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An explicit formula for the Fourier coefficients of Siegel-Eisenstein series

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1 Introduction

Siegel-Eisenstein series is one of the most important subjects in number theory (cf. Böcherer [B], Kudla [Ku], Shimura [Sh3], and Siegel [Si]). In this note, we give an explicit formula for the Fourier coefficients of Siegel-Eisenstein series of any degree. We will state our main result explicitly. Let k be an even integer and

$$E_{n,k}(Z,s)=\sum_{\{C,D\}}\det(CZ+D)^{-k}\mid\det(CZ+D)\mid^{-2s}\det\mathrm{Im}(Z)^{s}$$

Siegel Eisenstein series of degree n and of weight k, where $\{C,D\}$ runs over a complete set of representatives of the equivalence classes of coprime symmetric pairs of degree n, and $\mathrm{Im}(Z)$ denotes the imaginary part of a complex symmetric matrix Z of degree n. We note that the function $E_{n,k}(Z,0)$ of Z is the classical Eisenstein series defined by Siegel. To explain the Fourier expansion of the Eisenstein series, for a commutative ring R, we denote by $M_{mn}(R)$ the set of (m,n) matrices with entries in R, and especially write $M_n(R)=M_{nn}(R)$. We often identify an element a of R and the matrix (a) of degree 1 whose component is a. If m or n is 0, we understand an element of $M_{mn}(R)$ is the *empty matrix* and denote it by ϕ. Let $GL_n(R)$ be the group consisting of all invertible elements of $M_n(R)$, and $S_n(R)$ the set of symmetric matrices of degree n with entries in R. Further for an integral domain R, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathcal{H}_n(R)$ is the set of symmetric matrices (a_{ij}) of degree n with entries in the quotient field of R such that a_{ii} $(i=1,...,n)$ and $2a_{ij}$ $(1\leq i\neq j\leq n)$ belong to R. We note that $\mathcal{H}_n(R)=S_n(R)$ if R contains the inverse of 2. Further if $\mathbf{R}$ is the field of real numbers, we denote by $S_n(\mathbf{R})_+$ the subset of $S_n(\mathbf{R})$ consisting of all positive definite matrires. For two square matrices X and Y we write $X\perp Y=\begin{pmatrix}X&0\\0&Y\end{pmatrix}$. We often write $x\perp Y$ instead of $(x)\perp Y$ if (x) is a matrix of degree 1.

Let $\mathbf{Q}_p$ be the field of p-adic numbers, $\mathbf{Z}_p$ the ring of p-adic integers, and $\mathbf{Z}_p^*$ the group of p-adic units. For a complex number x put $e(x)=exp(2\pi i\ x)$, and

for a p-adic number x put $e_p(x) = e(x')$, where x' denotes the fractional part of x. Further for a square matrix X, let $tr(X)$ denote the trace of X. Then for a half-integral matrix B of degree n over $\mathbf{Z}$, we define the Siegel series $b(B;s)$ by

$$b(B,s) = \sum_{R \in S_n(\mathbf{Q})/S_n(\mathbf{Z})} e(tr(BR))\mu(R)^{-s},$$

where $\mu(R)$ denotes the product of denominators of elementary divisors of R. Further for a half-integral matrix B of degree n over $\mathbf{Z}$, we define the confluent hypergeometric function $\Xi(g,h;\alpha,\beta)$ defined on $S_n(\mathbf{R})_+ \times S_n(\mathbf{R}) \times \mathbf{C} \times \mathbf{C}$ by

$$\Xi(g,h;s,s') = \int_{S_n(\mathbf{R})} e(-tr(hx)) \det(x+ig)^{-s} \det(x-ig)^{-s'} dx.$$

