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MIXED HODGE THEORY AND PREHOMOGENEOUS VECTOR SPACES.

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0.0. In [Sat], M.Sato obtained a formula which describes the Fourier transform of a complex power of a relatively invariant polynomial of a prehomogeneous vector space over the real number field, up to an ambiguity of certain exponential factors. In [Gyo2], I formulated conjectures which would give a finite field analogue of the theorem of M.Sato, without any ambiguity. Recently, J.Denef and I jointly have succeeded to prove these conjectures [DG] based on Laumon's product formula [Lau]. The purpose of the present paper is to give an alternative approach based on the mixed Hodge theory. Our main result is Theorem 11, which includes as a special case Conjecture A of [Gyo2] up to an ambiguity of a constant factor of absolute value one. Thus our result is less precise than [DG]. The result of the present paper was obtained around 1986 with help of M.Kashiwara, and thus seems more or less out of date, but I think it is still of some interest. The content was announced and outlined in [Gyo2].

0.1. Our argument roughly goes as follows. Fix an isomorphism $(1-q)^{-1}\mathbf{Z}/\mathbf{Z} \xrightarrow{\sim} \mathrm{Hom}(\mathbf{F}_q^\times, \overline{\mathbf{Q}_l}^\times)$, $\alpha \mapsto \chi_\alpha$, where l is a prime number $\neq p$, and $\overline{\mathbf{Q}_l}$ an algebraic closure of the l-adic number field.

First, we calculate the weight filtration of $\mathcal{F}(Df^\alpha)$ $(\alpha \in \mathbf{Q})$ in the sense of the mixed Hodge theory due to M.Saito, where $D = D_{\mathbf{C}^n} = \mathbf{C}\left\langle x_1, \cdots, x_n, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right\rangle$ and $\mathcal{F}$ is the formal Fourier transformation $x_j \mapsto \sqrt{-1}\frac{\partial}{\partial y_j}$, $\frac{\partial}{\partial x_j} \mapsto \sqrt{-1}y_j$.

Second, using the result of the first step and by the Riemann-Hilbert correspondence, we calculate the weight filtration of $\mathcal{F}^+(j_*\mathbf{C}f^\alpha[n])$, where $\mathcal{F}^+$ is the Sato-Fourier transformation and $j : V \setminus f^{-1}(0) \to V$ is the inclusion mapping.

Third, using the result of the second step and by the reduction modulo p, we calculate the weight filtration of $\mathcal{F}_\psi(j_*f^*L_{\chi_\alpha}[n])$, where $\mathcal{F}_\psi$ is the Deligne-Fourier transformation, and L_χ is the Kummer torsor associated to χ.

Finally, we deduce the desired result from the result of the third step, using the trace formula of Grothendieck, the 'Weil conjecture' proved by Deligne, and a result of Katz-Laumon on $\mathcal{F}_\psi$.

0.2. In this paper, we obtain arithmetic result starting from the mixed Hodge theory. The author expects that, following the opposite course, we might be able to study the mixed Hodge structure starting from the arithmetic result of [DG]. Cf. [Maz], [Kat].

Notation

N1. We denote by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of rational integers, the rational number field, the real number field, and the complex number field, respectively. For a prime number p, $\mathbf{Q}_p$ denotes the p-adic number field and $\mathbf{Z}_p$ its integer ring.

N2. We always assume that a commutative ring, say R, contains 1_R, a homomorphism $R \to R'$ sends 1_R to $1_{R'}$, and 1_R acts trivially on an R-module. For a (not necessarily commutative) ring A with the identity element, $A^\times$ denotes the multiplicative group of the invertible elements. For a commutative ring R, the subscript R of a symbol corresponding to a scheme or a morphism between schemes (resp. a sheaf) means that it belongs to the category of R-schemes (resp. sheaves of R-modules). If $(-)_{R'}$ has been already defined and R is an R'-algebra (i.e., a ring homomorphism $R' \to R$ is given), then $(-)_R := (-)_{R'} \otimes_{R'} R$, unless otherwise stated. If the ring R can be understood from the context, we omit the subscript. If X is an affine scheme over R, $R[X]$ denotes its coordinate ring. If X is an R'-scheme and R is an R'-algebra, $X(R)$ denotes the set of R-rational points.

N3. For a morphism F between two spaces, the sheaf theoretic pull-back F^* is sometimes denoted by F^{-1} to avoid a confusion. For a complex non-singular variety (always assumed to be of pure dimension), let $\mathcal{D} = \mathcal{D}_X$ (resp. $\mathcal{O} = \mathcal{O}_X$) denote the sheaf of algebraic differential operators (resp. regular functions). Let $Mod(\mathcal{D}_X)$ denote the category of $\mathcal{D}_X$-modules, and $Mod_{qc}(\mathcal{D}_X)$ (resp. $Mod_{rh}(\mathcal{D}_X)$) its full subcategory of $\mathcal{D}_X$-modules which are quasi-coherent over $\mathcal{O}_X$ (resp. regular holonomic). Let $D(\mathcal{D}_X)$ denote the derived category of $Mod(\mathcal{D}_X)$, and $D^b_{qc}(\mathcal{D}_X)$ (resp. $D^b_{rh}(\mathcal{D}_X)$) the full subcategory of $D(\mathcal{D}_X)$ consisting of bounded complexes whose cohomologies are quasi-coherent (resp. regular holonomic). Let $Mod(\mathbf{C}_X)$ denote the category of $\mathbf{C}_X$-modules, $D(\mathbf{C}_X)$ its derived category, and $D^b_c(\mathbf{C}_X)$ the full subcategory of $D(\mathbf{C}_X)$ consisting of bounded complexes whose cohomologies are (algebraically) constructible. Let $\mathcal{O}^{an} = \mathcal{O}^{an}_X$ be the sheaf of holomorphic functions on the underlying analytic manifold X^{an} of X, and $\mathcal{M}^{an} := \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}^{an}$ for $\mathcal{M} \in Mod(\mathcal{D}_X)$. For $\mathcal{M} \in Mod_{rh}(\mathcal{D}_X)$, the de Rham complex is defined by $\mathrm{DR}(\mathcal{M}) = \mathrm{DR}_X(\mathcal{M}) := R\underline{\mathrm{Hom}}_{\mathcal{D}^{an}}(\mathcal{O}^{an}, \mathcal{M}^{an})$, where $\underline{\mathrm{Hom}}$ denotes the sheaf of local homomorphisms. Besides, put ${}^p\mathrm{DR}_X = \mathrm{DR}_X[\dim X]$. For a morphism $F : X \to X'$, we define functors ${}^D F_*$, ${}^D F_!$, ${}^D F^*$ ${}^D F^!$ between $D^b_{rh}(\mathcal{D}_X)$ and $D^b_{rh}(\mathcal{D}_{X'})$ so that ${}^p\mathrm{DR}_{X'} \circ {}^D F_* = RF_* \circ {}^p\mathrm{DR}_X$, ${}^p\mathrm{DR}_{X'} \circ {}^D F_! = RF_! \circ {}^p\mathrm{DR}_X$, ${}^p\mathrm{DR}_X \circ {}^D F^* = F^* \circ {}^p\mathrm{DR}_{X'}$, and ${}^p\mathrm{DR}_X \circ {}^D F^! = F^! \circ {}^p\mathrm{DR}_{X'}$. If $f \in \Gamma(X, \mathcal{O}_X)$, and $i : f^{-1}(0) \to X$ is the inclusion mapping, we define the functors ${}^D\psi_f$, ${}^D\phi_{f,1}$ etc. of $D^b_{rh}(\mathcal{D}_X)$ to itself so that ${}^p\mathrm{DR}_X \circ {}^D\psi_f = i_*\psi_f[-1] \circ {}^p\mathrm{DR}_X$, ${}^p\mathrm{DR}_X \circ {}^D\phi_{f,1} = i_*\phi_{f,1}[-1] \circ {}^p\mathrm{DR}_X$, etc. where ψ_f and ϕ_f are the nearby cycle functor and the vanishing cycle functor, respectively [Del1]. If X is an affine variety, we put $D = D_X = \Gamma(X, \mathcal{D}_X)$. Let $Mod(D_X)$ be the category of D_X-modules, which is equivalent to $Mod_{qc}(\mathcal{D}_X)$. An object of $Mod_{qc}(\mathcal{D}_X)$ corresponding to an object of $Mod(D_X)$ is denoted by the script of the same letter. Thus $Mod(D_X) \xrightarrow{\sim} Mod_{qc}(\mathcal{D}_X)$ by $M \mapsto \mathcal{M}$. Using this category equivalence, we define ${}^DF_*$, ${}^DF_!$, ${}^DF^*$, ${}^DF^!$, ${}^D\psi_f$, ${}^D\phi_{f,1}$, supp, etc. for D-modules (satisfying appropriate conditions) as well. We denote the functors in the category of mixed Hodge modules given in [Sai2] by the same symbol with the superscript ^{MH} on the left, e.g., ${}^{MH}F_*$, ${}^{MH}\psi$, etc.

N4. We write, for example $L(\alpha)$ in place of L_α on occasion to avoid multiple indices, or conversely, L_α in place of $L(\alpha)$ on occasion to avoid a confusion with the Tate twist.

