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SURFACES IN MINKOWSI 3-SPACE AND HARMONIC MAPS

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INTRODUCTION

Over the past few years substantial progress has been made in the study of harmonic maps from a compact Riemann surface into a compact Riemannian symmetric space. In particular, F. E. Burstall, F. Ferus, F. Pedit and U. Pinkall initiated the theory of finite-type harmonic maps [3], [4]. Viewing maps from a torus as doubly periodic ones from a whole plane, the above authors have constructed such harmonic maps from commuting Hamiltonian flows on certain finite dimensional subspace of a suitable loop algebra [3]. That such a picture obtains has its roots in the classification of constant mean curvature tori in Euclidean 3-space due to U. Pinkall and I. Sterling. The link between harmonic maps and the classification problem as above is that Gauss maps of constant mean curvature surfaces are harmonic maps into a 2-sphere. The situation is similar for the study of constant negative Gaussian curvature surfaces in Euclidean 3-space. M. Melko and I. Sterling have developed a detailed study on such surfaces via the theory of finite-type harmonic maps which is arranged for maps from a Lorentz surface [8], [9].

On the other hand, the theory of finite-type harmonic maps into a noncompact Riemannian symmetric space has not been constructed yet .

The purpose of this talk is to construct such theory to noncompact target manifolds of dimension 2. We shall study harmonic maps from a Riemann or Lorentz surface into a hyperbolic 2-space.

As in the geometry of surfaces in Euclidean 3-space, the harmonicity of the Gauss map for a spacelike surface in Minkowski 3-space is equivalent to the constancy of the mean or Gaussian curvature. So we are able to apply our results on harmonic maps to the geometry of spacelike surfaces. In particular, we can construct such surfaces which are not graphs.

1. FUNDAMENTAL EQUATIONS OF SPACELIKE SURFACES

We start with some preliminaries on geometry of spacelike surfaces in Minkowski 3-space.

Let $\mathbf{E}^3_1$ be a *Minkowski 3-space* with Lorentz metric $\langle\ ,\ \rangle$. The metric $\langle\ ,\ \rangle$ is expressed as $\langle\ ,\ \rangle = -d\xi_1^2 + d\xi_2^2 + d\xi_3^2$ in terms of natural coordinates.

Let M be a connected 2-manifold and $\varphi : M \to \mathbf{E}^3_1$, an immersion. The immersion φ is said to be *spacelike* if the induced metric of M is positive definite. Hereafter we may assume that M is an orientable spacelike surface in $\mathbf{E}^3_1$ (immersed by φ). It is worth-while to remark that there exists no closed spacelike surface in $\mathbf{E}_1^3$.

The induced Riemannian metric I of a spacelike surface M determines a conformal structure on M. We treat M as a Riemann surface with respect to this conformal structure and φ as a conformal immersion. Let $z = x + \sqrt{-1}y$ be a local complex coordinate of M. The induced metric I (*the first fundamental form*) of M can be written as

$$I = e^{\omega} dz d\bar{z} = e^{\omega}(dx^2 + dy^2). \tag{1.1}$$

Now, let N be a local unit normal vector field to M. The vector field N is timelike, that is, $\langle N, N\rangle = -1$ since M is spacelike.

The second fundamental form II of M is defined by

$$II = \langle d\varphi,\ dN\rangle. \tag{1.2}$$

The Gaussian curvature K of M is given by

$$K = -\det(II \,.\, I^{-1}). \tag{1.3}$$

The Gauss-Codazzi equation have the following form:

$$\omega_{z\bar{z}} = \frac{1}{2}H^2 e^{\omega} - 2|Q|^2 e^{-\omega} \tag{1.4}$$

$$H_{\bar{z}} = 2e^{-\omega}\bar{Q}_z, \quad H_z = 2e^{-\omega}Q_{\bar{z}}. \tag{1.5}$$

where $Q := -\langle \varphi_{zz}, N\rangle$, $H = -2e^{-\omega}\langle \varphi_{z\bar{z}}, N\rangle$. It is easy to see that $Q^{\#} = Qdz^2$ is globally defined 2-differential on M. The differential $Q^{\#}$ is called the *Hopf differential of* M.

The Gauss-Codazzi equations (1.4)–(1.5) actually show somewhat more. In fact, the equation (1.5) show that the constancy of the mean curvature H is equivalent to the holomorphicity of the Hopf differential $Q^{\#}$.

Remark. It is easy to deduce that the zero of $Q^{\#}$ coincides with an umbilic point. Hence a spacelike surface is totally umbilic if and only if its Hopf differential $Q^{\#}$ vanishes. A totally umbilic spacelike surface is congruent to an open portion of a hyperbolic 2-space:

$H^2(r) = \{\xi \in \mathbf{E}_1^3 \mid \langle \xi, \xi\rangle = -r^2,\ \xi_1 > 0\}$ of radius $r > 0$.(hence if M is complete, M is congruent to $H^2(r)$). Note that there is no totally umbilic spacelike surfaces of (constant) positive curvature.