Then by Maass [Ma1] we have

$$E_{n,k}(X+iY,s) = (\det Y)^s$$

$$+(\det Y)^s \sum_{j=1}^{n} \sum_{B \in \mathcal{H}_j(\mathbf{Z})} \sum_{q} b(B, 2s+k)\Xi(Y[q], B; s+k, s)e(tr(B[{}^t q]X)),$$

where q runs over a complete set of representatives of $GL_j(\mathbf{Z})-$ equivalence classes of primitive $(n,j)-$ matrices with entries in $\mathbf{Z}$. For further information on the Fourier expansion of $E_{n,k}(X+iY,s)$, see Mizumoto [Mi]. See also Shimura [Sh2] for the Fourier coefficients of Siegel Eisenstein series with respect to $\Gamma_0(N)$. We now focus our attention on the Siegel series because the analytic properties of the confluent hypergeometric function have been deeply investigated by Shimura [Sh1]. To investigate the Siegel series, for a prime number p and a half-integral matrix B of degree n over $\mathbf{Z}_p$ define the local Siegel series $b_p(B,s)$ by

$$b_p(B,s) = \sum_R e_p(tr(BR))p^{-\mathrm{ord}_p(\mu(R))s},$$

where R runs over a complete set of representatives of $S_n(\mathbf{Q}_p)/S_n(\mathbf{Z}_p)$ and $\mu(R)$ is the product of denominators of elementary divisors of R. The we easily see that for a half-integral matrix B of degree n over $\mathbf{Z}$ we have

$$b(B,s) = \prod_p b_p(B,s).$$

Then our main theorem gives an explicit form of $b_p(B;s)$ for any non-degenerate half-integral matrix B over $\mathbf{Z}_p$ (cf. Theorem 1). As a result we show that the Siegel series $b(B;s)$ of a half-integral matrix B over $\mathbf{Z}$ can be determined completely in terms of certain invariants of B (cf. Theorem 2). It contains all the results concerning this subject. Our main result for the case $n = 2$ and 3 has been treated by Kaufhold [Kau] and by the author [Kat2], respectively, though they have not been formulated in the form in the present paper (cf. Section 4). We should remark that our result is new even in the case of $n = 4$.

2 Main result

For a non-zero element $a \in \mathbf{Q}_p$ we put $\chi_p(a) = 1, -1$, or 0 according as $\mathbf{Q}_p(a^{1/2}) = \mathbf{Q}_p$, $\mathbf{Q}_p(a^{1/2})$ is an unramified quadratic extension of $\mathbf{Q}_p$, or $\mathbf{Q}_p(a^{1/2})$ is a ramified quadratic extension of $\mathbf{Q}_p$. Further let $(\ ,\ )_p$ be the Hilbert symbol over $\mathbf{Z}_p$ and h_p the Hasse invariant (for the definition of Hasse invariant, see [Ki5]). Let B be a non-degenerate half-integral matrix of degree n with entries in $\mathbf{Z}_p$. If n is odd, define $\eta_p(B)$ by

$$\eta_p(B) = h_p(B)(\det B, (-1)^{(n-1)/2} \det B)_p (-1,-1)_p^{(n^2-1)/8},$$

and if n is even, define $\xi_p(B)$ by

$$\xi_p(B) = \chi_p((-1)^{n/2} \det B),$$

and call it the sign of B. Further put

$$\xi'_p(B) = 1 + \xi_p(B) - \xi_p(B)^2.$$

From now on we often write $\xi(B)$ instead of $\xi_p(B)$, and so on if there is no fear of confusion.

Let B be a non-degenerate half-integral matrix of degree n with entries in $\mathbf{Z}_p$. We define several invariants of B as follows.