1. Monodromy filtration

Let ψ be an object of some abelian category (e.g., a module), N a nilpotent endomorphism of ψ, and $\{W_m\}$ the monodromy filtration associated to N, shifted by $w-1$ ($w \in \mathbf{Z}$). Cf. [Del2]. Namely $\{W_m\}$ is the finite increasing filtration of ψ characterized by the two properties $NW_m \subset W_{m-2}$ and $N^j : \mathrm{gr}^W_{w-1+j}\psi \xrightarrow{\simeq} \mathrm{gr}^W_{w-1-j}\psi$ for any $j \geq 0$. More explicitly,

$$W_{w-1+m} = \sum_{i \geq 0, -m} N^i(\ker N^{m+1+2i}). \tag{1.1}$$

The primitive part of $\mathrm{gr}^W_{w-1+m}\psi$ ($m \geq 0$) is by definition

$$\begin{aligned} P\,\mathrm{gr}^W_{w-1+m}\psi &:= \ker(N^{m+1} \mid \mathrm{gr}^W_{w-1+m}\psi \to \mathrm{gr}^W_{w-3-m}\psi) \\ &= \frac{W_{w-1+m} \cap (N^{m+1})^{-1}W_{w-4-m}}{W_{w-2+m}}. \end{aligned}$$

Let $m > 0$. By (1.1),

$$\begin{aligned} (N^{m+1})^{-1}W_{w-4-m} &= (N^{m+1})^{-1}\sum_{i \geq m+3} N^i(\ker N^{-m-2+2i}) \\ &= \ker N^{m+1} + \sum_{i \geq 2} N^i(\ker N^{m+2i}) \subset W_{w-1+m}. \end{aligned}$$

Hence

$$\begin{aligned} P\,\mathrm{gr}^W_{w-1+m}\psi &= \frac{\ker N^{m+1} + \sum_{i\geq 2} N^i(\ker N^{m+2i})}{\sum_{i \geq 0} N^i(\ker N^{m+2i})} \\ &= \frac{\ker N^{m+1} + \sum_{i>0} N^i(\ker N^{m+2i})}{\ker N^m + \sum_{i>0} N^i(\ker N^{m+2i})} \twoheadrightarrow \frac{\ker N^{m+1} + N\psi}{\ker N^m + N\psi}. \end{aligned}$$

The last surjection is the natural one, which is easily seen to be also injective. Thus we get

$$P\,\mathrm{gr}^W_{w-1+m}\psi = \frac{\ker N^{m+1} + N\psi}{\ker N^m + N\psi} \tag{1.2}$$

for $m > 0$. Similar argument shows (1.2) holds for $m = 0$ as well, where N^0 should be understood as the identity. Let m_0 be the integer such that $N^{m_0} = 0$ and $N^{m_0 - 1} \neq 0$. Here we understand $m_0 = 0$ (resp. $= 1$) if $\psi = 0$ (resp. $\psi \neq 0$ and $N = 0$). If $m_0 > 1$,

$$\psi = \ker N^{m_0} + N\psi \supsetneqq \ker N^{m_0 - 1} + N\psi = \ker N^{m_0 - 1}.$$

Hence in any case,

$$\max\{m \geq 0 \mid P\operatorname{gr}^W_{w-1+m}\psi \neq 0\} = m_0 - 1. \tag{1.3}$$

Here we understand $\max \phi = -1$.

2. $\mathbf{C}[s,t]$-Modules

Let $\mathbf{C}[s,t,t^{-1}]$ be the algebra defined by the relations $ts = (s+1)t$ and $tt^{-1} = t^{-1}t = 1$. Let M be a $\mathbf{C}[s,t]$-module (or more generally, an object of suitable abelian category with a $\mathbf{C}[s,t]$-action) such that

$$M \subset M[t^{-1}] := \mathbf{C}[s,t,t^{-1}] \otimes_{\mathbf{C}[s,t]} M, \quad \text{and} \tag{2.1}$$

$$\text{there exists } 0 \neq b(s) \in \mathbf{C}[s] \text{ such that } b(s)M \subset tM. \tag{2.2}$$

For two integers $k \leq l$, let $b_{k,l}(s) = \prod_{\gamma \in \mathbf{C}}(s-\gamma)^{m(\gamma;k,l)}$ be the monic generator of the ideal $\ker(\mathbf{C}[s] \to \operatorname{End}_{\mathbf{C}}(t^k M/t^l M))$. Given $\gamma \in \mathbf{C}$. Since $b_{kl}(s)$ divides $\prod_{k \leq j < l} b(s+j)$, there exists integers $k_0 < l_0$ such that $b_{kl}(\gamma) \neq 0$ whenever $k \geq l_0$ or $l \leq k_0$. If $k' \leq k \leq k_0$ and $l_0 \leq l \leq l'$, then $b_{k'l'}(s)$ divides $b_{k'k}(s)b_{kl}(s)b_{ll'}(s)$, and $b_{k'k}(\gamma)b_{ll'}(\gamma) \neq 0$. Hence $0 \leq m(\gamma;k',l') \leq m(\gamma;k,l)$. Taking $k_0 \ll 0$ and $l_0 \gg 0$, we may assume that $m(\gamma;k,l)$ is independent of k and l for any $k \leq k_0$ and $l_0 \leq l$. Put

$$m(\gamma) := \lim_{\substack{k \to -\infty \\ l \to +\infty}} m(\gamma;k,l).$$

2.3. *$m(\gamma)$ depends only on (γ mod $\mathbf{Z}$).*

Proof. Since $b_{kl}(s) = b_{k-1,l-1}(s+1)$, $m(\gamma) = m(\gamma - 1)$. □

2.4. *$s - \gamma$ acts on $t^k M/t^l M$ as an automorphism if $k \geq l_0$ or $l \leq k_0$.*

Proof. Take $a(s)$, $c(s) \in \mathbf{C}[s]$ so that $a(s)b_{kl}(s) + c(s)(s-\gamma) = 1$. Then $c(s)(s-\gamma) = 1$ on $t^k M/t^l M$. □

2.5. $\ker((s-\gamma)^m \mid t^k M/t^l M)$ *is independent of $m \geq m(\gamma)$, if $k \leq k_0$ and $l \geq l_0$.*

Proof. Let $b_{kl}(s) = (s-\gamma)^{m(\gamma)}d(s)$. For $i > 0$, take $a(s)$, $c(s) \in \mathbf{C}[s]$ so that $a(s)(s-\gamma)^i + c(s)d(s) = 1$. If $(s-\gamma)^{m(\gamma)+i}x = 0$ for some $x \in t^k M/t^l M$, then $0 = a(s)(s-\gamma)^i \cdot (s-\gamma)^{m(\gamma)}x + c(s)d(s) \cdot (s-\gamma)^{m(\gamma)}x = (s-\gamma)^{m(\gamma)}x$. □

2.6. *Let M_1 be a module over a polynomial ring* $\mathbf{Z}[T]$, *and* $M_1 \supset M_2 \supset M_3 \supset M_4$ $\mathbf{Z}[T]$*-submodules. Assume that the* T*-actions on* M_1/M_2 *and* M_3/M_4 *are invertible. Then the natural morphism* $M_2/M_4 \to M_2/M_3 \to M_1/M_3$ *induce isomorphisms* $\ker(T \mid M_2/M_4) \xrightarrow{\simeq} \ker(T \mid M_2/M_3) \xrightarrow{\simeq} \ker(T \mid M_1/M_3)$.

Proof. Consider T-actions on the exact sequences $0 \to M_3/M_4 \to M_2/M_4 \to M_2/M_3 \to 0$ and $0 \to M_2/M_3 \to M_1/M_3 \to M_1/M_2 \to 0$. Then apply the snake lemma. □

By (2.4)–(2.6), we can see that

2.7. $\psi = \psi_\gamma := \ker((s-\gamma)^m \mid t^k M/t^l M)$ does not depend on $m \geq m(\gamma)$, $k \leq k_0$ nor $l \geq l_0$.
Let $N = N_\gamma$ denote the endomorphism of ψ induced by $s - \gamma$.

2.8. If $m(\gamma) > 0$, then $N^{m(\gamma)} = 0$ and $N^m \neq 0$ for $0 \leq m < m(\gamma)$.
The proof of (2.8) is easy and omitted. Applying (1.3) to the above (ψ, N) and understanding $\max \phi = -1$, we get the monodromy filtration $\{W_m\}$ of ψ shifted by $w - 1$, and we get

2.9. $\max\{m \geq 0 \mid P\,\mathrm{gr}^W_{w-1+m}\,\psi \neq 0\} = m(\gamma) - 1$.
(Here we include the case $m(\gamma) = 0$.)

2.10. Remark. If $(\deg b_{k,l}(s))/(l-k)$ is independent of k and l, then $m(\gamma) = \mathrm{card}\{\alpha \in \mathbf{C} \mid b(\alpha) = 0,\ \alpha \equiv \gamma \mod \mathbf{Z}\}$ (including multiplicity).

3. $\mathcal{D}$-MODULES

Let $\mathbf{C}[s]$, $\mathbf{C}[s,t]$, $\mathbf{C}[s,t,t^{-1}]$ be as in §2. Put $\mathcal{D}[s] = \mathcal{D} \otimes_{\mathbf{C}} \mathbf{C}[s]$ etc.

3.1. $\mathcal{D}$-Module $\mathcal{D}[s](f^s\underline{u})$. Let X be a connected non-singular variety over $\mathbf{C}$, $0 \neq f \in \Gamma(X, \mathcal{O}_X)$, $X_0 := X \setminus f^{-1}(0)$, $\mathcal{M}$ a coherent $\mathcal{D}_{X_0}$-module, and $\underline{u} = (u_1, \cdots, u_p)$ a p-tuple of elements of $\Gamma(X_0, \mathcal{M})$. Consider the left $\mathcal{D}_X[s]$-submodule $\mathcal{I}$ of $\mathcal{D}_X[s]^p$ consisting of $(P_1(s), \cdots, P_p(s)) \in \mathcal{D}_X[s]^p$ such that $\sum_{i=1}^p (f^{m-s}P_i(s)f^s)u_i = 0$ holds in $\mathbf{C}[s] \otimes_{\mathbf{C}} \mathcal{M}$ whenever $m \in \mathbf{Z}$ is sufficiently large. Put $\mathcal{N} := \mathcal{D}_X[s]^p/\mathcal{I}$. Denote by $(f^s\underline{u})_i$ the element $((0, \cdots 0, 1, 0, \cdots, 0) \mod \mathcal{I})$, where 1 appears at the i-th place. Put $f^s\underline{u} = ((f^s\underline{u})_1, \cdots, (f^s\underline{u})_p)$. Then $\mathcal{N} = \sum_{i=1}^p \mathcal{D}_X[s](f^s\underline{u})_i$. For $\alpha \in \mathbf{C}$, put $\mathcal{N}(\alpha) := \mathcal{N}/(s-\alpha)\mathcal{N}$ and $f^\alpha\underline{u} = ((f^\alpha\underline{u})_1, \cdots, (f^\alpha\underline{u})_p) := (f^s\underline{u} \mod (s-\alpha)\mathcal{N})$. Then $\mathcal{N}(\alpha) = \sum_{i=1}^p \mathcal{D}_X(f^\alpha\underline{u})_i$. We often write $\mathcal{N} = \mathcal{D}_X[s](f^s\underline{u})$, $\mathcal{N}(\alpha) = \mathcal{D}_X(f^\alpha\underline{u})$, $\sum_{i=1}^p \mathcal{D}_{X_0}u_i = \mathcal{D}_{X_0}\underline{u}$ etc.