Next, we shall define the Gauss map of a spacelike surfaces. Let M be a spacelike surface with local unit normal vector field N to M. We choose N a future-pointing one (See H. I. Choi and A. Treibergs [5] and B. O'Neill [11]). For each $p \in M$ the point $\psi(p)$ of $\mathbf{E}_1^3$ canonically corresponding to the vector N_p lies in a unit hyperbolic 2-space since N is timelike. The resulting smooth mapping $\psi : M \to H^2$ is called the *Gauss map* of M. The constancy of mean or Gaussian curvature is characterised by the harmonicity of the Gauss map.

Proposition 1.1. *The Gauss map of a spacelike surface is harmonic if and only if the mean curvature is constant.*

Proposition 1.2. *Let M be a spacelike surface. Assume that the Gaussian curvature K is nowhere zero on M and has a constant sign. then the second fundamental form II gives M another (semi-) Riemannian metric. With respect to this metric II, the Gauss map of M is harmonic if and only if K is constant.*

It is known that to find spacelike surfaces of constant negative curvature is the almost same as to find surfaces of constant mean curvature surfaces. So we restrict our attention to constant mean or positive Gaussian curvature surfaces. On such surfaces special local coordinates are available.

Proposition 1.3. *Let M be a spacelike surface of constant positive curvature* 1. *Then there exist local coordinates (u, v) around an arbitrary point of M such that;*

$$I = du^2 + 2\cos\phi du dv + dv^2, \quad II = -2\sin\phi du dv. \tag{1.6}$$

With respect to this coordinate system, the Gauss-Codazzi equation (GC) *of the surface is written as following form:*

$$\phi_{uv} = -\sin\phi. \tag{1.7}$$

The above local coordinates (u, v) are called *asymptotic Chebyshev coordinates.* Because the parameter curves of this coordinate system are asymptotic. The function ϕ in (1.7) is the angle between two asymptotic directions.

Proposition 1.4. *Let M be a spacelike surface of constant mean curvature* $\frac{1}{2}$. *Then there exist local coordinates (u, v) over a region free of umbilics such that*

$$I = e^{\omega}(du^2 + dv^2), \; II = e^{\frac{\omega}{2}}(\cosh\frac{\omega}{2}du^2 + \sinh\frac{\omega}{2}dv^2). \tag{1.8}$$

With respect to this coordinate system, the Gauss-Codazzi equation (GC) *of the surface is written as following form:*

$$\omega_{uu} + \omega_{vv} = \sinh\omega. \tag{1.9}$$

It is easy to see that the parameter curves of the above coordinates are principal, that is, they are lines of curvature. The coordinates described in the above proposition is called *isothermal principal coordinates* or *isothermal curvature-line coordinates.*

To close this section, we shall mention the completeness of spacelike constant mean curvature surfaces. It is known that every spacelike surface of constant mean curvature which is complete as a Riemannian 2-manifold with respect to the induced metric is entire. More precisely, Such a surface is a graph of a function defined on a whole plane **C** or a unit open disk **D**.
(See T.Y. Wan and T.K.-K. Au [13], [14]). Thus we need another method to construct constant mean curvature surfaces which are not graphs. On the contrary, there are no known result on systematic construction of spacelike surfaces of constant positive Gaussian curvature.

These motivate us to establish the theory of finite-type harmonic maps into hyperbolic 2-spaces.

2. The split-quaternion algebra

To reformulate the Gauss-Codazzi equations in a form familiar to the theory of integrable systems, we shall use 2 by 2 matrix-formalism ([1], [17]).

Our idea for this purpose is to identify the Minkowski 3–space $\mathbf{E}^3_1$ with the imaginary part Im $\mathbf{H}'$ of the split-quaternion algebra $\mathbf{H}'$.

Let us denote the algebra of split-quaternions by $\mathbf{H}'$ and its natural basis by $\{\mathbf{1},\ \mathbf{i},\ \mathbf{j}',\ \mathbf{k}'\}$. The multiplication of $\mathbf{H}'$ are defined by as follows:

$$\mathbf{i}\mathbf{j}' = -\mathbf{j}'\mathbf{i} = \mathbf{k}',\ \mathbf{j}'\mathbf{k}' = -\mathbf{k}'\mathbf{j}' = -\mathbf{i},\ \mathbf{k}'\mathbf{i} = -\mathbf{i}\mathbf{k}' = \mathbf{j}', \tag{2.1}$$

$$\mathbf{i}^2 = -\mathbf{1},\ \mathbf{j}'^2 = \mathbf{k}'^2 = \mathbf{1}.$$

Hereafter we identify $\mathbf{H}'$ with a semi-Euclidean space $\mathbf{E}^4_2$:

$$\mathbf{E}^4_2 = (\mathbf{R}^4(\xi_0, \xi_1, \xi_2, \xi_3),\ -d\xi_0^2 - d\xi_1^2 + d\xi_2^2 + d\xi_3^2).$$

Let $G = \{\xi \in \mathbf{H}' \mid \xi\bar{\xi} = 1\}$ be the multiplicative group of timelike unit split-quaternions. The Lie algebra $\mathfrak{g}$ of G is the imaginary part Im $\mathbf{H}'$ of $\mathbf{H}'$.

