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COHOMOLOGICAL RADON TRANSFORM AND MIXED TWISTED EHRHART POLYNOMIAL

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Let $f_1, \dots, f_r$ be Laurent polynomials in $\mathbf{C}[x_1^{\pm}, \dots, x_n^{\pm}]$, $\alpha_1, \dots, \alpha_r$ $\beta_1, \dots, \beta_n$ be rational numbers in $\frac{1}{m}\mathbf{Z}$. For a multivalved function $\eta = f_1^{-\alpha_1} \cdots f_r^{-\alpha_r} x_1^{\beta_1} \cdots x_n^{\beta_n}$, the corresponding local system on $U = \mathbf{C}^N - \{f_1 \cdots f_r \neq 0\}$ is denoted by $\mathbf{C}(\eta)$. In this note, we study the Hodge structure of $H_c^i(U, \mathbf{C}(\eta))$. Let $\mathbf{T}(x)$ and $\mathbf{T}(t)$ be n-dimensional tori $\operatorname{Spec}\mathbf{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ and $\operatorname{Spec}[t_1^{\pm}, \dots, t_n^{\pm}]$ respectively and $\pi : \mathbf{T}(t) \to \mathbf{T}(x)$ be the morphism defined by $x_i = t_i^m$ $(i = 1, \dots, n)$. Define finite coverings Y and $\tilde{Y}$ of $\mathbf{T}(x)$ and $\mathbf{T}(t)$ by $Y = Spec[x_1^{\pm}, \dots, x_n^{\pm}, w_1^{\pm}, \dots, w_r^{\pm}]/(w_i^m - f_i)_{i=1,\dots,r}$ and $\tilde{Y} = Y \times_{\mathbf{T}(x)} \mathbf{T}(t)$ respectively. The natural homomorphisms are denoted by $\phi : Y \to \mathbf{T}(x)$ and $\tilde{\phi} : \tilde{Y} \to \mathbf{T}(t)$ respectively. For elements $g = (g_1, \dots, g_r) \in \mu_m^r$ and $h = (h_1, \dots, h_n) \in \mu_m^n$, we define automorphisms of Y and $\mathbf{T}(t)$ over $\mathbf{T}(x)$ by

$$\mu_m^r \ni g \mapsto (w_i \mapsto g_i w_i) \in \operatorname{Aut}(Y/\mathbf{T}(x))$$
$$\mu_m^r \ni h \mapsto (t_j \mapsto h_j t_j) \in \operatorname{Aut}(\mathbf{T}(t)/\mathbf{T}(x))$$

respectively. Via the above homomorphisms, μ_m^r and μ_m^n are identified with subgroups of $\operatorname{Aut}(Y/\mathbf{T}(x))$ and $\operatorname{Aut}(\mathbf{T}(t)/\mathbf{T}(x))$ respectively. The group $\mu_m^r \times \mu_m^n$ is identified with a subgroup of $\operatorname{Aut}(\tilde{Y}/\mathbf{T}(x))$ by fiber product. For characters χ_1 and χ_2 of μ_m^r and μ_m^n, we define $(\phi_! \mathbf{C})(\chi_1^{-1})$, $\pi_! \mathbf{C}(\chi_2)$ and $\tilde{\phi}_! \mathbf{C}(\chi_1^{-1}, \chi_2)$ through the actions defined as before. Then we have

$$(\tilde{\phi}_!)(\chi_1^{-1}, \chi_2) \simeq (\phi_! \mathbf{C})(\chi_1^{-1}) \otimes (\pi_! \mathbf{C})(\chi_2).$$

From now on, we assume the following conditions:

(*) The variety $D_i = \{x \in \mathbf{T}(x) \mid f_i = 0\}$ is irreducible and $D_i \neq D_j$.
(**) $\alpha_1, \dots, \alpha_r, \alpha_1 + \cdots + \alpha_r$ are not an integers.

Let χ_1, χ_2 be characters corresponding to $(a_1, \dots, a_r) \in (\mathbf{Z}/m)^r$ and $(b_1, \dots, b_n) \in (\mathbf{Z}/m)^n$, where $a_i = m\alpha_i$ and $b_i = m\beta_i$. If we define $Y^0 = \phi^{-1}(U)$ and $\phi^0 = \phi \mid_{Y^0}$, then under the above conditions (*) and (**), we have $(\phi_! \mathbf{C})(\chi_1^{-1}) = j_!(\phi_!^0 \mathbf{C})(\chi_1^{-1})$, where j is the natural inclusion $j : U \to \mathbf{T}(x)$. Therefore we have an isomorphism $H_c^i(U, \mathbf{C}(\eta)) \simeq H_c^i(\mathbf{T}(x), (\tilde{\phi}_! \mathbf{C})(\chi_1^{-1}, \chi_2))$.