3.2. $\mathcal{D}[s,t]$-Module structure. Define a $\mathcal{D}[s,t]$-module structure of $\mathcal{N} = \mathcal{D}_X[s](f^s\underline{u})$ by $t\,(\sum_{i=1}^p P_i(s)(f^s\underline{u})_i) = \sum_{i=1}^p P_i(s+1)f(f^s\underline{u})_i$. Then $\mathcal{N}[f^{-1}]$ has a natural $\mathcal{D}_X[s,t,t^{-1}]$-module structure, $\mathcal{N} \subset \mathcal{N}[f^{-1}]$, and $\mathcal{N}[f^{-1}] = \mathcal{N}[t^{-1}]$.

3.3. b-Function. Assume $\mathcal{D}_{X_0}\underline{u}$ to be holonomic, and let $B(s,\underline{u}) = B_f(s,\underline{u})$ be the monic minimal polynomial of $s \in \mathrm{End}(\mathcal{N}/t\mathcal{N})$. Cf [Kas3].

3.4. Assumption A1. We assume that
(1) *$\mathcal{D}_{X_0}\underline{u}$ is regular holonomic,*
(2) *the zeros of $B(s,\underline{u}) \in \mathbf{C}[s]$ are rational numbers, and*
(3) *there exists $N \in \mathbf{Z}_{>0}$ and a complex of $\mathbf{Z}[N^{-1}]$-sheaves $K_{0,\mathbf{Z}[N^{-1}]}$ on X_0 such that* $\mathrm{DR}(\mathcal{D}_{X_0}\underline{u}) \simeq K_{0,\mathbf{C}}$. (Cf. (N2).)

These assumptions are satisfied if $\mathcal{M}$ is a regular holonomic $\mathcal{D}_{X_0}$-module such that $\mathrm{DR}(\mathcal{D}_{X_0}\underline{u})$ is a locally constant sheaf whose monodromy is finite and defined over $\mathbf{Q}$. Cf. [Gyo3, (5.14)]. (In the subsequent argument, we assume several conditions including the above one. We have in mind an application to the theory of prehomogeneous vector spaces, where all these conditions are satisfied.)

3.5. Vanishing cycle sheaf. Fix $\gamma \in \mathbf{C}$. Applying the construction of §2 to $M = \mathcal{D}_X[s,t](f^s\underline{u}) = \mathcal{D}_X[s](f^s\underline{u})$, we can define $m(\gamma) = m(\gamma,\underline{u})$, ${}^D\psi_{f,\mathrm{e}(\gamma)}(\mathcal{D}_{X_0}\underline{u}) := \psi_\gamma = \psi$, and a nilpotent endomorphism $N = N_\gamma$ of ψ, where $\mathrm{e}(\gamma) := \exp(2\pi\sqrt{-1}\gamma)$. Cf. (2.3). By (3.4, (2)), $m(\gamma) = 0$ unless $\gamma \in \mathbf{Q}$. Put

$$ {}^D\psi_f(\mathcal{D}_{X_0}\underline{u}) := \bigoplus_{\gamma\in\mathbf{Q}/\mathbf{Z}} {}^D\psi_{f,\mathrm{e}(\gamma)}(\mathcal{D}_{X_0}\underline{u}), \text{ and} $$

$$ b^{\exp}(t,\underline{u}) = b_f^{\exp}(t,\underline{u}) = \prod_{\gamma\in\mathbf{Q}/\mathbf{Z}} (t-\mathrm{e}(\gamma))^{m(\gamma)}. $$

Then

(1) *$b^{\exp}(t)$ is the minimal polynomial of the endomorphism T of ${}^D\psi_f(\mathcal{D}_{X_0}\underline{u})$ induced by the s-action on M.* (Cf. (2.8).)

Here $b^{\exp}(x,\underline{u})$ is determined only by $\mathcal{D}_{X_0}\underline{u}$, and is independent of the special choice of the generator system $\underline{u}$. Thus we sometimes write $b^{\exp}(t,\mathcal{D}_{X_0}\underline{u})$ or $b^{\exp}(t,{}^p\mathrm{DR}_{X_0}(\mathcal{D}_{X_0}\underline{u}))$ for $b^{\exp}(t,\underline{u})$. We follow the similar convention for $m(\gamma,\underline{u})$; $m(\gamma,\underline{u}) = m(\gamma,\mathcal{D}_{X_0}\underline{u}) = m(\gamma,{}^p\mathrm{DR}_{X_0}(\mathcal{D}_{X_0}\underline{u}))$. By [Mal], [Kas2], we have

(2) $\mathrm{DR}({}^D\psi_f(\mathcal{D}_{X_0}\underline{u})) = i_*\psi_f(\mathrm{DR}(\mathcal{D}_{X_0}\underline{u}))[-1]$, *and the right hand side is* $i_*\psi_f K_{0,\mathbf{Z}[N^{-1}]}[-1] \otimes_{\mathbf{Z}[N^{-1}]} \mathbf{C}$ *by* (3.4, (3)), *where $i : f^{-1}(0) \to X$ is the inclusion mapping and ψ_f is the nearby cycle functor* [Del1], and

(3) *T corresponds to the Picard-Lefschetz monodromy of $i_*\psi_f K_0$.*

Hence

(4) $b^{\exp}(t,\underline{u}) \in \mathbf{Q}[t]$.

In other words,

(5) *if $\mathrm{e}(\gamma)$ and $\mathrm{e}(\gamma')$ ($\gamma,\gamma' \in \mathbf{Q}$) are conjugate over $\mathbf{Q}$, then $m(\gamma) = m(\gamma')$.*

By (3)

(6) *if $\deg B_{f^n}(s,\underline{u})/n$ is independent of $n \in \mathbf{Z}_{>0}$, $b_f^{\exp}(t,\underline{u})$ has the following simple expression. If $B_f(s,\underline{u}) = \prod(s+\alpha_j)$, then $b_f^{\exp}(t,\underline{u}) = \prod(t-\mathrm{e}(\alpha_j))$.* (Cf. (2.10).)

3.6. Applying (2.9) to ${}^{D}\psi_{f,1}(\mathcal{D}_X(f^\alpha \underline{u})) = {}^{D}\psi_{f,\mathrm{e}(\alpha)}(\mathcal{D}_X \underline{u})$, we get a monodromy filtration $\{W_m\}$ shifted by $w-1$, and we get

$$\max\{m \geq 0 \mid P\,\mathrm{gr}^W_{w-1+m}({}^{D}\psi_{f,1}(f^\alpha \underline{u})) \neq 0\} = m(-\alpha-1)-1.$$

By (3.5, (5)), the right hand side is $m(\alpha)-1$.

4. Mixed Hodge modules

Here we study some (mixed) Hodge modules. As for the Hodge modules (resp. mixed Hodge modules), the basic reference is [Sai1] (resp. [Sai2]). A brief account can be found in [Tan]. We fix a positive integer c throughout §4.

4.1. Locally constant sheaf $H(c)_{\mathbf{Z}}$. Put $\zeta = \zeta_c = \mathrm{e}(1/c)$, and $\Xi(c) := \{d/c \in c^{-1}\mathbf{Z} \mid 0 < d \leq c, (c,d) = 1\}$. For any $\beta \in \Xi(c)$, there is a unique element $\sigma_\beta \in \mathrm{Gal}(\mathbf{Q}(\zeta_c)/\mathbf{Q})$ such that $\sigma_\beta(\mathrm{e}(1/c)) = \mathrm{e}(\beta)$. Then $\mathrm{Gal}(\mathbf{Q}(\zeta_c)/\mathbf{Q}) = \{\sigma_\beta \mid \beta \in \Xi(c)\}$. Define a locally constant sheaf of $\mathbf{Z}$-modules on $\mathbf{C}^\times = \{t \in \mathbf{C} \mid t \neq 0\}$ by

$$H_{c,\mathbf{Z}} = H(c)_{\mathbf{Z}} = \{\sum_{\beta \in \Xi(c)} \sigma_\beta(u) t^\beta \mid u \in \mathbf{Z}[\zeta_c]\}.$$

Here we take the single-valued branches t^β ($\beta \in \Xi(c)$) locally on $\mathbf{C}^\times$ as follows. First take any single-valued branch of $t^{1/c}$ locally on $\mathbf{C}^\times$. Then put $t^{d/c} = (t^{1/c})^d$ in the domain where $t^{1/c}$ is defined. Let T be the generator of $\pi_1(\mathbf{C}^\times)$ defined by the oriented circle $\{\mathrm{e}(t) \mid t : 0 \to 1\}$. Consider the natural action of $\pi_1(\mathbf{C}^\times)$ on the set of single-valued branches of t^β (=monodromy action). Then $T(t^{1/c}) = \zeta_c t^{1/c}$ and $T(t^{d/c}) = \zeta_c^d t^{d/c} = \sigma_{d/c}(\zeta_c) t^{d/c}$. Hence $H(c)_{\mathbf{Z}}$ is well-defined. For any $\gamma \in \Xi(c)$, define a locally constant sheaf of $\mathbf{Z}[\zeta]$-modules on $\mathbf{C}^\times$ by

$$L_{\gamma,\mathbf{Z}[\zeta]} = L(\gamma)_{\mathbf{Z}[\zeta]} := \mathbf{Z}[\zeta] t^\gamma$$

in the same way as above.