First let $p \neq 2$. Let ϵ be an element of $\mathbf{Z}_p^*$ not contained in $\mathbf{Z}_p^{*2}$. Then B is equivalent, over $\mathbf{Z}_p$ to

$$p^{e_1}U_1 \perp \ldots\ldots \perp p^{e_s}U_s,$$

where $e_1, \ldots, e_s$ are non-negative integers such that $e_1 > e_2 > \ldots > e_s$ and

$$U_i = \overbrace{1\perp 1 \ldots . \perp 1}^{n_i - 1} \perp u_i$$

with $u_i \in \{1, \epsilon\}$. We call the above form a canonical Jordan form of B and denote it by $J(B)$. We remark that $e_1, \ldots, e_s, n_1, \ldots, n_s$ and $u_1, \ldots, u_s$ are uniquely determined by B, and so is $J(B)$. Let b_i be the i-th diagonal component of $J(B)$ and put $J_i = b_i \perp \ldots \perp b_n$. Then for each $i = 1, 2, \ldots, n$ put $d_{p,i}(B) = \mathrm{ord}_p(\det J_i)$, and $\delta_{p,i} = 2[(d_{p,i}(B)+1)/2]$ or $d_{p,i}(B)$ according as $n - i + 1$ is even or odd. We note that $d_{p,i}(B) = \mathrm{ord}_p(a_1 \ldots a_{n-i+1})$, where $a_1, \ldots, a_n$ are elementary divisors of B such that $a_1 \mid a_2 \ldots . \mid a_n$. Further we define invariants $\xi_{p,i}(B)$ $(i = 1, \ldots, [n/2])$ and $\eta_{p,i}(B)$ $(i = 1, \ldots, [(n-1)/2])$ of B by

$$\xi_{p,i}(B) = \begin{cases} \xi(J_{2i-1}) & \text{if } n \text{ is even} \\ \xi(J_{2i}) & \text{if } n \text{ is odd,} \end{cases}$$

and

$$\eta_{p,i}(B) = \begin{cases} \eta(J_{2i}) & \text{if } n \text{ is even} \\ \eta(J_{2i-1}) & \text{if } n \text{ is odd.} \end{cases}$$

Further put $\sigma_{p,i}(B) = \tau_{p,i}(B) = 0$ for any $i = 1, \ldots, n$.

Next let $p = 2$. Put $H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$. Then by [W] B is equivalent, over $\mathbf{Z}_2$, a unique matrix of the following form:

$$\perp_{i=0}^{r} 2^{r-i}(U_i \perp V_i),$$

where

$$U_i = \perp_{j=1}^{k_i} c_{ij} \text{ with } k_i \geq 0, c_{ij} \in \mathbf{Z}_2^*$$

and

$$V_i = \phi, H\perp H\perp ...\perp H, \text{ or} H\perp H\perp ...\perp H\perp Y$$

satisfying the following conditions:
(c.1) $c_{i1} = 1$ or -3 if $k_i = 1$ and $(c_{i1}, c_{i2}) = (1, \pm 1), (1, \pm 3), (-1, -1)$,or $(-1, 3)$ if $k_i = 2$,
(c.2) $k_{i-2} = k_i = 0$ if $V_i = H\perp H\perp ...\perp H\perp Y$,
(c.3) $-\det U_i \equiv 1 \bmod 4$ if $k_i = 2$ and $V_{i-1} = H\perp H\perp ...\perp H\perp Y$,
(c.4) $(-1)^{k_i - 1} \det U_i \equiv 1 \bmod 4$ if $k_i, k_{i+1} > 0$,
(c.5) $U_i \neq \begin{pmatrix} -1 & 0 \\ 0 & -c_{i2} \end{pmatrix}$ if $k_{i-1} > 0$,
(c.6) $U_i = \phi, (\pm 1), \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$, or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ if $k_{i+2} > 0$.
The matrix satisfying the conditions (c.1) $\sim$ (c.6) is called a canonical form and denote it by $J(B)$. Put $K = H$ or Y. Write the canonical form of B as

$$J(B) = 2^{m_1} b_1 \perp .... \perp 2^{m_s} b_s$$

where $m_1 \geq ... \geq m_s$, and b_j is K or a unimodular matrix of the form U_j with some $0 \leq j \leq r$. Put $n_i = \deg b_i$. We remark that under the above condition we have $m_i > m_{i+1}$ if b_i and b_{i+1} are both diagonal matrices. On the other hand, put $J(B) = (c_{ij})_{1 \leq i,j \leq n}$, and $J_j = (c_{\alpha,\beta})_{j \leq \alpha,\beta \leq n}$ for any $j = 1, ..., n$. Then J_j satisfies exactly one of the following 5 conditions:

(1) $J_j = 2^{m_l} b_l \perp 2^{m_{l+1}} b_{l+1} \perp .. \perp 2^{m_s} b_s$ with some $1 \leq l \leq s$ such that $\deg b_l = 1$.

(2) $J_j = 2^{m_l} b_l \perp 2^{m_{l+1}} b_{l+1} \perp .. \perp 2^{m_s} b_s$ with some $1 \leq l \leq s$ such that b_l is a diagonal unimodular matrix of degree 2.

(3) $J_j = 2^{m_l} b_l \perp 2^{m_{l+1}} b_{l+1} \perp .. \perp 2^{m_s} b_s$ with some $1 \leq l \leq s$ such that $b_l = K$.

(4) J_j satisfies none of the conditions (1),(2),(3), but J_{j-1} is of type (2)

(5) J_j satisfies none of the conditions (1),(2),(3),(4) but J_{j-1} is of type (3)

For each $i = 1, ..., n$ we define $d_{2,i}(B)$ by

$$d_{2,i}(B) = \begin{cases} \operatorname{ord}_2(2^{2[(n-i+1)/2]} \det J_i) & \text{if } J_i \text{ is of type (1), (2), or (3)} \\ \operatorname{ord}_2(2^{2[(n-i+1)/2]}(2^{m_l} \perp J_{i+1})) & \text{if } J_i \text{ is of type (4) or (5)} \\ & \text{and } J_{i-1} = 2^{m_l} b_l \perp 2^{m_{l+1}} b_{l+1} \perp .. \perp 2^{m_s} b_s \\ & \text{with some } 1 \leq l \leq s \text{ such that } \deg b_l = 2 \end{cases},$$

and put $\delta_{2,i} = 2[d_{2,i}(B)/2]$ or $d_{2,i}(B)$ according as $n - i + 1$ is even or odd. For a while write $d_i(B) = d_{2,i}(B)$. We then define further invariants $\xi_{2,i}(B)$ $(i =$

$1, ..., [n/2]$) and $\eta_{2,i}(B)$ ($i = 1, ..., [(n-1)/2]$) of B as follows. First let n be even. Then put

$$\xi_{2,i}(B) = \begin{cases} \xi(J_{2i-1}) & \text{if } J_{2i-1} \text{ is of type (1), (2), or (3)} \\ 1 & \text{if } J_{2i-1} \text{ is of type (5) and } d_{2i-1}(B) \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\eta_{2,i}(B) = \begin{cases} \eta(J_{2i}) & \text{if } J_{2i} \text{ is of type (1), (2), (3) or if } J_{2i} \text{ is of type (4) and } d_{2i+1}(B) \text{ is even} \\ (-1)^{((n-2i+1)^2-1)/8} h(J_{2i+1}) \xi(J_{2i+1})^{m_{l_{2i-1}}} & \text{if } J_{2i} \text{ is of type (5) and } \xi(J_{2i+1}) \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Next let n be odd. Then put

$$\xi_{2,i}(B) = \begin{cases} \xi(J_{2i}) & \text{if } J_{2i} \text{ is of type (1), (2), or (3)} \\ 1 & \text{if } J_{2i} \text{ is of type (5) and } d_{2i}(B) \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\eta_{2,i}(B) = \begin{cases} \eta(J_{2i-1}) & \text{if } J_{2i-1} \text{ is of type (1), (2), (3) or if } J_{2i-1} \text{ is of type (4) and } d_{2i}(B) \text{ is even} \\ (-1)^{((n-2i+2)^2-1)/8} h(J_{2i}) \xi(J_{2i})^{m_{l_{2i-2}}} & \text{if } J_{2i-1} \text{ is of type (5) and } \xi(J_{2i}) \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Finally let n be any positive integer. We then define $\sigma_{2,i}(B)$ and $\tau_{2,i}(B)$ by