We compute $(\phi_!\mathbf{C})(\chi_1^{-1})$ via a family of Fermat hypersurfaces. Let $\mathbf{T}(u) = \operatorname{Spec}\mathbf{C}[u_1^{\pm},\dots u_r^{\pm}]$ and $\mathcal{Y}$ be a hypersurface of $\mathbf{T}(u)\times\mathbf{T}(x)$ defined by

$$\mathcal{Y} : u_1^m f_1(x)+\cdots+u_r^m f_r(x)=1.$$

The composite of the natural inclusion $\mathcal{Y}\to\mathbf{T}(u)\times\mathbf{T}(x)$ and the second projection $\mathbf{T}(u)\times\mathbf{T}(x)\to\mathbf{T}(x)$ is denoted by $\psi:\mathcal{Y}\to\mathbf{T}(x)$. This is a family of Fermat hypersurfaces parameterized by $(x_1,\dots,x_n)\in\mathbf{T}(x)$. We define Fermat hypersurface F^{r-1} by $F^{r-1}:\xi_1^m+\cdots+\xi_r^m=1$ in $\mathbf{T}(\xi)=\operatorname{Spec}\mathbf{C}[\xi_1^{\pm},\dots,\xi_r^{\pm}]$. The group μ_m^r acts on F^r by

$$\mu_m^r\ni g\mapsto(\xi_i\mapsto g_i\xi_i)\in\operatorname{Aut}(F^{r-1})$$

Through this action, we define $H^{r-1}(F^{r-1},\mathbf{C})(\chi_1)$ for a character χ_1 of μ_m^r.

Proposition 1. *Under the condition (*) and (**), we have a motivic isomorphism:*

$$(R^i\psi_!\mathbf{C})(\chi_1)\simeq\begin{cases}(\phi_!\mathbf{C})(\chi_1)\otimes H^{r-1}(F^{r-1},\mathbf{C}) & (i=r-1)\\ 0 & (i\neq r-1).\end{cases}$$

Proof. Let $\mathcal{Y}^0$ be the inverse image $\psi^{-1}(U)$ of U and ψ^0 be the restriction of ψ to $\mathcal{Y}^0$. First we prove

$$(R^i\psi_!^0\mathbf{C})(\chi_1)\simeq\begin{cases}(\phi_!^0\mathbf{C})(\chi_1)\otimes H^{r-1}(F^{r-1},\mathbf{C}) & (i=r-1)\\ 0 & (i\neq r-1).\end{cases}\tag{1}$$

We define an isomorphism ν between $\mathcal{Y}^0\times_U Y^0$ and $F^{r-1}\times Y^0$ by

$$\nu:\mathcal{Y}^0\times_U Y^0\ni(u_i,w_i)\mapsto(\xi_i=u_iw_i,w_i)\in F^{r-1}\times Y^0.$$

Then if we define a homomorphism

$$\mu:\mu_m^r\times\mu_m^r\ni(g,h)\mapsto(gh,h)\in\mu_m^r\times\mu_m^r,$$

the following diagram commutes.

$$\begin{array}{ccc}\mathcal{Y}^0\times_U\mathcal{Y}^0 & \xrightarrow{(g,h)} & \mathcal{Y}^0\times Y^0\\ \nu\downarrow & & \nu\downarrow\\ F^{r-1}\times Y^0 & \xrightarrow{\mu(g,h)} & F^{r-1}\times Y^0\end{array}$$

We denote ψ_1 and ψ_2 the natural morphisms $\psi_1:\mathcal{Y}^0\times_U Y^0\to U$ and $\psi_2:F^{r-1}\times Y^0\to U$, respectively. Then we have the following isomorphisms:

$$\begin{aligned}(R^i\psi_!^0\mathbf{C})(\chi_1)&\simeq(R^i\psi_{1!}^0\mathbf{C})(\chi_1,1)\\ &\simeq(R^i\psi_{2!}^0\mathbf{C})(\chi_1,\chi_1^{-1})\\ &\simeq H_c^i(F^{r-1},\mathbf{C})(\chi_1)\otimes(\phi_!^0\mathbf{C})(\chi_1^{-1})\end{aligned}$$

By the condition for $a_1,\dots,a_r$, we have $H^i(F^{r-1},\mathbf{C})(\chi_1)=0$ if $i\neq r-1$. Therefore we have the identity (1). Since $(\phi_!\mathbf{C})(\chi_1^{-1})_x=0$ for $x\in D=f_1\cdots f_r=0$, it is enough to prove that $(R^i\psi_!\mathbf{C})(\chi_1)_x=0$ for $x\in D$. But if $f_i(x)=0$, the $1\times\cdots\times\overset{i}{\mu_m}\times\cdots\times 1$ acts trivially on the stalk $(R^i\psi_!\mathbf{C})_x$ by proper base change theorem. Therefore we have the required vanishing.