4.2. Variation of Hodge structures $(H(c)_{\mathbf{Q}}, F)$ and polarization S. For $\beta \in \Xi(c)$, let $\beta' \in \Xi(c)$ be the element such that $\beta + \beta' \in \mathbf{Z}$. (If $c > 1$, $\beta' = 1 - \beta$. If $c = 1$, $\beta = \beta' = 1$.) Then $\sigma_{\beta'}(u) = \overline{\sigma_\beta(u)}$ ($u \in \mathbf{Z}[\zeta]$), and

$$H(c)_{\mathbf{R}} = \{\sum_{\beta \in \Xi(c)} u_\beta t^\beta \mid u_\beta \in \mathbf{C},\ u_{\beta'} = \overline{u_\beta}\}.$$

Consider the tensor product of sheaves $H(c)_{\mathcal{O}} := H(c)_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathcal{O} = \bigoplus_{\beta \in \Xi(c)} \mathcal{O} t^\beta = \bigoplus_{\beta \in \Xi(c)} \mathcal{D} t^\beta$, where $\mathcal{O} = \mathcal{O}_{\mathbf{C}^\times}$ and $\mathcal{D} = \mathcal{D}_{\mathbf{C}^\times}$. Define a decreasing filtration $\{F^p\}_{p \in \mathbf{Z}}$ of $H(c)_{\mathcal{O}}$ by $F^p = H(c)_{\mathcal{O}}$ if $p \leq 0$ and $= 0$ if $p > 0$. Then $(H(c)_{\mathbf{Q}}, F)$ is a variation of Hodge structures of weight 0 on $\mathbf{C}^\times$. See [Sai1, 5.4] or [Tan, 1.1]. Define a $\mathbf{C}$-bilinear form S on $\sum_{\beta \in \Xi(c)} \mathbf{C} t^\beta$ by $S(t^\beta, t^\gamma) := \delta_{\beta',\gamma}$. Then

$$S(\sum_\beta \sigma_\beta(u) t^\beta, \sum_\gamma \sigma_\gamma(v) t^\gamma) = \sum_\beta \sigma_\beta(u\overline{v}) \in \mathbf{Z}$$

for $u, v \in \mathbf{Z}[\zeta]$, and

$$S(\sum_\beta u_\beta t^\beta, \sum_\beta u_\beta t^\beta) = \sum_\beta u_\beta u_{\beta'} = \sum_\beta |u_\beta|^2$$

for $\sum_\beta u_\beta t^\beta \in H(c)_{\mathbf{R}}$. Hence S gives a polarization of $(H(c)_{\mathbf{Q}}, F)$. Cf. [Tan, 1.3, (p5)] and [Del4, (2.1.15)]. Let $MH(X, \mathbf{Q}, w)^p$ be the category of polarizable Hodge modules of weight w [Sai1, 5.1.6]. Cf. [Tan, 1.3]. Put $\mathcal{H}(c) := (H(c)_{\mathcal{O}}, F, H(c)_{\mathbf{Q}}[1]) \in MH(\mathbf{C}^\times, \mathbf{Q}, 1)^p$, where $\mathcal{O} = \mathcal{O}_{\mathbf{C}^\times}$.

4.3. Assumption A2. Besides (A1), assume further that
(1) $f : X_0 \to \mathbf{C}^\times$ *is smooth, and*
(2) $\mathcal{D}_{X_0}\underline{u}$ *is underlying a pure Hodge module* ${}^{MH}\mathcal{D}_{X_0}\underline{u} \in MH(X_0, \mathbf{Q}, w)^p$.

4.4. Hodge module $\mathcal{M}(c)$. Let $\Delta : X_0 \to X_0 \times X_0$ be the diagonal embedding, and put

$$\mathcal{M}(c) = \mathcal{M}_{X_0}(c) := {}^{MH}\Delta^*({}^{MH}f^*\mathcal{H}(c) \boxtimes {}^{MH}\mathcal{D}_{X_0}\underline{u})[-n],$$

where $n = \dim X$. By [Sai2, 2.25], $\mathcal{M}(c) \in MH(X_0, \mathbf{Q}, w)$. By (3.4, (3)), the underlying perverse sheaf of $\mathcal{M}(c)$ is

$$(f^*H(c)_{\mathbf{Z}} \otimes_{\mathbf{Z}} K_{0,\mathbf{Z}[N^{-1}]}[\dim X]) \otimes_{\mathbf{Z}[N^{-1}]} \mathbf{Q}.$$

Let $j : X_0 \to X$ be the inclusion mapping. By the definition of the category $MHM(X)$ of mixed Hodge modules [Sai2, 4.2.11], ${}^{MH}j_*\mathcal{M}(c) \in MHM(X)$ can be defined. Its underlying $\mathcal{D}_X$-module is $\bigoplus_{\beta\in\Xi(c)} {}^Dj_*(\mathcal{D}_{X_0}(f^\beta\underline{u})) = \bigoplus_{\beta\in\Xi(c)} \mathcal{D}_X(f^{\beta-k}\underline{u}))$ $(k \gg 0)$. Cf. [Gyo3, (5.16, (2))].

4.5. Weight filtration of ${}^{MH}j_*\mathcal{M}(c)$. We can describe the weight filtration of ${}^{MH}j_*\mathcal{M}(c)$ using the description given in the proof of [Sai2, 2.11] as follows. (See also [Sai2, 2.8].)

(1) *For any* $m \in \mathbf{Z}$, *the weight filtration* $W_m({}^{MH}j_*\mathcal{M}(c))$ *is of the form* $\bigoplus_{\beta\in\Xi(c)} W_m({}^Dj_*(\mathcal{D}_{X_0}(f^\beta\underline{u})))$ *with some* $\mathcal{D}_X$*-submodule* $W_m({}^Dj_*(\mathcal{D}_{X_0}(f^\beta\underline{u})))$ *of* ${}^Dj_*(\mathcal{D}_{X_0}(f^\beta\underline{u}))$.

(2) $\max\{m \geq 0 \mid \mathrm{gr}^W_{w+m}({}^Dj_*(\mathcal{D}_{X_0}(f^\beta\underline{u}))) \neq 0\} = m(\beta)$.

Proof. Put $\mathcal{M} = \mathrm{image}({}^{MH}j_!\mathcal{M}(c) \to {}^{MH}j_*\mathcal{M}(c))$. Then $\mathcal{M} \in MH(X, w)^p$. Apply [Sai2, (2.11.10)] to the present situation, using (3.6). □

Note that $m(\beta)$ is independent of $\beta \in \Xi(c)$. (Cf (3.5,(5)).) Put $m(\Xi(c)) := m(\beta)$ $(\beta \in \Xi(c))$. Then

(3) $\max\{m \geq 0 \mid \mathrm{gr}^W_{w+m}({}^{MH}j_*(\mathcal{M}(c))) \neq 0\} = m(\Xi(c))$.

4.6. (The description of the weight filtration given in this and the next paragraphs is due to M.Kashiwara, which I learned from him in 1986. Here I include this result by permission of M.Kashiwara, to whom I am very grateful. In the present paper, (4.6) and (4.7) will not be used.) Let $k \ll 0$ and $l \gg 0$. Put $\mathcal{M}_k := t^k\mathcal{D}_X[s,t](f^s\underline{u})$. Then $\mathcal{M}_k/(s-\beta)\mathcal{M}_k = \mathcal{D}_X(f^{\beta+k}\underline{u}) = {}^Dj_*(\mathcal{D}_{X_0}(f^\beta\underline{u}))$. Put

(1)
$$W'_m({}^Dj_*(\mathcal{D}_{X_0}(f^\beta\underline{u}))) := \text{image}\left(\{u \in \mathcal{M}_k \mid (s-\beta)^m u \in \mathcal{M}_l\} \to \frac{\mathcal{M}_k}{(s-\beta)\mathcal{M}_k}\right).$$

Then

(2)
$$\begin{aligned}\mathrm{gr}^{W'}_m({}^Dj_*(\mathcal{D}_{X_0}(f^\beta\underline{u}))) &= P\,\mathrm{gr}^W_{w+m-2}({}^D\psi_{f,1}(\mathcal{D}_{X_0}(f^\beta\underline{u}))) \quad \text{by (1.2)}\\ &= \mathrm{gr}^W_m({}^Dj_*(\mathcal{D}_{X_0}(f^\beta\underline{u}))) \quad \text{by [Sai2, (2.11.10)]}.\end{aligned}$$

Hence

(3)
$$W_m({}^Dj_*(\mathcal{D}_{X_0}(f^\beta\underline{u}))) = W'_m({}^Dj_*(\mathcal{D}_{X_0}(f^\beta\underline{u}))),$$

and the underlying $\mathcal{D}_X$-module of $W_m({}^{MH}j_*(\mathcal{M}(c)))$ is $\bigoplus_{\beta\in\Xi(c)} W'_m({}^Dj_*(\mathcal{D}_{X_0}(f^\beta\underline{u})))$.

4.7. Microlocal structure of the weight filtration. Let $\pi : T^*X \to X$ be the cotangent bundle of X, and $\mathcal{E} = \mathcal{E}_X$ the sheaf of microdifferential operators on T^*X. Since $\mathcal{E}$ is flat over $\pi^{-1}\mathcal{D}$, we can see

$\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}W_m({}^Dj_*(\mathcal{D}_{X_0}(f^\beta\underline{u})))$ is equal to the right hand side of (4.6, (1)) with $\mathcal{M}_j$ replaced with $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}_j$.