The Lie algebra $\mathfrak{g}$ is naturally identified with a Minkowski 3-space

$$\mathbf{E}^3_1 = (\mathbf{R}^3(\xi_1, \xi_2, \xi_3),\ -d\xi_1^2 + d\xi_2^2 + d\xi_3^2)$$

as a metric linear space.

Since $\mathbf{H}' = \mathbf{C}\oplus\mathbf{k}'\mathbf{C}$ is a right complex linear space, $\mathbf{H}'$ has a matricial expression in the linear space $\mathrm{M}_2\mathbf{C}$ of all complex 2 by 2 matrices.

$$\alpha + \mathbf{k}'\,\beta \quad\longleftrightarrow\quad \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

for any $\alpha,\ \beta \in \mathbf{C}$. More explicitely, the correspondence

$$\xi = \xi_0\mathbf{1} + \xi_1\mathbf{i} + \xi_2\mathbf{j}' + \xi_3\mathbf{k}' \quad\longleftrightarrow\quad \begin{pmatrix} \xi_0 + \sqrt{-1}\xi_1 & \xi_3 - \sqrt{-1}\xi_2 \\ \xi_3 + \sqrt{-1}\xi_2 & \xi_0 - \sqrt{-1}\xi_1 \end{pmatrix} \tag{2.2}$$

gives a matricial expression of $\mathbf{H}'$ in $\mathrm{M}_2\mathbf{C}$. Under the identification (2.2), the group G of timelike unit split-quaternions corresponds to an indefinite special unitary group:

$$\mathrm{SU}_1(2) = \{\begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \mid -|\alpha|^2 + |\beta|^2 = 1\}.$$

The semi-Euclidean metric of $\mathbf{H}'$ corresponds to the following scalar product on $\mathrm{M}_2\mathbf{C}$.

$$\langle X,\ Y\rangle = \frac{1}{2}\{\mathrm{tr}(XY) - \mathrm{tr}(X)\mathrm{tr}(Y)\} \tag{2.3}$$

for all $X,\ Y \in \mathrm{M}_2\mathbf{C}$. This scalar product $\langle\ ,\ \rangle$ is the Killing form of $\mathrm{M}_2\mathbf{C}$ upto constant multiple. The metric of G induced by (2.3) is a bi-invariant Lorentz metric of constant curvature -1. Hence the Lie group G is identified with an anti de-Sitter 3-space H^3_1 of constant curvature -1.

Next, we introduce a Lorentz analogue of the Hopf-fibering. The $\mathrm{Ad}(G)$-orbit of $\mathbf{i} \in \mathfrak{g}$ is a hyperbolic plane:

$$H^2 = \{\xi \in \mathbf{E}^3_1 \mid \langle \xi, \xi \rangle = -1, \ \xi_1 > 0\}.$$

The Ad-action of G on H^2 is transitive and isometric. The isotropy subgroup of G at $\mathbf{i}$ is $\mathrm{U}(1) = \{\xi_0 \mathbf{1} + \xi_1 \mathbf{i} \mid \xi_0^2 + \xi_1^2 = 1\}$. The natural projection $\pi : G \to H^2$, given by $\pi(g) = \mathrm{Ad}(g)\mathbf{i}$ for all $g \in G$, defines a principal circle bundle over H^2. This fibering is also called the *Hopf-fibering*. The isotropy subgroup at $\mathbf{i}$ will be denoted by K. The Lie algebra $\mathfrak{k}$ of K is spanned by $\mathbf{i}$. The tangent space of H^2 at the origin $\mathbf{i}$ is given by $\mathfrak{m} = \mathbf{R}\mathbf{j}' \oplus \mathbf{R}\mathbf{k}'$.

Let τ be an involution of $\mathfrak{g}$ defined by $\tau = \mathrm{Ad}(\mathbf{i})$. The pair $(\mathfrak{g}, \ \tau)$ is a orthogonal symmetric Lie algebra for the Riemannian symmetric space $H^2 = G/K$.