Let $\tilde{\mathcal{Y}}$ be the fiber product $\mathcal{Y}\times_{\mathbf{T}(x)}\mathbf{T}(t)$ and $\mu_m^r\times\mu_m^n$ acts on $\tilde{\mathcal{Y}}$ induced by the actions of μ_m^r and μ_m^n on $\mathcal{Y}$ and $\mathbf{T}(t)$ respectively.

Corollary 2. *Under the same condition of Proposition 1. We have the following isomorphism*

$$H_c^{i+r-1}(\tilde{\mathcal{Y}}, \mathbf{C})(\chi_1, \chi_2) \simeq H_c^i(\mathbf{T}(x), (\tilde{\phi}_!\mathbf{C})(\chi_1^{-1}, \chi_2)) \otimes H^{r-1}(F^{r-1}, \mathbf{C})(\chi_1)$$

Moreover, this isomorphism is compatible with the Hodge filtrations.

Proof. Let $\tilde{\psi} : \tilde{\mathcal{Y}} \to \mathbf{T}(x)$ be the natural morphism. Since

$$\begin{aligned}(R^q\tilde{\psi}_!\mathbf{C})(\chi_1, \chi_2) &\simeq (R^q\psi_!\mathbf{C})(\chi_1) \otimes (\pi_!\mathbf{C})(\chi_2) \\ &\simeq \begin{cases} 0 & (q \neq r-1) \\ (\phi_!\mathbf{C})(\chi_1^{-1}) \otimes (\pi_!\mathbf{C})(\chi_2) \otimes H^{r-1}(F^{r-1}, \mathbf{C})(\chi_1) & (q = r-1), \end{cases}\end{aligned}$$

the spectral sequence

$$E_2^{p,q} = H_c^p(\mathbf{T}(x), (R^q\tilde{\psi}_!\mathbf{C})(\chi_1, \chi_2)) \Rightarrow E_\infty^{p+q} = H_c^{p+q}(\tilde{\mathcal{Y}}, \mathbf{C})(\chi_1, \chi_2)$$

degenerates at E_2-term and we get the theorem.

Remark. So far, we did not assume the resonance condition nor the non-degenerate condition. (See the definitions below.)

Using this identity, we will state the Cohomological interpretation of Radon transform.

Definition. Let $\Delta_1, \ldots, \Delta_r$ be convex polytopes. The polytope $\tilde{\Delta}$ is defined by the convex hull of $(1, 0, \ldots, 0, \Delta_1), \ldots, (0, \ldots, 0, 1, \Delta_r)$ and 0 in $\mathbf{R}^{r+n}$. A sequence of rational number $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_n$ is said to be non resonant if $\partial\tilde{\Delta} \cap (\mathbf{Z}^{r+n} + (\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_n)) = \emptyset$.

If the Newton polygon of $f_1, \ldots, f_r$ is $\Delta_1, \ldots, \Delta_r$ respectively, the Newton polygon of $v_1 f_1 + \cdots + v_r f_r - 1$ is $\tilde{\Delta}$.

Theorem 3 (Cohomological Radon transformation). *Under the same condition of Proposition 1 and assume that*

(1) $u_1^m f_1 + \cdots + u_r^m f_r - 1$ *is non degenerated with respect to its Newton polygon.*
(2) $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_n$ *is non resonant with respect to* $\Delta_1, \ldots, \Delta_r$.

Then $H_c^i(U, \mathbf{C}(\eta)) = 0$ *if* $i \neq n$ *and*

$$H_c^n(U, \mathbf{C}(\eta)) \simeq H_c^{n+r-1}(U, \mathbf{C}(\eta)) \otimes H^{r-1}(F^{r-1}, \mathbf{C})(\chi_1)^{\otimes(-1)}$$

as Hodge structures. Moreover, $H_c^n(U, \mathbf{C}(\eta))$ *is pure of weight* n.

Proof. Under the condition, we have

$$H_c^{i+r-1}(\tilde{\mathcal{Y}}, \mathbf{C})(\chi_1, \chi_2) \simeq H_c^i(U, \mathbf{C}(\eta)) \otimes H^{r-1}(F^{r-1}, \mathbf{C})(\chi_1)$$

We rewrite

$$u_1^m f_1 + \cdots + u_r^m f_r - 1 = u_1^m (f_1 + (\frac{u_2}{u_1})^m f_2 + \cdots + (\frac{u_r}{u_1})^m f_r) - 1$$

and

$$u_1^{a_1}\cdots u_r^{a_r}t_1^{b_1}\cdots t_n^{b_n}=u_1^{a_1+\cdots+a_r}(\frac{u_2}{u_1})^{a_2}\cdots(\frac{u_r}{u_1})^{a_r}t_1^{b_1}\cdots t_n^{b_n}.$$