Now let us consider the case where X is a complex non-singular algebraic variety on which a connected algebraic group G acts. Assume that f is relatively G-invariant, and that Λ is an irreducible component of the characteristic variety $\mathrm{ch}(\mathcal{D}_X f^\beta)$ on which G acts prehomogeneously. Let $b_\Lambda(s)$ be the local b-function of f on Λ [SKKO] (cf. [Gyo4, 6.1]). Put $b_\Lambda(s) = \prod_{j=1}^{d_\Lambda}(s+\alpha_j)$ and $w_\Lambda(\beta) := \#\{j \mid \beta+\alpha_j \in \mathbf{Z}\}$. Let us apply (2.9) and (4.6, (2)) to the $\mathcal{E}$-modules appearing in (1). Then we get

(2)
$$0 = W_0 = \cdots = W_{w_\Lambda(\beta)-1} \subsetneq W_{w_\Lambda(\beta)} = \cdots = \mathcal{E}_X f^{\beta-k} := \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}(\mathcal{D}_X f^{\beta-k})$$

on Λ, where $W_m := \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}W_m({}^Dj_*(\mathcal{D}_{X_0}f^\beta))$ for $m \in \mathbf{Z}$. (Note that

(3)
$$\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\,{}^Dj_*(\mathcal{D}_{X_0}f^\beta) = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}(\mathcal{D}_X f^{\beta-k}) \quad (k \ll 0)$$

is simple holonomic on Λ [SKKO], and hence (2) is equivalent to

(4)
$$\mathrm{gr}^W_{w_\Lambda(\beta)} \neq 0$$

on Λ.) In terms of characteristic variety, (2) can be also expressed as

(5)
$$\mathrm{ch}(W_m({}^Dj_*(\mathcal{D}_{X_0}f^\beta))) \supset \Lambda \Leftrightarrow w_\Lambda(\alpha) \leq m,$$

or

(6)
$$\mathrm{ch}(\mathrm{gr}^W_m({}^Dj_*(\mathcal{D}_{X_0}f^\beta))) \supset \Lambda \Leftrightarrow w_\Lambda(\alpha) = m.$$

5. Fourier transformation of D-modules

5.1. Let $V = \mathbf{C}^n$ $(n > 0)$, $V^\vee$ be its dual space, $x = (x_1, \cdots, x_n)$ a linear coordinate of V and $y = (y_1, \cdots, y_n)$ its dual coordinate of $V^\vee$. Then $D_V = \mathbf{C}\left\langle x_1, \cdots, x_n, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right\rangle$ and $D_{V^\vee} = \mathbf{C}\left\langle y_1, \cdots, y_n, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}\right\rangle$. (Cf. (N3).) Define an algebra isomorphism $\mathcal{F} : D_V \to D_{V^\vee}$ by

$$\mathcal{F}(x_j) = \sqrt{-1}\frac{\partial}{\partial y_j} \quad \text{and} \quad \mathcal{F}(\frac{\partial}{\partial x_j}) = \sqrt{-1}y_j.$$

For a D_V-module M, define a $D_{V^\vee}$-module structure on M using this isomorphism $\mathcal{F}$ (cf. [Gyo1, 2.7.1]). We shall denote this $D_{V^\vee}$-module by $\mathcal{F}(M)$, and call it *the Fourier transform* of M.

5.2. Assumption A3. *Let $f \in \mathbf{C}[V]$, $f^\vee \in \mathbf{C}[V^\vee]$, $\Omega = V \setminus f^{-1}(0)$, $\Omega^\vee = V^\vee \setminus f^{\vee -1}(0)$, $j : \Omega \to V$ and $j^\vee : \Omega^\vee \to V^\vee$ be the inclusion mappings. Besides* (A1) *and* (A2), *assume that there exists a simple $D_{\Omega^\vee}$-module $0 \neq M^\vee = M^\vee(\alpha) \in Mod_{rh}(\Omega^\vee)$ such that $\mathcal{F}({}^Dj_*D(f^\alpha \underline{u})) = {}^Dj^\vee_! M^\vee$.*

5.3. Put $\mathcal{F}(W_{w+m}({}^Dj_*D(f^\alpha \underline{u}))) =: M^\vee_m$. Then $M^\vee_m \subset {}^Dj^\vee_! M^\vee$, $j^{\vee -1}M^\vee_m \subset M^\vee$, and

$$(1) \qquad \begin{aligned} &j^{\vee -1}M^\vee_m \neq 0 \Leftrightarrow j^{\vee -1}M^\vee_m = M^\vee \Leftrightarrow j^{\vee -1}({}^Dj^\vee_! M^\vee / M^\vee_m) = 0 \\ &\Leftrightarrow {}^Dj^\vee_! M^\vee / M^\vee_m = 0 \Leftrightarrow M^\vee_m = {}^Dj^\vee_! M^\vee. \end{aligned}$$

(The first and the second lines are equivalent since ${}^Dj^\vee_! M^\vee$ does not have a non-zero quotient supported by $f^{\vee -1}(0)$. Note that ${}^Dj^\vee_! M^\vee \in Mod(D_{V^\vee})$, since $Rj^\vee_!({}^p\mathrm{DR}(M^\vee))$ is a perverse sheaf [BBD, 4.1.3].) Hence

$$(2) \qquad \begin{aligned} &\mathcal{F}(\mathrm{gr}^W_{w+m}({}^Dj_*D(f^\alpha \underline{u})))|\Omega^\vee \neq 0 \\ &\Leftrightarrow j^{\vee -1}(M^\vee_m / M^\vee_{m-1}) \neq 0 \\ &\Leftrightarrow j^{\vee -1}M^\vee_m \neq 0 \quad \text{and} \quad j^{\vee -1}M^\vee_{m-1} = 0 \\ &\Leftrightarrow M^\vee_{m-1} \subsetneq M^\vee_m = {}^Dj^\vee_! M^\vee \\ &\Leftrightarrow W_{w+m-1}({}^Dj_*D(f^\alpha \underline{u})) \subsetneq W_{w+m}({}^Dj_*D(f^\alpha \underline{u})) = {}^Dj_*D(f^\alpha \underline{u}) \\ &\Leftrightarrow m = m(\alpha). \end{aligned}$$

(The second and the third lines are equivalent because of the simplicity of $M^\vee$. The equivalence of the third and the fourth lines follows from (1). The last equivalence follows from (4.5,(2)).)

6. Fourier transformation of sheaves on $\mathbf{C}^n$

6.1. Sato-Fourier transformation. ([BMV], [Bry]). Let notation be as in §4. Let $Z = \{z \in \mathbf{C} \mid \mathrm{Re}(z) \leq 0\}$, $\tilde{i} : Z \to \mathbf{C}$ be the inclusion mapping, $L^+ = \tilde{i}_* \mathbf{C}_Z$,

$pr : V^\vee \times V \to V$ and $pr^\vee : V^\vee \times V \to V^\vee$ be the projections, and $\langle\ \rangle : V^\vee \times V \to \mathbf{C}$ the natural pairing. For $K \in D^b(\mathbf{C}_V)$, put $\mathcal{F}^+(K) := Rpr^\vee_!(pr^*(K) \otimes \langle\ \rangle^* L^+)[n]$, which is called the *Sato-Fourier transform* of K [SKK], [KasSch], [BMV], [Bry], [HK].

6.2. Define $h : \mathbf{C}^\times \times V \to V$ by $h(t,v) = tv$. Let c be a positive integer, $\alpha \in \Xi(c)$, and put

$$D^b_{mon,\alpha}(V) := \{K \in D^b(\mathbf{C}_V) \mid h^*K = L_\alpha \boxtimes K\}.$$

(See (4.1) for $L_\alpha = L_{\alpha,\mathbf{C}}$.) If a subvariety $X \subset V$ is $\mathbf{C}^\times$-stable, we can similarly define $D^b_{mon,\alpha}(X)$.

6.3. Put $\tau^+(\alpha) = R\Gamma_c(\mathbf{C}^\times, L_\alpha \otimes L^+)[1]$. Since $\mathbf{C}^\times \cap Z$ is homeomorphic to $(0,\infty) \times [\frac{\pi}{2}, \frac{3\pi}{2}]$, $\tau^+(\alpha) = H^1_c(\mathbf{C}^\times, L_\alpha \otimes L^+) \simeq \mathbf{C}$.

6.4. Let $Q := \{(v^\vee, v) \in V^\vee \times V \mid \langle v^\vee, v\rangle = 1\}$, $e : Q \to V^\vee \times V$ be the inclusion mapping, consider the natural morphism $\gamma : e_!e^!\mathbf{C}_{V^\vee \times V} \to \mathbf{C}_{V^\vee \times V}$, and let ω be its mapping cone; $\omega := \mathrm{cone}(\gamma)$.

6.5. Radon transformation. For any $K \in D^b_{mon,\alpha}(V)$,

$$\mathcal{F}^+(K) \simeq \mathcal{F}^+(K) \otimes \tau^+ \simeq Rpr^\vee_!(pr^*(K) \otimes \omega)[n] =: \mathcal{R}(K).$$

(This isomorphism is not canonical.) The right member is called *the Radon transform* of K. We omit the proof, since the proof of (7.7) below is essentially the same, and the latter is included in [DG].

