数理解析研究所講究録
999 巻 1997 年 172-173

November 14, 1996

A REMARK ON INVARIANT DIFFERENTIAL OPERATORS ON PREHOMOGENEOUS VECTOR SPACES

AKIHIKO GYOJA

Department of Fundamental Sciences, Faculty of Integrated Human Studies,
Kyoto University, Kyoto 606-01, Japan
e-mail : gyoja@ math.h.kyoto-u.ac.jp

0. Let G be a complex reductive group acting linearly and prehomogeneously on a vector space $V=\mathbb{C}^n$. Let $f\in\mathbb{C}[V]$ be a relative invariant, and $f^\vee$ a relative invariant on the dual G-module $V^\vee$ such that $f\otimes f^\vee$ is absolutely G-invariant. Let $\mathcal{A}\subset\mathbb{C}[x_1,\cdots,x_n,\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}]$ be the subalgebra generated by $f(x)$ and $f^\vee(\partial_x)$. H.Rubenthaler asked *whether the Euler operator* $\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}$ *belongs to* $\mathcal{A}$ *or not.* At present I can not answer this question. Instead I will give an intimately related result. See Theorem below.

1. Put $V:=\mathbb{C}^n=\{(x_1,\cdots,x_n)\}$ and let $V^\vee:=\{(y_1,\cdots,y_n)\}$ be its dual. For functions g, h on $V\times V^\vee$, their Poisson bracket is defined by

$$\{g,h\}=\sum_{i=1}^n\left(\frac{\partial g}{\partial y_i}\frac{\partial h}{\partial x_i}-\frac{\partial g}{\partial x_i}\frac{\partial h}{\partial y_i}\right).$$

If $g=g(y)$ and $h=h(x)$, we have

$$\{g,h\}=\left(h_i\frac{\partial}{\partial y_i}\right)g,$$

where $h_i:=h_{x_i}$. Thus the operation $g\mapsto\{g,h\}$ can be regarded as the polarization $y\to h$ (cf. [W]). Therefore, if $g(y)=y_1^{d_1}\cdots y_k^{d_k}$ ($d_i\in\mathbb{Z}_{>0}$), we have

$$\begin{aligned}
&\{\cdots\{\{g,\overbrace{h\},h\},\cdots,h}^{|d|-1}\}\\
(1)\qquad &=\sum_{j=1}^k\binom{|d|-1}{d_1,\cdots,d_j-1,\cdots,d_k}d!\cdots d_k!h_1^{d_1}\cdots h_j^{d_j-1}\cdots h_k^{d_k}\cdot y_j\\
&=(|d|-1)!\sum_{j=1}^n g_{y_j}(\operatorname{grad}h)y_j,
\end{aligned}$$

where $|d|=d_1\cdots+d_k$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Example. If $g(y) = y_1^2 y_2^3$, the coefficient of y_1 can be obtained as follows. Consider any path from $y_1^2 y_2^3$ to y_1 in the diagram below. Multiply all the weights attached to the edges. Then sum up them for all such paths.

$$\begin{array}{ccccccc}
y_1^2y_2^3 & \xrightarrow{3h_2} & y_1^2y_2^2 & \xrightarrow{2h_2} & y_1^2y_2 & \xrightarrow{h_2} & y_1^2 \\
\downarrow{\scriptstyle 2h_1} & & \downarrow{\scriptstyle 2h_1} & & \downarrow{\scriptstyle 2h_1} & & \downarrow{\scriptstyle 2h_1} \\
y_1y_2^3 & \xrightarrow{3h_2} & y_1y_2^2 & \xrightarrow{2h_2} & y_1y_2 & \xrightarrow{h_2} & y_1 \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_1} & & \\
y_2^3 & \xrightarrow{3h_2} & y_2^2 & \xrightarrow{2h_2} & y_2 & &
\end{array}$$

2. Let G be a complex reductive group acting linearly and prehomogeneously on V. Then G acts prehomogeneously on $V^\vee$ as well (cf. [G]). Let f (resp. $f^\vee$) be a relative G-invariant polynomial function on V (resp. $V^\vee$) such that $f \otimes f^\vee$ is absolutely invariant. Then f and $f^\vee$ are homogeneous of the same degree (cf. [G]). Put $d := \deg f = \deg f^\vee$. By (1), we have

$$\sigma(\{\cdots[[f^\vee(\partial), \overbrace{f(x)], f(x)], \cdots, f(x)]}^{d-1}\}) = (d-1)! \sum_{j=1}^{n} f^\vee_{y_j}(\operatorname{grad} f(x)) y_j, \tag{2}$$

where σ denotes the principal symbol. (Note that $\sigma[\ ,\ ] = \{\sigma(\ ), \sigma(\ )\}$.)

It is known that $f^\vee(\partial) f^{s+1} = b(x) f^s$ with some $b(s) = b_0 s^d + b_1 s^{d-1} + \cdots = b_0 \prod_{j=1}^{d} (s + \alpha_j) \in \mathbb{C}[s]$, where $b_0 \neq 0$. Cf. [G, 1.7].

3. Theorem. *If the Hessian of* $\log f$ *is not identically zero, then*

(3)
$$[\cdots[[f^\vee(\partial), \overbrace{f(x)], f(x)], \cdots, f(x)]}^{d-1} = (d-1)! b_0 f(x)^{d-2} \left(\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + \frac{b_1}{b_0} - d \right).$$

Proof. Since $(\operatorname{grad} \log f^\vee) \circ (\operatorname{grad} \log f)$ is the identity map [G, 1.18, (3)], and since $f^\vee(\operatorname{grad} \log f(x)) = b_0 f(x)^{-1}$ [G, 1.8, (1)], we can see that the right hand side of (2) is equal to the principal symbol of the right hand side of (3). Hence we get (3) up to constant. Thus it is enough to apply both members of (3) to f^{s+1} and then to see the results are the same. From the left member, we get $\delta^{d-1} b(s) \cdot f(x)^{s+d-1}$, where $\delta\varphi(s) = \varphi(s+1) - \varphi(s)$. It is easy to see the right member yields the same. □

Remark. In the above theorem, if V is an irreducible G-module, then we have $b(s) = (-1)^d b(-s - \frac{n}{d} - 1)$, $\{\alpha_j\}_j = \{\frac{n}{d} + 1 - \alpha_j\}_j$ and hence

$$\frac{b_1}{b_0} = \frac{\dim V + \deg f}{2}.$$

References

[G] A.Gyoja, *Theory of prehomogeneous vector spaces without regularity condition*, Publ. RIMS **27** (1991), 861 922.

[W] H.Weyl, *The classical groups, 2nd edition* (1946), Princeton University Press.