3. The harmonic map equation

In this section we shall derive a correspondence between harmonic maps from a (simply connected) region of a plane to a hyperbolic plane $H^2 = H^2(1)$ and certain kind of flat connections. (so-called zero curvature representation).

We note that since the harmonic map equation is conformally invariant, it suffices, at least locally, to consider harmonic maps from the Euclidean or Lorentz plane into hyperbolic 2-spaces.

First of all, we shall start with a notational convention.

$$\mathbf{R}^2_\nu = (\mathbf{R}^2(x,y), \ dx^2 + (-1)^\nu dy^2), \ \nu = 0, \ 1. \tag{3.1}$$

$$(u,v) = \begin{cases} (x + \sqrt{-1}y, \ x - \sqrt{-1}y) & \nu = 0 \\ (x+y, \ x-y) & \nu = 1. \end{cases} \tag{3.2}$$

Note that $dx^2 + (-1)^\nu dy^2 = dudv$ holds for both cases.

Throught this note $\mathfrak{D}_\nu$ denotes a simply connected region of $\mathbf{R}^2_\nu$.

The following well-known result is the starting point of our approach.

Proposition 3.1. *A smooth map $\psi : \mathfrak{D}_\nu \to H^2 \subset \mathbf{E}^3_1$ is harmonic if and only if*

$$\psi_{uv} = \rho\psi$$

for some function ρ on $\mathfrak{D}_\nu$.

Let $\psi : \mathfrak{D}_\nu \to H^2$ be a smooth map and $\pi : G \to H^2 \subset \mathfrak{g}$ the Hopf fibering described as before.

Since the hyperbolic 2-space H^2 has the absolutely parallelism, any smooth map ψ has a smooth lift $\Psi : \mathfrak{D}_\nu \to G$ unique upto the right K-action. The pulled-back bundle $\psi^*\ G$ is necessary a trivial bundle $\mathfrak{D}_\nu \times K$. So the gauge transformation group $\mathcal{G}$ of $\psi^*\ G$ is identified with $\mathrm{C}^\infty(\mathfrak{D}_\nu, K)$.

It follows that the mappings ψ and Ψ are related by

$$\psi = \mathrm{Ad}(\Psi)\,\mathbf{i} = \Psi\,\mathbf{i}\,\Psi^{-1}. \tag{3.3}$$

A smooth lift Ψ of ψ satisfying (3.3) is said to be a *framing of* ψ. We wish to describe the harmonicity of ψ in terms of a framing.

Let μ_G be the Maurer-Cartan form of G. Then the pulled-back 1-form $\alpha = \Psi^*\mu_G = \Psi^{-1}d\Psi$ of μ_G by Ψ satiesfies

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0. \tag{3.4}$$

This identity (3.4) implies the integrability condition for the existence of a smooth map $\Psi : \mathfrak{D}_\nu \to G$ such that $\alpha = \Psi^*\mu_G$. (Frobenius theorem). By definition, α is a $\mathfrak{g}$-valued 1-form. The $\mathfrak{g}$-valued 1-form α has type decomposition along the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$;

$$\alpha = \alpha_0 + \alpha_1. \tag{3.5}$$

Here α_0 and α_1 denotes the $\mathfrak{k}$-valued part and $\mathfrak{m}$-valued part respectively. Further α_0 and α_1 are decomposed as follows:

$$\alpha_0 = \alpha_0'\,du + \alpha_0''\,dv, \quad \alpha_1 = \alpha_1'\,du + \alpha_1''dv. \tag{3.6}$$

Define α' and α'' by

$$\alpha' := \alpha_0' + \alpha_1', \quad \alpha'' := \alpha_0'' + \alpha_1''. \tag{3.7}$$

Note that $\alpha'\,du$ and $\alpha''\,dv$ are (1,0)-part and (0,1)-part of α respectively if the index $\nu = 0$. By usual computations we get the following proposition.

Proposition 3.2. *Let $\psi : \mathfrak{D}_\nu \to H^2$ be a smooth map with a framing Ψ. Then ψ is harmonic if and only if*

$$d(*\alpha_1) + [\alpha_0 \wedge *\alpha_1] = 0 \tag{3.8}$$

for $\alpha = \Psi^{-1}d\Psi$.

Let $A^1(\mathfrak{D}_\nu;\mathfrak{g})$ be the space of all $\mathfrak{g}$-valued one-forms and $\mathcal{A}$ the affine space of all connection one-forms on a product bundle $\mathfrak{D}_\nu \times G$. The space $\mathcal{A}$ is an affine space associated to the linear space $A^1(\mathfrak{D}_\nu;\mathfrak{g})$. We shall choose the trivial flat connection as the origin of $\mathcal{A}$, then the space $\mathcal{A}$ is identified with $A^1(\mathfrak{D}_\nu;\mathfrak{g})$. Hereafter we shall identify $\mathcal{A}$ with $A^1(\mathfrak{D}_\nu;\mathfrak{g})$ in this way.