6.6. Assumption A4. (1) $K_{0,\mathbf{C}} \in D^b_{mon,\alpha_0}(\Omega)$ *for some* $\alpha_0 \in \mathbf{C}$.
(2) f *and* $f^\vee$ *are homogeneous polynomials.*

6.7. Define a filtration $\{W_m\}$ of the perverse sheaf $Rj_*(f^*L_{-\alpha} \otimes K_{0,\mathbf{C}})[n]$ (cf. [BBD]) by ${}^p\mathrm{DR}\,(W_m({}^Dj_*D_\Omega(f^\alpha\underline{u}))) =: W_m(Rj_*(f^*L_{-\alpha} \otimes K_{0,\mathbf{C}})[n])$. Then

$$\begin{aligned} & m = m(\alpha) \\ & \Leftrightarrow \mathcal{F}(\mathrm{gr}^W_{w+m}({}^Dj_*D_\Omega(f^\alpha\underline{u})))|\Omega^\vee \neq 0 \quad \text{by (5.3, (2))} \\ & \Leftrightarrow \mathcal{F}^+(\mathrm{gr}^W_{w+m}(Rj_*(f^*L_{-\alpha} \otimes K_{0,\mathbf{C}})[n]))|\Omega^\vee \neq 0 \quad \text{by (A4) in (6.6), and [HK]} \\ & \Leftrightarrow \mathcal{R}(\mathrm{gr}^W_{w+m}(Rj_*(f^*L_{-\alpha} \otimes K_{0,\mathbf{C}})[n]))|\Omega^\vee \neq 0 \quad \text{by (6.5).} \end{aligned}$$

7. Fourier transformation of l-adic étale sheave

7.1. Let V, $V^\vee$, $\langle\ \rangle$, f, $f^\vee$ etc. be as in §5 and §6. In this section, we assume that these objects are defined over a finite field $\mathbf{F}_q$. Let $p = \mathrm{char}(\mathbf{F}_q)$, l be a prime number $\neq p$, and for a variety X over $\mathbf{F}_q$, $D^b_c(X, \overline{\mathbf{Q}_l})$ the triangulated category defined by Deligne. See [Del2, (1.1.1)–(1.1.3)] and also [Lau, (0.5)–(0.6)].

7.2. Artin-Schreier torsor. Fix a non-trivial additive character $\psi \in \mathrm{Hom}(\mathbf{F}_q, \overline{\mathbf{Q}_l}^\times)$. Let L_ψ be the Artin-Schreier torsor on $\mathbf{A}^1$. The Frobenius endomorphism Frob_q acts on $L_{\psi,x}$ ($x \in \mathbf{F}_q$) as the multiplication by $\psi(x)$. See [Del3, 1.4].

7.3. Lang torsor. For $\chi \in \mathrm{Hom}(\mathbf{F}_q^\times, \overline{\mathbf{Q}_l}^\times)$, let L_χ be the Kummer torsor of order $q-1$ on $\mathbf{G}_m = \mathbf{A}^1 \backslash \{0\}$. The Frobenius endomorphism Frob_q acts on $L_{\chi,x}$ ($x \in \mathbf{F}_q^\times$) as the multiplication by $\chi(x)$. Fixing an isomorphism $\frac{1}{1-q}\mathbf{Z}/\mathbf{Z} \simeq \mathrm{Hom}(\mathbf{F}_q^\times, \overline{\mathbf{Q}_l}^\times)$, $\alpha \mapsto \chi_\alpha$, we sometimes denote $L_\alpha = L(\alpha)$ for L_{χ_α}. See [Del3, 1.4].

7.4. Deligne-Fourier transformation. ([KL]). Define a functor $\mathcal{F}_\psi : D_c^b(V_{\mathbf{F}_q}, \overline{\mathbf{Q}_l}) \to D_c^b(V_{\mathbf{F}_q}^\vee, \overline{\mathbf{Q}_l})$ by $\mathcal{F}_\psi(-) = Rpr_!^\vee(pr^*(-) \otimes \langle\ \rangle^* L_\psi)[n]$, which is called *the Deligne-Fourier transformation.*

7.5. Define $h : \mathbf{G}_{m,\mathbf{F}_q} \times V_{\mathbf{F}_q} \to V_{\mathbf{F}_q}$ by $h(t,v) = tv$. For $\chi \in \mathrm{Hom}(\mathbf{F}_q^\times, \overline{\mathbf{Q}_l}^\times)$, put

$$D^b_{mon,\chi}(V_{\mathbf{F}_q}) = \{K \in D_c^b(V_{\mathbf{F}_q}, \overline{\mathbf{Q}_l}) \mid h^*K = L_\chi \boxtimes K\}.$$

7.6. Put $\tau(\chi, \psi) = R\Gamma_c(\mathbf{G}_{m,\mathbf{F}_q}, L_\chi \otimes L_\psi)[1]$. Then we get a (non-canonical) isomorphism $\tau(\chi, \psi) = H_c^1(\mathbf{G}_{m,\mathbf{F}_q}, L_\chi \otimes L_\psi) \simeq \overline{\mathbf{Q}_l}$ (cf. [Del3, 4.2]).

7.7. Radon transformation. (Cf. (6.5).) Consider the natural morphism $\gamma : e_! e^! \overline{\mathbf{Q}_l} \to \overline{\mathbf{Q}_l}$. Put $\omega := \mathrm{cone}(\gamma)$. For $K \in D^b_{mon,\chi}(V_{\mathbf{F}_q})$, define its *Radon transform* by $Rpr_!^\vee(pr^*K \otimes \omega)[n] =: \mathcal{R}(K)$. In [DG], we get

$$\mathcal{F}_\psi(K) \simeq \mathcal{F}_\psi(K) \otimes \tau(\overline{\chi}, \overline{\psi}) \simeq \mathcal{R}(K) \text{ (non-canonically).} \tag{1}$$

8. Specialization from $D_c^b(V(\mathbf{C}), \mathbf{C})$ to $D_c^b(V_{\mathbf{F}_q} \overline{\mathbf{Q}}_l)$

8.1. Since $H_{c,\mathbf{Q}_l} = H_{c,\mathbf{Z}_l} \otimes \mathbf{Q}_l$ and $K_{0,\mathbf{Q}_l} = K_{0,\mathbf{Z}_l} \otimes \mathbf{Q}_l$, we can consider their 'reduction modulo p' ($p \gg 0$). See [BBD, 6.1]. (See (A1, (3)) in (3.4) for K_0.) We denote the resulting $\mathbf{Q}_l$-sheaves on $V_{\mathbf{F}_q}$ by the same letter and thus each of such symbols has two meanings.

8.2. Assumption A5. Assume that
(1) $f \in \mathbf{Q}[V]$, $f^\vee \in \mathbf{Q}[V^\vee]$, *and*
(2) $K_{0,\mathbf{Q}_l}[n] \in D_c^b(V_{\mathbf{F}_q}, \mathbf{Q}_l)$ *is pure of weight* w.

8.3. $W_m(Rj_*(f^*H_{c,\mathbf{Q}_l} \otimes K_{0,\mathbf{Q}_l})[n]) \in D_c^b(V(\mathbf{C}), \mathbf{Q}_l)$ *(=the perverse sheaf underlying the weight filtration of the mixed Hodge module, whose coefficient ring is extended to* $\mathbf{Q}_l$*) specializes (by the comparison theorem and by '*$\otimes\mathbf{F}_q$*') to the perverse sheaf of the same name in* $D_c^b(V_{\mathbf{F}_q}, \mathbf{Q}_l)$ *(=the weight filtration of the mixed perverse sheaf) if* $\mathrm{char}(\mathbf{F}_q) \gg 0$ *and if* l *is a prime which does not divide* qN*. Here* N *is the integer appeared in* (3.4, (3)).

Proof. Let $\tilde{V}$ be a smooth scheme over $\mathbf{Z}[N_1^{-1}]$ ($N_1 \in \mathbf{Z}_{>0}$), $\pi : \tilde{V} \to V$ a projective morphism such that $\tilde{f} := f \circ \pi$ is normal crossing relative to $\mathbf{Z}[N_1^{-1}]$ and π induces an isomorphism $\tilde{\Omega} := \tilde{V} \setminus \tilde{f}^{-1}(0) \to \Omega$ [Hir]. Let $\tilde{j} : \tilde{\Omega} \to \tilde{V}$ be the inclusion morphism. Then it is easy to see that (8.3) holds if f and j are replaced with $\tilde{f}$ and $\tilde{j}$. Put $\tilde{K} := R\tilde{j}_*(\tilde{f}^* H_{c,\mathbf{Q}_l} \otimes \pi^* K_{0,\mathbf{Q}_l})$ and $K := Rj_*(f^* H_{c,\mathbf{Q}_l} \otimes K_{0,\mathbf{Q}_l})$. Then $R\pi_* \tilde{K} = K$. From the exact sequence of perverse sheaves

$$0 \to W_{\leq m}(\tilde{K}[n]) \to \tilde{K}[n] \to W_{>m}(\tilde{K}[n]) \to 0,$$

we get an exact sequence

$${}^pH^0(R\pi_*(W_{\leq m}(\tilde{K}[n]))) \xrightarrow{\alpha} K[n] \to {}^pH^0(R\pi_*(W_{>m}(\tilde{K}[n]))),$$

and consequently, image$(\alpha) = W_{\leq m}(K[n])$ in the both sense of (8.1). Thus we get (8.3). $\square$

8.4. Continue to assume (A1)–(A5). By (6.7) and by the scalar restriction, we get (1)
$$m = m(\Xi(c)) \Leftrightarrow \mathcal{R}(\mathrm{gr}^W_{w+m}(Rj_*(f^*H_{c,\mathbf{Q}_l} \otimes K_{0,\mathbf{Q}_l})[n]))|\Omega^\vee \neq 0 \text{ in } D^b_c(V^\vee(\mathbf{C}), \mathbf{Q}_l).$$

Since the Radon transformation $\mathcal{R}(-)$ is compatible with the 'reduction modulo p' ($p \gg 0$), we may understand the right member of the above equivalence in $D^b_c(V^\vee_{\mathbf{F}_q}, \mathbf{Q}_l)$. Since $H_{c,\overline{\mathbf{Q}_l}} = \bigoplus_{\mathrm{ord}\,\chi = c} L_\chi$,

(2)
$$\begin{aligned}&\mathcal{R}(\mathrm{gr}^W_{w+m}(Rj_*(f^*L_\chi \otimes K_{0,\mathbf{Q}_l})[n]))|\Omega^\vee \neq 0 \text{ for some } \chi \text{ of order } c\\ &\Rightarrow \mathcal{R}(\ \mathrm{gr}^W_{w+m}(Rj_*(f^*H_{c,\mathbf{Q}_l} \otimes K_{0,\mathbf{Q}_l})[n]))|\Omega^\vee \neq 0\end{aligned}$$

Take l so that $\mathbf{Z}[\zeta_c] \otimes \mathbf{Q}_l$ is an integral domain. Then $\mathbf{Q}_l(\zeta_c) = \mathbf{Z}[\zeta_c] \otimes \mathbf{Q}_l$ and it acts on $H_{c,\mathbf{Q}_l}$ in several ways, giving $\mathbf{Q}_l(\zeta_c)$-sheaves L_χ for all χ of order c. Hence

(3)
$$\Rightarrow \mathcal{R}(\mathrm{gr}^W_{w+m}(Rj_*(f^*L_\chi \otimes K_{0,\mathbf{Q}_l})[n]))|\Omega^\vee \neq 0 \text{ for any } \chi \text{ of order } c,$$

for the specific l as above, and hence the same holds for any l.