$$\sigma_{2,i}(B) = \begin{cases} (2\delta_{i+1}(B) - \delta_i(B) - \delta_{i+2}(B) + 2)/2 & \text{if } n-i+1 \text{ is even, } J_i \text{ is of type (2) and } d_i(B) \text{ is odd} \\ & \text{or if } n-i+1 \text{ is even, } J_i \text{ is of type (3) and } \xi(J_{i+2}) = 0 \\ -(2\delta_i(B) - \delta_{i-1}(B) - \delta_{i+1}(B) + 2)/2 & \text{if } n-i+2 \text{ is even, } J_i \text{ is of type (4) and } d_{i-1}(B) \text{ is odd} \\ & \text{or if } n-i+2 \text{ is even, } J_i \text{ is of type (5) and } \xi(J_{i+1}) = 0 \\ 2 & \text{if } n-i+1 \text{ is odd, } J_i \text{ is of type (3) and } d_{i+1}(B) \text{ is even} \\ -2 & \text{if } n-i+2 \text{ is odd, } J_i \text{ is of type (5) and } d_i(B) \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tau_{2,i}(B) = \begin{cases} -(2\delta_i(B) - \delta_{i-1}(B) - \delta_{i+1}(B) + 2) & \text{if } n-i+2 \text{ is even, } J_i \text{ is of type (4) and } d_{i-1}(B) \text{ is odd} \\ & \text{or if } n-i+2 \text{ is even, } J_i \text{ is of type (5) and } \xi(J_{i+1}) = 0 \\ 2 & \text{if } n-i+1 \text{ is odd, } J_i \text{ is of type (3) and } d_{i+1}(B) \text{ is even} \\ -2 & \text{if } n-i+2 \text{ is odd, } J_i \text{ is of type (5) and } d_i(B) \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Now for a non-degenerate half-integral matrix B of degree n over $\mathbf{Z}_p$ define a polynomial $\gamma_p(B;X)$ in X by

$$\gamma_p(B;X) = \begin{cases} (1-X)\prod_{i=1}^{n/2}(1-p^{2i}X^2)(1-p^{n/2}\xi_p(B)X)^{-1} & \text{if } n \text{ is even} \\ (1-X)\prod_{i=1}^{(n-1)/2}(1-p^{2i}X^2) & \text{if } n \text{ is odd} \end{cases},$$

where $\xi_p(B) = \chi_p((-1)^{n/2}\det B)$. Then it is shown by [Ki4] that there exists a polynomial $F(B;X)$ in X such that

$$F(B;p^{-s}) = \frac{b_p(B,s)}{\gamma_p(B;p^{-s})}.$$

Thus it suffices to deal with $F(B;p^{-s})$ for our purpose.

Theorem 1. *Let B be a non-degenerate half-integral symmetric matrix of degree n over $\mathbf{Z}_p$, and $\delta_{p,i}(B)$ $(i=1,..,n)$, $\sigma_{p,i}(B)$, $\tau_{p,i}(B)$ $(i=1,...,n)$, $\xi_i(B)$ $(i=1,..,[n/2])$, $\eta_i(B)$ $(i=1,...,[(n-1)/2])$ as above. Write $\delta_i = \delta_{p,i}(B)$ and the others, and further put $\xi_{[(n+2)/2]} = \xi'_{[(n+2)/2]} = \eta_{[(n+1)/2]} = 1$ and $\delta_{n+1} = 0$. Then we have*

