets G/H where H is taken up to conjugacy. By means of induction, restriction, and conjugation of G-sets this gives rise to a Mackey functor denoted Ω^G, which is call the **Burnside functor** for G.

Lemma 1.3 ([TW95] 8.9) *Suppose that K is a field which is a splitting field for $\mu_K(G)$. If* $\mathrm{char}(K) = 0$ *then*

$$\Omega^G \cong \bigoplus_{H \in C(G)} S_{H,K}.$$

For three Mackey functors M, N, L a **pairing** $M \times N \to L$ of Mackey functors is a family of R-bilinear maps

$$M(H) \times N(H) \to L(H): \qquad (m, n) \mapsto m \cdot n$$

such that the following axioms hold: for subgroups H, K of G with $H \leq K$

(P1) $R^H_K(ab) = R^H_K(a)R^H_K(b), \quad a \in M(H),\, b \in N(H),$

(P2) $c^H_g(ab) = c^H_g(a)c^H_g(b), \quad a \in M(H),\, b \in N(H),$

(P3) $I^H_K(a)b' = I^H_K(aR^H_K(b')), \quad a \in M(K),\, b' \in N(H),$

(P4) $a'I^H_K(b) = I^H_K(R^H_K(a')b), \quad a' \in M(H),\, b \in N(K).$

A **Green functor** A is a Mackey functor with a pairing $A \times A \to A$ such that for each $H \in S(G)$ the R-linear map $A(H) \times A(H) \to A(H)$ makes $A(H)$ into associative R-algebra with unity $1_{A(H)}$ such that:

(G) $R^H_K(1_{A(H)}) = 1_{A(K)}, \quad K \leq H \leq G.$

Let M be a Mackey functor and let A a Green functor. If there exists a pairing $l_A : A \times M \to M$ such that $M(H)$ becomes a unitary left $A(H)$-module via the R-homomorphism $l_A(H) : A(H) \times M(H) \to M(H)$ then we said that M is a **left A-module** [Is89], [Lu96]. One can define similarly the notion of **right A-module** with a pairing r_A.

Let A be a Green functor for G. Let M_A, ${}_AN$ be A-modules and L a Mackey functor for G over R. A **A-pairing** $p : M \times N \to L$ [Is89], [Lu96] is a pairing $p : M \times N \to L$ such that the following axiom hold:

(P5) For $H \in S(G)$ diagram

$$\begin{tikzcd} M(H) \times A(H) \times N(H) \arrow[r, "1_{M(H)} \times l_A(H)"] \arrow[d, "r_A(H) \times 1_{N(H)}"'] & M(H) \times N(H) \arrow[d, "p(H)"] \\ M(H) \times N(H) \arrow[r, "p(H)"] & L(H) \end{tikzcd}$$

is commutative.

2 Tensor product of Mackey functors

In this section, we recall the tensor product of Mackey functors. We refer to [Is89], [Le80], [Lu96] for detail.

Let M and N be Mackey functors for a finite group G over a commutative ring R. For $H \leq G$, we put

$$T(H) = < 1^H_D \otimes \mu \otimes \nu \,|\, \mu \in M(D),\, \nu \in N(D),\, D \in S(H) > \cong \bigoplus_{D \in S(H)} M(D) \otimes_R N(D),$$

where $1^H_D \otimes$ is a symbol. Let $I(H)$ be the R-submodule of $T(H)$ generated by the following elements;

(R1) $1^H_H \otimes (\mu_1 + \mu_2) \otimes \nu_0 = 1^H_H \otimes \mu_1 \otimes \nu_0 + 1^H_H \otimes \mu_2 \otimes \nu_0$,

(R2) $1^H_H \otimes \mu_0 \otimes (\nu_1 + \nu_2) = 1^H_H \otimes \mu_0 \otimes \nu_1 + 1^H_H \otimes \mu_0 \otimes \nu_2$,

(R3) $1^H_H \otimes \mu_0 \alpha \otimes \nu_0 = 1^H_H \otimes \mu_0 \otimes \alpha \nu_0$,

(R4) $1^H_{D'} \otimes t^{D'}_D(\mu) \otimes \nu' = 1^H_D \otimes \mu \otimes r^{D'}_D(\nu')$,

(R5) $1^H_{D'} \otimes \mu' \otimes t^{D'}_D(\nu) = 1^H_D \otimes r^{D'}_D(\mu') \otimes \nu$,

(R6) $1^H_{{}^hD} \otimes {}^h\mu \otimes {}^h\nu = 1^H_D \otimes \mu \otimes \nu$,

whenever $\mu_0, \mu_1, \mu_2 \in M(H)$, $\mu \in M(D)$, $\mu' \in M(D')$, $\nu_0, \nu_1, \nu_2 \in N(H)$, $\nu \in N(D)$, $\nu' \in N(D')$, $\alpha \in A(H)$, $h \in H$, $D \leq D' \leq H$. Moreover, for subgroups $K \leq H$ of G and an element g of G, the linear maps restriction, induction, and conjugation defined as follows.