Theorem 9. *Assume* (A1)–(A5) *and that* $\mathrm{char}(\mathbf{F}_q) \gg 0$. *Then* $\mathcal{F}_\psi(Rj_*(f^*L_\chi \otimes K_{0,\mathbf{Q}_l})[n])|\Omega^\vee$ *is pure of weight* $w+n+m$ *with* $m = m(\Xi(\mathrm{ord}\,\chi))$.

Proof. By [KL, (2.2.1)], $\mathcal{F}_\psi(\mathrm{gr}^W_{w+m}(-)) = \mathrm{gr}^W_{w+n+m}(\mathcal{F}_\psi(-))$. Hence (8.4, (1)–(3)), (8.3), and (7.7, (1)) yields the result. $\square$

10. Prehomogeneous vector spaces

10.1. Let us review [Gyo1] and [Gyo3]. Let G be a connected reductive group, $\rho : G \to GL(V)$ a finite dimensional rational representation, and assume that everything is defined over an algebraic number field K ($\subset \mathbf{C}$). Assume that $(G, \rho, V) \otimes \mathbf{C}$ is a prehomogeneous vector space, i.e., $G(\mathbf{C})$ has a dense orbit in $V(\mathbf{C})$. Let

$\rho^\vee : G \to GL(V^\vee)$ be the contragradient representation of ρ, $\phi \in \mathrm{Hom}(G, \mathbf{G}_m)$, $0 \neq f \in \mathbf{Q}[V]$, $0 \neq f^\vee \in \mathbf{Q}[V^\vee]$, and assume that $f(gv) = \phi(g)f(v)$ and $f^\vee(gv^\vee) = \phi(g)^{-1}f^\vee(v^\vee)$ for any $g \in G(\mathbf{C})$, $v \in V(\mathbf{C})$ and $v^\vee \in V^\vee(\mathbf{C})$. Put $\Omega = V \setminus f^{-1}(0)$ and $\Omega^\vee = V^\vee \setminus f^{\vee -1}(0)$. Then there exists a unique closed $G(\mathbf{C})$-orbit $O_1(\mathbf{C})$ (resp. $O_1^\vee(\mathbf{C})$) in $\Omega(\mathbf{C})$ (resp. $\Omega^\vee(\mathbf{C})$). Let $i : O_1 \to \Omega$ and $i^\vee : O_1^\vee \to \Omega^\vee$ be the inclusion morphisms. Put $n = \dim V = \dim V^\vee$ and $m = \dim O_1 = \dim O_1^\vee$. Define $F : \Omega \to V^\vee$ and $F^\vee : \Omega^\vee \to V$ by $F := \mathrm{grad} \log f$ and $F^\vee = \mathrm{grad} \log f^\vee$. Then F and $F^\vee$ induce smooth morphisms $F : \Omega \to O_1^\vee$ and $F^\vee : \Omega^\vee \to O_1$. Let $\pi^\vee : \tilde{O}_1^\vee \to O_1^\vee$ be the two fold covering defined in [Gyo1, 3.14]. Let $L(\omega^\vee)_{\mathbf{Z}}$ be the cokernel of $\mathbf{Z}_{O_1^\vee} \to \pi^\vee_* \mathbf{Z}_{\tilde{O}_1^\vee}$. Then $L(\omega^\vee)_{\mathbf{Z}}$ is a locally constant sheaf on $O_1^\vee(\mathbf{C})$.

Consider a $\mathcal{D}$-module $\mathcal{D}_{O_1^\vee} u^\vee$ satisfying the following condition.

10.2. Assumption A6. *$\mathcal{D}_{O_1^\vee} u^\vee$ is a regular holonomic $\mathcal{D}$-module such that $-Au^\vee = \chi(A)u^\vee$ ($A \in \mathrm{Lie}(G)$) and $\mathrm{DR}(\mathcal{D}_{O_1^\vee} u^\vee)$ is a locally constant sheaf of rank one.* (Note that $A \in \mathrm{Lie}(G)$ induces a vector field on $O_1^\vee$, which can be regarded as a differential operator of first order.) Put $K_0^\vee := \mathrm{DR}(\mathcal{D}_{O_1^\vee} u^\vee)$, $K_\alpha^\vee := f^{\vee *} L_\alpha \otimes K_0^\vee$, and $K_\alpha := F^* K_\alpha^\vee$.

Lemma 10.3. [Gyo3, 6.21]. *Assume* (A6). *Then* (1) $\mathcal{F}^+(Rj_* f^* K_\alpha[n]) = j_!^\vee i_*^\vee(K_\alpha^\vee \otimes L(\omega^\vee))[m]$, *and* (2) $\mathcal{F}^+(j_! f^* K_\alpha[n]) = Rj_*^\vee i_*^\vee(K_\alpha^\vee \otimes L(\omega^\vee))[m]$.

10.4. Assume that $\mathrm{char}(\mathbf{F}_q) \neq 2$ and let $\chi_{1/2}$ be the unique character of $\mathbf{F}_q^\times$ of order 2. For $v^\vee \in V^\vee(\mathbf{F}_q)$, let $h^\vee(v^\vee)$ be the discriminant of the quadratic form determined by $\left(\frac{\partial^2 \log f^\vee}{\partial y_i \partial y_j}(v^\vee)\right)$. (Cf. [Gyo2, §7].) By [DG, 3.5.4], we have

$$\mathrm{trace}(\mathrm{Frob}_q \mid L(\omega^\vee)_{\mathbf{Q}_l, v^\vee}) = \chi_{1/2}(h^\vee(v^\vee)) \times C_1, \text{ for } v^\vee \in O_1^\vee(\mathbf{F}_q)$$

for any Frobenius action on $L(\omega^\vee)_{\mathbf{Q}_l}$, where C_1 is some constant independent of $v^\vee$. In the situation of the following theorem, all the assumptions (A1)–(A6) are satisfied.

Theorem 11 *Assume that $f \in \mathbf{Q}[V]$, $f^\vee \in \mathbf{Q}^\vee[V^\vee]$ and $\mathrm{char}(\mathbf{F}_q) \gg 0$. Fix an isomorphism $(\frac{1}{1-q}\mathbf{Z})/\mathbf{Z} \xrightarrow{\simeq} \mathbf{F}_q^\times$, $\alpha \mapsto \chi_\alpha$, and put $L_\alpha := L_{\chi_\alpha}$. Then the following holds in $D_c^b(V_{\overline{\mathbf{F}}_q}, \overline{\mathbf{Q}}_l)$.*

(a) If we forget the Frobenius action, then

(1) $\mathcal{F}_{\overline{\psi}}(Rj_* f^* L_{-\alpha}[n]) \simeq j_!^\vee i_*^\vee(f^{\vee *} L_\alpha \otimes L(\omega^\vee))[m]$,

(2) $\mathcal{F}_{\overline{\psi}}(Rj_*(f^* L_{-\alpha} \otimes F^* L(\omega^\vee))[n]) \simeq j_!^\vee i_*^\vee f^{\vee *} L_\alpha[m]$,

(3) $\mathcal{F}_{\overline{\psi}}(Rj_*^\vee i_*^\vee(f^{\vee *} L_{-\alpha} \otimes L(\omega^\vee))[m]) \simeq j_! f^* L_\alpha[n]$,

(4) $\mathcal{F}_{\overline{\psi}}(Rj_*^\vee i_*^\vee f^{\vee *} L_{-\alpha}[m]) \simeq j_!(f^* L_\alpha \otimes F^* L(\omega^\vee)[n]$,

(1′) $\mathcal{F}_\psi(j_! f^* L_\alpha[n]) \simeq Rj_*^\vee i_*^\vee(f^{\vee *} L_{-\alpha} \otimes L(\omega^\vee))[m]$

(2′) $\mathcal{F}_\psi(j_!(f^* L_\alpha \otimes F^* L(\omega^\vee))[n]) \simeq Rj_*^\vee i_*^\vee f^{\vee *} L_{-\alpha}[m]$

(3′) $\mathcal{F}_\psi(j_!^\vee i_*^\vee(f^{\vee *} L_\alpha \otimes L(\omega^\vee))[m]) \simeq Rj_* f^* L_{-\alpha}[n]$

(4′) $\mathcal{F}_\psi(j_!^\vee i_*^\vee f^{\vee *} L_\alpha[m]) \simeq Rj_*(f^* L_{-\alpha} \otimes F^* L(\omega^\vee))[n]$

(b) *Assume that the Frobenius action on* L_α *and* $L(\omega)$ *are the natural ones. Then the left hand sides of* (1)–(4) *and* (1′)–(4′) *restrict to pure sheaves on* Ω *or* $\Omega^\vee$. *The weights are*
(1) $n + m(\alpha, \mathbf{C}_\Omega[n])$,
(2) $n + m(\alpha, F^*L(\omega^\vee)[n])$,
(3) $n + m(\alpha, i_*^\vee L(\omega^\vee)[m])$,
(4) $n + m(\alpha, i_*^\vee \mathbf{C}_{O_1^\vee}[m])$,
(1′) $n - m(\alpha, \mathbf{C}_\Omega[n])$,
(2′) $n - m(\alpha, F^*L(\omega^\vee)[n])$,
(3′) $n - m(\alpha, i_*^\vee L(\omega^\vee)[m])$,
(4′) $n - m(\alpha, i_*^\vee \mathbf{C}_{O_1^\vee}[m])$,
respectively. (For $m(\alpha, -)$, see the lines following (3.5, (1)).)