$$F(B;p^{-s}) = \sum_{(j_1,\dots,j_n)} \prod_{i=1}^{n/2} \Big[\frac{1-\xi_i p^{n/2+2-i+j_1+\dots+j_{2i-2}-j_{2i-1}-s}}{1-p^{n+3-2i+2j_1+\dots+2j_{2i-2}-2s}}$$

$$\times \{(-1)^{\xi_i+1}\xi'_i\eta_i(p^{n/2+1-i+j_1+\dots+j_{2i-2}-s})^{\delta_{2i-1}-\delta_{2i}+\xi_i^2+\sigma_{2i-1}}p^{(\delta_{2i-1}+\tau_{2i-1})/2}\}^{1-j_{2i-1}}\Big]$$

$$\times \prod_{i=1}^{n/2} \frac{\{(-1)^{\xi_{i+1}}\xi'_{i+1}\eta_i(p^{n/2-i+j_1+\dots+j_{2i-1}-s})^{\delta_{2i}-\delta_{2i+1}+2-\xi_{i+1}^2+\sigma_{2i}}p^{(2\delta_{2i}-\delta_{2i+1}+2+\tau_{2i})/2}\}^{1-j_{2i}}}{1-\xi_{i+1}p^{n/2+1-i+j_1+\dots+j_{2i-1}-s}}$$

or

$$F(B;p^{-s}) = \sum_{(j_1,\dots,j_n)} \Big[\prod_{i=1}^{(n-1)/2} \frac{1-\xi_i p^{(n+3)/2-i+j_1+\dots+j_{2i-1}-j_{2i}-s}}{1-p^{n+2-2i+2j_1+\dots+2j_{2i-1}-2s}}$$

$$\times \{(-1)^{\xi_i+1}\xi'_i\eta_{i+1}(p^{(n+1)/2-i+j_1+\dots+j_{2i-1}-s})^{\delta_{2i}-\delta_{2i+1}+\xi_i^2+\sigma_{2i}}p^{(\delta_{2i}+\tau_{2i})/2}\}^{1-j_{2i}}\Big]$$

$$\times \prod_{i=1}^{(n+1)/2} \frac{\{(-1)^{\xi_i}\xi'_i\eta_i(p^{(n+1)/2-i+j_1+\dots+j_{2i-2}-s})^{\delta_{2i-1}-\delta_{2i}+2-\xi_i^2+\sigma_{2i-1}}p^{(2\delta_{2i-1}-\delta_{2i}+2+\tau_{2i-1})/2}\}^{1-j_{2i-1}}}{1-\xi_i p^{(n+3)/2-i+j_1+\dots+j_{2i-2}-s}}$$

according as n is even or odd, where $(j_1, ..., j_n)$ runs over all elements of $\{0,1\}^n$. Here we make the convention that $j_1 + ... + j_{2i-2} = 2j_1 + ... + 2j_{2i-2} = 0$ if $i = 1$, and that $\prod_{i=1}^{(n-1)/2}(*) = 1$ if $n = 1$.

Theorem 2. *Let B be a half-integral matrix of degree n a and of rank r. Assume that B is equivalent over $\mathbf{Z}$ to $O_{n-r} \perp B_0$ with B_0 a non-degenerate half-integral matrix of degree r. Let $\{p_1, ..., p_t\}$ be the set of distinct prime factors of $2^{2[n/2]} \det B_0$. Then $b(B,s)$ can be expressed explicitly in terms of the invariants $\cup_{i=1}^{t}\{d_{p_i,j}(B_0)\ (j = 1, ..., r), \xi_{p_i,j}(B_0)\ (j = 1, ..., [r/2]), \eta_{p_i,j}(B_0)\ (j = 1, ..., [(r-1)/2])\}$ of B_0.*