Definition 3.3. A connection $\alpha \in \mathcal{A} = A^1(\mathfrak{D}_\nu;\mathfrak{g})$ is *admissible* provided that

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0, \quad d(*\alpha_1) + [\alpha_0 \wedge *\alpha_1] = 0. \tag{AD}$$

In particular, an admissible connection α is said to be *weakly regular* if $\langle \alpha_1', \alpha_1' \rangle \neq 0$ and $\langle \alpha_1'', \alpha_1'' \rangle \neq 0$ on $\mathfrak{D}_\nu$. The space of all admissible connections on $\mathfrak{D}_\nu$ is denoted by $\mathcal{A}_0$ and corresponding subspace of weakly regular admissible connections by $\mathcal{A}_1$.

It is easily checked that both $\mathcal{A}_0$ and $\mathcal{A}_1$ are invariant under the gauge group action as well as conformal change of $\mathfrak{D}_\nu$.

Proposition 3.4. *Let $\mathcal{H}_0$ be the space of all harmonic maps from $\mathfrak{D}_\nu$ to H^2 and $\mathcal{H}_1 = \{\psi \in \mathcal{H}_0 \mid \langle \psi_u, \psi_u \rangle \neq 0, \langle \psi_v, \psi_v \rangle \neq 0$ on $\mathfrak{D}_\nu\}$. Then there are following bijective correspondeces:*

$$\mathcal{H}_0 \longleftrightarrow \mathcal{A}_0/\mathcal{G}, \quad \mathcal{H}_1 \longleftrightarrow \mathcal{A}_1/\mathcal{G}.$$

Here $\mathcal{G}$ denotes the gauge transformation group of the bundle $\psi^* G = \mathfrak{D}_\nu \times K$. These correspondences are described by $\alpha = \Psi^{-1} d\Psi$ via a framing Ψ of ψ.

We shall call a harmonic map $\psi \in \mathcal{H}_1$, a *weakly regular harmonic map.*

The above proposition describes only a set-theoretical correspondence between the spaces of harmonic maps and the moduli space of admissible connections.

Remark. Here we shall explain a differential geometric interpretation of the weak-regularity for harmonic maps.

(1) Euclidean case:

For any weakly regular harmonic map $\psi \in \mathcal{H}_1$, there exists a spacelike immersion φ which is related by (2.9) unique up to translation (See [6], [12]). The weak-regularity condition is equivalent to the nonexistence of umbilics on φ. Hence the weak-regularity guarantees the existence of umbilic free spacelike nonzero constant mean curvature immersions whose Gauss map is ψ (and hence the existence of isothermal curvature-line coordinates).

(2) Lorentz case:

A branched spacelike surface φ parametrised by asymptotic coordinates (u, v) such that

$$|\partial_u \varphi| \neq 0, \ |\partial_v \varphi| \neq 0$$

is called a *weakly regular surface.* The weak-regularity for harmonic (Gauss) maps corresponds to the weak-regularity for surfaces.

To close this section, we shall introduce the so-called *spectre parameter.*

Definition 3.5. Let $\alpha \in \mathcal{A}$ be a connection. *A loop α_λ of connections through α* is defined by the following rule:

$$\alpha_\lambda = \alpha_0 + \lambda \alpha_1' \, du + \lambda^{-1} \alpha_1'' \, dv, \lambda \ \in S^1 (\text{if } \nu = 0), \ \mathbf{R}^* (\text{if } \nu = 1).$$

The following observation (originally due to Pohlmeyer) is fundamental for our approach.

Proposition 3.6. *A connection $\alpha \in \mathcal{A}$ is admissible if and only if a loop α_λ through α satiesfies*

$$da_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \tag{3.12}$$

for every λ.

It is clear that each α_λ is admissible whenever α is, and α_λ generates the same loop. For every α_λ satisfyning (3.12), there exists a one-parameter family of smooth maps $\Psi_\lambda : \mathfrak{D}_\nu \to G$ depending smoothly on λ such that $\Psi_\lambda^{-1} d\Psi_\lambda = \alpha_\lambda$. Throught this paper, we shall normalise Ψ_λ as $\Psi_\lambda(0,0) \equiv \mathbf{1}$. Such normalised one-parameter family of maps Ψ_λ is called an *extended framing.* To every harmonic maps $\psi : \mathfrak{D}_\nu \to H^2$, there is a naturally associated one-parameter family of harmonic maps $\{\psi_\lambda\}$ such that $\psi_1 = \psi$ parametrised by $\lambda \in S^1$ if $\nu = 0$ and by $\lambda \in \mathbf{R}^*$ if $\nu = 1$. We shall therefore refer to ψ_λ as a *loop of harmonic maps* (*through* ψ).