(T1) $\rho^H_K : T(H) \to T(K)\,;\, 1^H_D \otimes \mu \otimes \nu \mapsto \sum_{g \in [K \backslash H/D]} 1^K_{K \cap {}^gD} \otimes R^{{}^gD}_{K \cap {}^gD}({}^g\mu) \otimes R^{{}^gD}_{K \cap {}^gD}({}^g\nu)$,

(T2) $\tau^H_K : T(K) \to T(H)\,;\, 1^K_D \otimes \mu \otimes \nu \mapsto 1^H_D \otimes \mu \otimes \nu$

(T3) $\sigma^H_g : T(H) \to T({}^gH)\,;\, 1^H_D \otimes \mu \otimes \nu \mapsto 1^{{}^gH}_{{}^gD} \otimes c^H_g \mu \otimes c^H_g \nu$.

For $H \leq G$, we set

$$M \otimes N(H) = T(H)/I(H).$$

A **tensor product** of Mackey functors M and N consist of $M \otimes N$ with induction, restriction, and conjugation above. Also the Mackey functor $M \otimes N$ satisfy the universality of tensor product.

Lemma 2.1 ([Is89], [Le80], [Lu96], Yoshida) *There exists a unique pairing (resp. A-pairing) $\theta : M \times N \to M \otimes N$, such that for every pairng (resp. A-pairing) $\eta : M \times N \to T$, there exist a unique family of maps $\phi : M \otimes N \to T$ the diagram*

$$\begin{tikzcd}
M \times N \arrow[rr, "\theta"] \arrow[dr, "\eta"'] & & M \otimes N \arrow[dl, "\phi"] \\
& L &
\end{tikzcd}$$

is commutative.

Let

$$M \times N \xrightarrow{\otimes} M \otimes_A N$$

denote the universal A-pairing.

Lemma 2.2 *Let M be a Mackey functor for G over R. Then there exists a Ω^G-pairing*

$$\theta : \Omega^G \times M \to M.$$

Proof. See [Is89], [Le80], [Lu96], [TW95]. ∎

Hence for Mackey functors M_{Ω^G} and ${}_{\Omega^G}N$ we have Ω^G-pairing $M \times N \to M \otimes_{\Omega^G} N$.

3 Grothendieck ring of Mackey functors

In this section, we describe the Grothendieck ring of the category of Mackey functors for G over R.

Lemma 3.1 *Let M, N and L be Mackey functors for G over R. Then there exist isomorphisms of Mackey functors as follows;*

(i) $M \otimes_{\Omega^G} N \cong N \otimes_{\Omega^G} M$,

(ii) $(M \otimes_{\Omega^G} N) \otimes_{\Omega^G} L \cong M \otimes_{\Omega^G} (N \otimes_{\Omega^G} L)$,

(iii) $(M \oplus N) \otimes_{\Omega^G} L \cong (M \otimes_{\Omega^G} L) \oplus (N \otimes_{\Omega^G} L)$,

(iv) $M \otimes_{\Omega^G} (N \oplus L) \cong (M \otimes_{\Omega^G} N) \oplus (M \otimes_{\Omega^G} L)$,

(v) $M \otimes_{\Omega^G} \Omega^G \cong \Omega^G \otimes_{\Omega^G} M \cong M$.

Proof. (i) We shall construct a family of maps $\phi : N \otimes_{\Omega^G} M \to L$, such that the next diagram

$$\begin{tikzcd} M \times N \arrow[rr, "\otimes"] \arrow[dr, "\rho"'] & & N \otimes_{\Omega^G} M \arrow[dl, "\phi"] \\ & L & \end{tikzcd}$$

is commutative for all Ω^G-pairing and a Mackey functor L. For $H \in S(G)$ it suffices to define

$$\phi(H) : N \underset{\Omega^G}{\otimes} M(H) \to L(H)\,;\ \phi(n \times m) = \rho(m, n)$$

where $n \in N(H)$ and $m \in M(H)$. (ii)-(iv) are similar to (i).

(v) There is a family of maps $\phi : M \to L$, such that the next diagram

$$\begin{tikzcd} \Omega^G \times M \arrow[rr, "\theta"] \arrow[dr, "\rho"'] & & M \arrow[dl, "\phi"] \\ & L & \end{tikzcd}$$

is commutative for all Ω^G-pairing ρ and a Mackey fnctor L : that is,

$$\phi(H) \,:\, M(H) \to L(H)\,;\, m \mapsto \rho(1_H, m).$$

∎

The next result appears also in a general form in Luca's paper [Lu96] 4.1.11, but the proof differs from ours.