Proof. (a) (1) and (2) follows from (10.3) as in [Gyo2]. By the Verdier duality, (i) $\Leftrightarrow$ (i') for $i = 1, \cdots, 4$. By '$\mathcal{F}_{\overline{\psi}} = \mathcal{F}_{\psi}^{-1}$', (1′) $\Leftrightarrow$ (3) and (2′) $\Leftrightarrow$ (4).
(b) (1)–(4) follows from Theorem 9, from which (1′)–(4′) follows by the Verdier duality. □

Corollary 12 *Keep the notations and the assumptions of the above theorem.*
(a) There exist constants $C_1, \cdots, C_4$ such that

$$(1)\quad q^{-n} \sum_{v \in \Omega(\mathbf{F}_q)} \chi_\alpha(f(v))\psi(\langle v^\vee, v\rangle) = \begin{cases} C_1 \cdot \chi_\alpha(f^\vee(v^\vee)^{-1}) \cdot \chi_{1/2}(h^\vee(v^\vee)) & \text{for } v^\vee \in O_1^\vee(\mathbf{F}_q) \\ 0 & \text{for } v^\vee \in (\Omega^\vee \setminus O_1^\vee)(\mathbf{F}_q). \end{cases}$$

$$(2)\quad q^{-n} \sum_{v \in \Omega(\mathbf{F}_q)} \chi_\alpha(f(v))\chi_{1/2}(h^\vee(F(v)))\psi(\langle v^\vee, v\rangle) = \begin{cases} C_2 \cdot \chi_\alpha(f^\vee(v^\vee)^{-1}) & \text{for } v^\vee \in O_1^\vee(\mathbf{F}_q) \\ 0 & \text{for } v^\vee \in (\Omega^\vee \setminus O_1^\vee)(\mathbf{F}_q). \end{cases}$$

$$(3)\quad q^{-m} \sum_{v^\vee \in O_1^\vee(\mathbf{F}_q)} \chi_\alpha(f^\vee(v^\vee))\chi_{1/2}(h^\vee(v^\vee))\psi(\langle v^\vee, v\rangle) = C_3 \cdot \chi_\alpha(f(v)^{-1}) \quad \text{for } v \in \Omega(\mathbf{F}_q).$$

$$(4)\quad q^{-m} \sum_{v^\vee \in O_1^\vee(\mathbf{F}_q)} \chi_\alpha(f^\vee(v^\vee))\psi(\langle v^\vee, v\rangle) = C_4 \cdot \chi_\alpha(f(v)^{-1})\chi_{1/2}(h^\vee(F(v))) \quad \text{for } v \in \Omega(\mathbf{F}_q).$$

The constants C_1 *and* C_2 *(resp.* C_3 *and* C_4*) are independent of* $v^\vee \in O_1^\vee(\mathbf{F}_q)$ *(resp.* $v \in \Omega(\mathbf{F}_q)$*), but depend on the other parameters. (see* [DG] *for the precise formula.)*
(b) *The values of* $w_i := \log|C_i|/\log\sqrt{q}$ *are given by the following formula.*
(1) $w_1 = -m - m(\alpha, \mathbf{C}_\Omega[n])$.
(2) $w_2 = -m - m(\alpha, F^*L(\omega^\vee)[n])$.
(3) $w_3 = -m - m(\alpha, i_*^\vee L(\omega^\vee)[m])$.
(4) $w_4 = -m - m(\alpha, i_*^\vee \mathbf{C}_{O_1^\vee}[m])$.

Corollary 13. $b^{\exp}(t, \mathbf{C}_\Omega[n]) = b^{\exp}(t, F^*L(\omega^\vee)[n]) = b^{\exp}(t, i_*^\vee L(\omega^\vee)[m])$ $= b^{\exp}(t, i_*^\vee \mathbf{C}_{O_1^\vee}[m])$.

Proof. By [DG], we can write down explicitly the constants C_i $(1 \leq i \leq 4)$, and we get $C_1 = C_4$ and $C_2 = C_3$. Then by (1) and (4) of (12, (b)), and by (3.5, (7)), we get $b^{\exp}(t, \mathbf{C}_\Omega[n]) = b^{\exp}(t, i_*^\vee \mathbf{C}_{O_1^\vee}[m])$, and $b^{\exp}(t, F^*L(\omega^\vee)[n]) = b^{\exp}(t, i_*^\vee L(\omega^\vee)[m])$. By [Gyo3, 6.19], $(-1)^d B_f(-s-1, 1) = B_{f^\vee}(s, \mathcal{F}(f^0))$. Hence $b_f^{\exp}(s, \mathbf{C}_\Omega[n]) = b_f^{\exp}(s, 1) = b_{f^\vee}^{\exp}(s, \mathcal{F}(f^0)) = b_{f^\vee}^{\exp}(s, i_*^\vee L(\omega^\vee)[m])$.

References

[BBD] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Astérisque **100** (1983).

[Bry] J. L. Brylinski, *Transformations canoniques, dualité projective, Théorie de Lefschetz, transformations de Fourier et sommes trigonométriques*, Astérisque **140-141** (1986), 3–134.

[BMV] J. L. Brylinski, B .Malgrange, J. L. Verdier, *Transformation de Fourier géométrique I*, C. R. Acad. Sci. Paris **297** (1983), 55–58.

[Del1] P. Deligne, *Le formalisme des cycles évanescents, SGA7, exposé 13*, Lecture Notes in Math. **340** (1973), Springer, 82–115.

[Del2] ———, *La conjecture de Weil II*, Publ. Math. IHES **52** (1980), 137–252.

[Del3] ———, *Applications de la formule des traces aux sommes trigonométriques, SGA4$\frac{1}{2}$*, Springer Lecture Notes **569** (1977), 168–232.

[Del4] ———, *Théorie de Hodge, II*, Publ. Math. IHES **40** (1971), 5–58.

[DG] J. Denef, A. Gyoja, *Character sums associated to prehomogeneous vector spaces*, preprint.

[Gyo1] A. Gyoja, *Theory of prehomogeneous vector spaces without regularity condition*, Publ. RIMS **27** (1991), 861–922.

[Gyo2] ———, *Lefschetz principle in the theory of prehomogeneous vector spaces*, Advanced Studies in Pure Math. **21** (1992), 87-99.

[Gyo3] ———, *Theory of prehomogeneous vector spaces, II, a complement*, preprint.

[Gyo4] ———, *Local b-functions of prehomogeneous Lagrangians*, J. Math. Kyoto Univ. **33** (1993), 413–436.

[Hir] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero, I,II*, Ann. of Math. **79** (1964), 109–326.

[HK] R. Hotta, M. Kashiwara, *The invariant holonomic systems on a semisimple Lie algebra*, Invent. Math. **75** (1984), 327–358.

[Kas1] M. Kashiwara, *b-Functions and holonomic systems*, Invent. Math. **38** (1976), 33-53.

[Kas2] ———, *Vanishing cycle sheaves and holonomic systems of differential equations*, Lecture Notes in Math. **1016** (1983), Springer, 134–142.

[Kas3] ———, *On the holonomic systems of linear differential equations, II*, Invent. Math. **49** (1978), 121–135.

[KasSch] M. Kashiwara, P. Schapira, *Microlocal study of sheaves*, Astérisque **128** (1985).

[Kat] N. M. Katz, *Crystalline cohomology, Dieudonné modules, and Jacobi sums*, in 'Automorphic forms, representation theory and arithmetic' : papers presented at the Bombay Colloquium (1979), 165–246.

[KL] N. M. Katz, G. Laumon, *Transformation de Fourier et majoration de sommes exponentielles*, Publ. Math. IHES **62** (1985), 145–202.

[Lau] G. Laumon, *Transformation de Fourier, constantes d'équations fonctionelles et conjecture de Weil*, Publ. Math. IHES **65** (1987), 131–210.

[Mal] B. Malgrange, *Polynôme de Bernstein-Sato et cohomologie évanescente*, Astérisque **101-102** (1983), 243-267.

[Maz] B. Mazur, *Frobenius and the Hodge filtration*, Bull. AMS **78** (1972), 653–667.

[Sai1] M. Saito, *Modules de Hodge polarisables*, Publ. RIMS **24** (1988), 849–995.

[Sai2] ———, *Mixed Hodge modules*, Publ. RIMS **26** (1990), 221–333.

[Sat] M. Sato, *Theory of prehomogeneous vector spaces (Notes by T.Shintani in Japanese)*, Sugaku no Ayumi **15-1** (1970), 85–157.

[SKK] M. Sato, M. Kashiwara, T. Kawai, *Microfunctions and pseudo-differential equations*, Lecture Notes in Math. **287** (1973), Springer, 265–529.

[SKKO] M. Sato, M.Kashiwara, T.Kimura, T.Oshima, *Micro-local analysis of prehomogeneous vector spaces*, Invent. Math. **62** (1980), 117–179.

[Tan] T. Tanisaki, *Hodge modules, equivariant K-theory and Hecke algebras*, Publ. RIMS **23** (1987), 841–879.