4. Formulae for immersions

Every harmonic map $\psi : \mathfrak{D}_\nu \to H^2$ corresponds to a gauge class $[\alpha]$ of admissible connections. In this section we give a choice of representative in $[\alpha]$ which is suitable for our purpose–construction of constant mean or positive Gaussian curvature surfaces–.

First, we consider the Lorentz case. ($\nu = 1$).

Let α be a weakly regular admissible connection, then we can choose a global conformal change of coordinates such that

$$\langle \alpha_1', \alpha_1' \rangle = \langle \alpha_1'', \alpha_1'' \rangle \equiv 1/4$$

Under the reparametrisation according to (4.1), any weakly regular connection may be brought to the following *canonical form*:

Proposition 4.1. *Every weakly regular admissible connection is gauge transformed into the following form:*

$$\alpha' = -\frac{1}{4}(\partial_u \phi)\mathbf{i} + \frac{1}{2} e^{-\frac{\phi}{2}\mathbf{i}}\, \mathbf{k}', \tag{4.2}$$
$$\alpha'' = \frac{1}{4}(\partial_v \phi)\mathbf{i} - \frac{1}{2} e^{\frac{\phi}{2}\mathbf{i}}\, \mathbf{k}'.$$

Here ϕ is a smooth function on $\mathfrak{D}_1$ satisfying

$$\phi_{uv} + \sin\phi = 0. \tag{4.3}$$

A weakly regular connection in a form (4.2) is called a *Sine-Gordon connection.*

Corollary 4.2. *Let ϕ be a Sine-Gordon field on $\mathfrak{D}_1$, that is, a solution of* (4.3) *on $\mathfrak{D}_1$. then the equation* (4.4) *defines a weakly regular admissible connection.*

This corolllary says that there exists a bijective correspondence between Sine-Gordon fields and normalised admissible connections.

Next we shall present a (Sym-type) representation formula for a spacelike surface of constant Gaussian curvature 1. Let α be a Sine-Gordon connection, $\alpha_{(t)} = \alpha_{\pm e^t}$ its loop, and $\Psi_{(t)} : \mathfrak{D} \times \mathbf{R} \to G$, the extended framing for $\alpha_{(t)}$. Then we define a one-parameter family of smmoth maps into $\mathbf{E}^3_1$ by

$$\varphi_{(t)} = 2\frac{\partial}{\partial t}\Psi_{(t)} \cdot \Psi_{(t)}^{-1}. \tag{4.4}$$

By a straightforward computation, we get the following

Proposition 4.3 (Sym-type formula). *Let $\psi : \mathfrak{D}_1 \to H^2$ be a weakly regular harmonic map and $\varphi_{(t)}$ as defined by* (4.4). *Then $\varphi_{(t)}$ describes a loop of weakly regular spacelike constant Gaussian curvature* 1 *surfaces. The first and second fundamental forms of each $\varphi_{(t)}$ are as follows :*

$$I_{(t)} = \lambda^2 du^2 + 2\cos\phi du dv + \lambda^{-2} dv^2, \tag{4.5}$$

$$II_{(t)} \equiv II = -2\sin\phi du dv, \tag{4.6}$$

where $\lambda = \pm e^t$. The local coordinates (u,v) is asymptotic coordinates of each $\varphi_{(t)}$. In particular (u,v) is Chevyshev for $t = 0$. The surface $\varphi_{(t)}$ is branched at the points such that $\phi(u,v) \in \pi\mathbf{Z}$. The Gauss mapping of each $\varphi_{(t)}$ is $\psi_{(t)} = \mathrm{Ad}(\Psi_{(t)})\,\mathbf{i}$. Furthermore the principal directions of $\varphi_{(t)}$ are given by $\mathrm{Ad}(\Psi_{(t)})\,\mathbf{j}'$ and $\mathrm{Ad}(\Psi_{(t)})\,\mathbf{k}'$.

Proposition 4.4. *Every weakly regular admissible connection over $\mathfrak{D} = \mathfrak{D}_0$ is gauge transformed into the following form:*

$$\alpha' = -\frac{\sqrt{-1}}{4}(\partial_z \omega)\ \mathbf{i} + \frac{1}{4}(\cosh\frac{\omega}{2}\mathbf{j}' + \sinh\frac{\omega}{2}\mathbf{k}'), \tag{4.7}$$

$$\alpha'' = \frac{\sqrt{-1}}{4}(\partial_{\bar z} \omega)\ \mathbf{i} + \frac{1}{4}(\cosh\frac{\omega}{2}\mathbf{j}' - \sinh\frac{\omega}{2}\mathbf{k}').$$

Here ω is a smooth function on $\mathfrak{D}$ satisfying

$$\omega_{z\bar z} = \frac{1}{4}\sinh\omega. \tag{4.8}$$

We shall call a weakly regular connection in a form (4.7), a *Sinh-Laplace connection.*

Corollary 4.5. *Let ω be a Sinh-Laplace field on $\mathfrak{D}$, that is, a solution of* (4.8). *Then the equation* (4.7) *defines a weakly regular connection.*

As Corollary 4.2, There exists a bijective correspondence between Sinh-Laplace fields and normalised admissible connections.