Lemma 3.2 *Let K be a splitting field for $\mu_K(G)$ and* $\mathrm{char}(K) = 0$.
(i) Let M and N be Mackey functors for G over K. Then

$$\overline{M \otimes N}(H) \cong \overline{M}(H) \otimes_K \overline{N}(H)$$

for every subgroup H.
(ii) Let A be a Green functor and let M (resp. N) be a right (resp. left) A-module. Then

$$\overline{M \otimes_{A(H)} N}(H) \cong \overline{M}(H) \otimes_{\overline{A}(H)} \overline{N}(H)$$

Proof. (i) We may assume M and N be simple Mackey functors $S^G_{P,V}$, $S^G_{Q,W}$ from Lemma 1.2. Let $f(H)$ be a map from $\overline{S_{P,V}}(H) \times \overline{S_{Q,W}}(H)$ to $\overline{S_{P,V} \otimes_{\Omega^G} S_{Q,W}}(H)$ defined by

$$f(H) = \begin{cases} 1^H_H \otimes s \otimes t & \text{if } P,\ Q, \text{ and } H \text{ are conjugate} \\ 0 & \text{otherwise} \end{cases}$$

where $s \in S^G_{P,V}(H)$ and $t \in S_{Q,W}(H)$. Then $f(H)$ is a K-bilinear map by the definition of tensor product.

We construct a map $\phi :\, N \otimes_{\Omega^G} M \to L$, such that the next diagram

$$\begin{tikzcd} \overline{S_{P,V}}(H) \times \overline{S_{Q,W}}(H) \arrow[rr, "f(H)"] \arrow[dr, "\rho"'] & & \overline{S_{P,V} \otimes_{\Omega^G} S_{Q,W}}(H) \arrow[dl, "\phi"] \\ & L & \end{tikzcd}$$

is commutative for a K-homomorphism ρ and a K-module L : that is,

$$\phi(1^H_D \otimes p \otimes q) = \begin{cases} \rho(p, q) & \text{if } D \text{ and } H \text{ are conjugate} \\ 0 & \text{otherwise} \end{cases}$$

The lemma follows by Lemma 1.1 and the universality of tensor product of K-modules. ∎

Let $G_0(\mathrm{Mack}_R(G))$ be a Grothendieck group of the category of Mackey functors $\mathrm{Mack}_R(G)$ for a finite group G over a commutative ring R with addition defined by the direct sum. Then $G_0(\mathrm{Mack}_R(G))$ has a commutative ring structure from Lemma 3.1 with multiplication defined by the tensor product of Mackey functors which we call the **Grothendieck ring** of Mackey functors. The Grothendieck ring $G_0(\mathrm{Mack}_R(G))$ has a basis

$$\{S_{H,V} | H \in C(G),\, V \in \mathrm{Irr}_R(WH)\}$$

from Lemma 3.1 (v) and the unit element Ω^G.

The main result of this paper is the following.

Theorem 3.3 *Let K be a field which is a splitting field for the representations of WH for every subgoup $H \leq G$ and $\mathrm{char}(K) = 0$. Then there is an isomorphism of rings*

$$G_0(\mathrm{Mack}_K(G)) \cong \bigoplus_{H \in C(G)} G_0(K[WH]).$$

Proof. We shall define a map

$$\psi : G_0(\mathrm{Mack}_R(G)) \to \bigoplus_{H \in C(G)} G_0(K[WH])$$

by $M \mapsto (\overline{M}(H))_H$. Here we use the symbol M to denote also the element of $G_0(\mathrm{Mack}_R(G))$ determined by M, and likewise $\overline{M}(H)$ denote the element of $G_0(K[WH])$ which this $K[WH]$-module determines. By Lemma 1.1 the matrix of ψ is the identity matrix. It follows that ψ is an isomorphism of abelian groups. By Lemma 1.3 ψ preserves the identity. Since K is a splitting field for $\mu_K(G)$ and $|G|^{-1} \in K$, we obtain the desired result from Lemma 3.2. ∎

4 Primitive idempotents

Let $Cl(G)$ be the set of the representatives of canjugacy classes of G and let $C_G(x)$ be the centralizer of $x \in G$ in G. We denote by $\mathrm{Irr}_K(G)$ the irreducible characters of G over a field K. We need the next lemma.

Lemma 4.1 *For an element x of G, we put*

$$e_{G,x} = |C_G(x)|^{-1} \sum_{\chi \in \mathrm{Irr}_K(G)} \chi(x^{-1})\chi.$$

Then $\{e_{G,x} | x \in Cl(G)\}$ is the set of primitive idempotents of the character ring of G over K.

For $H \leq G$ and $x \in WH$, we set

$$E_{H,x} = |C_{WH}(x)|^{-1} \sum_{\chi \in \mathrm{Irr}_K(WH)} \chi(x^{-1}) S^{WH}_{H,V_\chi} \quad \text{in } G_0(\mathrm{Mack}_K(G))$$

where V_χ is irreducible $K[WH]$-module corresponding χ.

Corollary 4.2 *There sxist the set of primitive idempotents*

$$\{E_{H,x} | x \in Cl(WH),\, H \in C(G)\}$$

of the Grothendieck ring $\boldsymbol{C} \otimes G_0(\mathrm{Mack}_K(G))$.

Proof. Let ψ be the isomorhism in Theorem 3.3. It is easy to see that $\psi(E_{H,x}) = e_{WH,x}$ from Lemma 1.1. Thus we obtain the corollary by Lemma 4.1 and Theorem 3.3. ∎

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