Now we introduce a set of new variables $s_2=\frac{u_2}{r_1},\dots,s_r=\frac{u_r}{u_1}$. Then we apply the non resonance condition for $f=f_1+s_2f_2+\cdots+s_rf_r$ and $(\alpha_1+\cdots+\alpha_r,\alpha_2,\dots,\alpha_r,\beta_1,\dots,\beta_n)$. The Newton polygon $\Delta(f)$ of f in $\mathbf{R}^{r-1}\times\mathbf{R}^n$ is the convex hull of $(0,\dots,0,\Delta_1),(1,0,\dots,0,\Delta_2),(0,\dots,0,1,\Delta_r)$. Therefore the non resonance condition for $(\alpha_1+\cdots+\alpha_r,\alpha_2,\dots,\alpha_r,\beta_1,\dots,\beta_n)$ with respect to $\Delta(f)$ is expressed as

$$\hat{\Delta}\cap(\mathbf{Z}^{r+n}+(\alpha_1+\cdots+\alpha_r,\alpha_2,\dots,\alpha_r,\beta_1,\dots,\beta_n))=\emptyset,$$

where $\hat{\Delta}$ is the convex hull of $(1,\Delta(f))$ and 0. It is easy to see that $\hat{\Delta}$ is the convex hull of $(1,0,\dots,0,\Delta_1),(1,1,0,\dots,0,\Delta_2),\dots,(1,0,\dots,0,1,\Delta_r)$ and 0. Bye the linear change of base, $(p_1,\dots p_r,q_1,\dots q_n)\mapsto(p_1-p_2\cdots-p_r,p_2,\dots,p_r,q_1,\dots,q_n)$, $\hat{\Delta}$ maps onto $\tilde{\Delta}$ and $(\alpha_1+\cdots+\alpha_r,\dots,\alpha_r,\beta_1,\dots,\beta_n))$ maps to $(\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_n)$. As a consequence, it is equivalent to the resonance condition for $\Delta_1,\dots,\Delta_r$ and $(\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_n)$.

Remark. In physics, the dummy variables $s_2,\dots,s_r$ are called ghost field.

Definition (Mixed twisted Ehrhart polynomial). Let $E(\Delta_1,\dots,\Delta_r,\alpha,\beta,t)$ be the formal power series defined by

$$E(\Delta_i,\alpha,\beta,t)=[\sum_{k=0}^{\infty}\#\{k\tilde{\Delta}\cap(\mathbf{Z}^{r+n}+(\alpha,\beta))\}](1-t)^{r+n-1}.$$

In the case where $r=1$, we have the following theorem.

Theorem 4.

(1) *(α_i) is non-resonant if and only if $l(j\Delta,\alpha_i)=l^*(j\Delta,\alpha_i)$ for all j.*
(2) *If (α_i) is non-resonant, $H^i(Z_F,\mathbf{C})(\alpha_i)$ is of pure of weight n and the twisted Ehrhart polynomial coincides with*

$$(-t)^n(\sum_{i=0}^{n}\dim Gr_F^iH^n(Z_f,\mathbf{C})(\alpha_i)t^{-i}).$$

Proof. This is a twisted version of the theorem due to Danilov and Khovanski. For the detail of the proof, see [T] or the paper in preparation.

We return to the case where r is general. Since

$$\#\{k\hat{\Delta}\cap(\mathbf{Z}^{r+n}+(\alpha_1+\cdots+\alpha_r,\alpha_r,\dots,\alpha_r,\beta))\}=\#\{k\tilde{\Delta}\cap(\mathbf{Z}^{r+n}+(\alpha,\beta))\},$$

$E(\Delta_1,\dots,\Delta_r,\alpha,\beta,t)$ turns out to be a polynomial by Theorem 4. It is called the mixed twisted Ehrhart polynomial. By Theorem 3 and Theorem 4, we have the following theorem.

Theorem 5. *Assume the condition of Theorem 3. Let $H^{p,q}$ be the (p,q)-component of the Hodge decomposition of $H_c^n(U, \mathbf{C}(\eta))$. Then we have*

$$\sum_{p=0}^{n} \dim(H^{p,q}) t^p = E(\Delta_i, \alpha, \beta, t) t^{-e},$$

where $e = < \alpha_1 > + \cdots + < \alpha_r > - < \alpha_1 + \cdots + \alpha_r >$.

Remark. $E(\Delta_i, \alpha, \beta, 1)$ is equal to the Minkowski's mixed volume of $\{\Delta_i\}_i$.

Reference

[T] ワークショップ. 超曲面の特異点、基本群、有限被覆写像 報告集, P116〜118. 1995年. 東工大.