Proposition 4.6 (Bobenko-type formula). *Let $\psi : \mathfrak{D} \to H^2$ be a weakly regular harmonic map and $\Psi_{(t)} := \Psi_{e^{2\sqrt{-1}t}} : \mathfrak{D} \times S^1 \to G$, the extended framing of ψ. Then*

$$\varphi_{(t)} = -2\{\frac{\partial}{\partial t}\Psi_{(t)} \cdot \Psi_{(t)}^{-1}\} + \psi_{(t)}, \quad \psi_{(t)} = \mathrm{Ad}(\Psi_{(t)})\ \mathbf{i} \tag{4.9}$$

describes a loop of spacelike umbilic free constant mean curvature $1/2$ immersions. The first and second fundamental forms of each $\varphi_{(t)}$ are

$$I_{(t)} \equiv I = e^{\omega} dz d\bar z, \tag{4.10}$$

$$II_{(t)} = \frac{1}{2}(e^{\omega} + \cos(2t))dx^2 - \sin(2t)dxdy + \frac{1}{2}(e^{\omega} - \cos(2t))dy^2.$$

The local coordinate system (x, y) defined by $z = x + \sqrt{-1}y$ is an isothermal coordinate system of each $\varphi_{(t)}$. In particular (x, y) is principal for $t = 0$. The Gauss mapping of each $\varphi_{(t)}$ is $\psi_{(t)}$.

Remark. Recently, T. Taniguchi [12] has obtained a Bobenko-type formula for spacelike constant mean curvature surfaces. His formula is slightly different from ours (4.9). Our formula describes a loop of such surfaces.(Compare Example 2.3 in [6] and Example 4.6 in [12]).

5. Harmonic maps of finite-type

In Section 3, we have introduced the spectre parameter for connection one forms. And we presented the so-called zero-curvature representation for the harmonic map equation (Proposition 3.6). A loop of admissible connections takes value in an appropriate twisted loop algebra. In this section we shall only describe the Lorentz case for simplicity. First of all, we shall explain the following materials we need.

Let $\tilde{\mathfrak{g}}^{\mathbf{C}} := \mathfrak{g}^{\mathbf{C}}[\lambda, \lambda^{-1}] = \{\xi(\lambda) = \sum_{a=m}^{n} \lambda^a \xi_a \mid \xi_a \in \mathfrak{g}^{\mathbf{C}}\}$ be the linear space of all Laurent polynomials with cofficients in the complexification $\mathfrak{g}^{\mathbf{C}}$ of $\mathfrak{g}$. We can naturally extend the bracket $[\ ,\ ]$ of $\mathfrak{g}$ to $\tilde{\mathfrak{g}}^{\mathbf{C}}$ as follows:

$$[\xi(\lambda),\ \eta(\lambda)] = [\sum_a \xi_a \lambda^a, \sum_b \eta_b \lambda^b] = \sum_a \sum_b [\xi_a, \eta_b] \lambda^{a+b}.$$

With respect to this bracket, $\tilde{\mathfrak{g}}^{\mathbf{C}}$ forms an infinite dimensional Lie algebra. Also the involution τ of $\mathfrak{g}$ is naturally extend to $\tilde{\mathfrak{g}}^{\mathbf{C}}$.

$$\tau(\xi)(\lambda) = \tau(\sum_a \xi_a \lambda^a) = \sum_a \tau(\xi_a)(-1)^a \lambda^a.$$

The extended involution τ gives the eigenspace decomposition of $\tilde{\mathfrak{g}}^{\mathbf{C}}$. We denote $\tilde{\mathfrak{g}}^{\mathbf{C}}_\tau$, the $(+1)$-eigenspace of $\tilde{\mathfrak{g}}^{\mathbf{C}}$. The Lie algebra $\tilde{\mathfrak{g}}^{\mathbf{C}}_\tau$ is called *the twisted polynomial loop algebra of* $\mathfrak{g}^{\mathbf{C}}$. The twisted polynomial loop algebra $\tilde{\mathfrak{g}}^{\mathbf{C}}_\tau$ is explicitely given by

$$\tilde{\mathfrak{g}}^{\mathbf{C}}_\tau = \{\xi(\lambda) = \sum_a \xi_a \lambda^a \mid \xi_{2l} \in \mathfrak{k}^{\mathbf{C}},\ \xi_{2l+1} \in \mathfrak{m}^{\mathbf{C}}\}. \tag{5.1}$$

In order to solve the zero curvature equation (3.12), we need to restrict ourselves to suitable real forms of $\tilde{\mathfrak{g}}^{\mathbf{C}}_\tau$. We define the conjugation operator ι as follows:

$$\iota(\xi)(\lambda) = \overline{\xi(\bar{\lambda})}. \tag{5.2}$$

It is easy to see that $\xi(\lambda) \in \tilde{\mathfrak{g}}^{\mathbf{C}}_\tau$ is invariant under the involution ι if and only if $\lambda \in \mathbf{R}^*$.

We denote $\tilde{\mathfrak{g}}_\tau$ the real form of $\tilde{\mathfrak{g}}^{\mathbf{C}}_\tau$ corresponding to the conjugation ι. This real form is explicitely given by:

$$\tilde{\mathfrak{g}}_\tau = \{\xi(\lambda) = \sum_a \xi_a \lambda^a : \mathbf{R}^* \to \mathfrak{g} \mid \xi_{2l} \in \mathfrak{k},\ \xi_{2l+1} \in \mathfrak{m}\}. \tag{5.3}$$

To get a loop of harmonic maps from $\mathfrak{D}_1$ into H^2, we have to solve the zero-curvature equation (3.6) over the real form $\tilde{\mathfrak{g}}_\tau$ as above.

We restrict our interest to the following finite dimensional linear subspace $\Lambda^N_\tau \mathfrak{g}$ of $\tilde{\mathfrak{g}}_\tau$.

$$\Lambda^N_\tau \mathfrak{g} = \{\xi(\lambda) = \sum_{a=-N}^{N} \xi_a \lambda^a \in \tilde{\mathfrak{g}}_\tau\}. \tag{5.4}$$

Now we shall introduce the Lax operators L'_N, L''_N of $\Lambda^N_\tau\mathfrak{g}$.

$$L'_N(\xi)(\lambda) = \frac{1}{2}\xi_{N-1} + \lambda\xi_N,\ L''_N(\xi)(\lambda) = -\frac{1}{2}\xi_{1-N} - \lambda^{-1}\xi_{-N}. \tag{5.5}$$

Evidentrly, the Lax operators L'_N, L''_N preserve $\Lambda^N_\tau\mathfrak{g}$ if N is odd.

Further, we shall introduce the notion of normal admissible loop.

Definition 5.1. An element $\xi(\lambda) \in \tilde{\mathfrak{g}}_\tau$ is a *normal admissible loop* if $\xi(\lambda) \in \Lambda^N_\tau\mathfrak{g}$ for odd N and satisfy the following condition.

$$\xi_{-N} = -\frac{1}{2}e^{\omega_0}\,\mathbf{k}',\ \ \xi_N = -\frac{1}{2}e^{-\omega_0}\,\mathbf{k}'$$

The following is our main result.

Theorem 5.2. *Let $\xi(\lambda) \in \Lambda^N_\tau\mathfrak{g}$ be a normal admissible loop. Then the following Lax equations have a unique smooth solution χ^λ over a region of $\mathbf{R}^2_1$.*

$$\frac{\partial}{\partial u}\chi^\lambda = [\chi^\lambda,\ L'_N(\chi^\lambda)],\ \ \frac{\partial}{\partial v}\chi^\lambda = [\chi^\lambda,\ L''_N(\chi^\lambda)].$$

Furthermore, the loop α_λ defined by

$$\alpha_\lambda := L'_N(\chi^\lambda)\ du + L''_N(\chi^\lambda)\ dv$$

is a loop of Sine-Gordon connections.

Sketch of the proof. Define the vector fields Q'_N and Q''_N on $\Lambda^N_\tau\mathfrak{g}$ by

$$Q'_N(\eta(\lambda)) = [\eta(\lambda),\ L'_N(\eta(\lambda))],\ \ Q''_N(\eta(\lambda)) = [\eta(\lambda),\ L''_N(\eta(\lambda))].$$

Direct calculations show that the vector fields Q'_N and Q''_N are commutes. Since Q'_N and Q''_N are vector fields defined on a finite dimensional linear space $\Lambda^N_\tau\mathfrak{g}$, there exist local flows F'_N and F''_N of Q'_N and Q''_N respectively. By the commutativity of Q'_N and Q''_N, we can define the following smooth map χ^λ

$$\chi^\lambda(u,v) = F'_N(u) \circ F''_N(v).$$

This mapping χ^λ is a desired one. $\square$

We call a harmonic map constructed by the solution χ^λ, a *harmonic map of finite type*. Using a finite-type harmonic map ψ_λ constructed by the solution χ^λ, we can construct a loop of spacelike constant positive curvature surfaces which are not graphs.

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