3 Outline of the proof

In this section we explain the outline of the proof of Theorem 1. For the details, see Katusrada [Kat 8]. Let a be an element of a commutative ring R. Then for an element X of $M_{mn}(R)$ we often use the same symbol X to denote the class of X mod $aM_{mn}(R)$. Now let m, n be non-negative integers such that $m \geq n \geq 1$. For $A \in S_m(\mathbf{Z}_p)$, $B \in S_n(\mathbf{Z}_p)$ define the local density $\alpha_p(B,A)$ by

$$\alpha_p(B,A) = \lim_{e \to \infty} p^{(-mn+n(n+1)/2)e} \# \mathcal{A}_e(B,A),$$

where

$$\mathcal{A}_e(B,A) = \{X = (x_{ij}) \in M_{m,n}(\mathbf{Z}_p)/p^e M_{m,n}(\mathbf{Z}_p);\ A[X] - B \in p^e \mathcal{H}_n(\mathbf{Z}_p)\}.$$

Put $H_k = \overbrace{H \perp ... \perp H}^{k}$. Then it is well known that we have

$$b_{n,p}(B;k) = \alpha_p(B, H_k)$$

for $k \geq n+1$. Thus to investigate the $b_p(B;s)$, first we give several induction formulae for local densities. For a non-degenerate half-integral matrix B let $i(B)$ denote the least integer l such that $p^l B^{-1}$ is half-integral. Then the following theorem can be derived from a generalization of [Kat3], Proposition 2.2' and a modification of [Ki2], Theorem 1.

Theorem 3. (1) *Let p be any prime number. Let $b_1 \in \mathbf{Z}_p \backslash \{0\}$ and $B_2 \in S_{n-1}(\mathbf{Z}_p) \cap GL_{n-1}(\mathbf{Q}_p)$. Assume that $\mathrm{ord}_p(b_1) \geq i(B_2) - 1 + 2\delta_{2p}$.*

$$\alpha_p(p^2 b_1 \perp B_2, H_k) = p^{-2k+n+1}\alpha_p(b_1 \perp B_2, H_k) + (1-p^{-k})(1+p^{1-k})\alpha_p(B_2, H_{k-1}).$$

(2) *Let $b_1, b_2 \in \mathbf{Z}_2 \backslash \{0\}$ and $B_3 \in S_{n-2}(\mathbf{Z}_2) \cap GL_{n-2}(\mathbf{Q}_2)$. Assume that $\mathrm{ord}_p(b_1) = \mathrm{ord}_p(b_2) \geq i(B_3) + 1$. Then we have*

$$\alpha_2(2^2 b_1 \perp b_2 \perp B_3, H_k) = 2^{-2k+n+1}\alpha_2(b_1 \perp b_2 \perp B_3, H_k)$$

$$+(1-2^{-k})(1+2^{1-k})2^{-1} \sum_{u=1,5} \alpha_2(b_2 u \perp B_3, H_{k-1}).$$

(3) *Let* $K = H$ *or* Y, *and* $B_3 \in S_{n-2}(\mathbf{Z}_2) \cap GL_{n-2}(\mathbf{Q}_2)$. *Assume that* $m \geq i(B_3) + 1$. *Then we have*

$$\alpha_2(2^{m+2}K \perp B_3, H_k) = -2^{2(-2k+n+1)+1}\alpha_2(2^m K \perp B_3, H_k)$$
$$+2^{-2k+n+2}\alpha_2(2^{m+1}U \perp B_3, H_k) + 2^{-2k+n+1}\alpha_2(2^m V \perp B_3, H_k)$$
$$+(1-2^{-k})(1-2^{2-2k})(1+2^{2-k})\alpha_2(B_3, H_{k-2}),$$

where $U = H$ *or* $1/2 \perp 3/2$ *according as* $K = H$ *or* Y, *and* $V = 1 \perp -1$ *or* $1 \perp 3$ *according as* $K = H$ *or* Y.

(4) *Let* $u_1, u_2 \in \mathbf{Z}_2 \backslash \{0\}$ *and* $B_3 \in S_{n-2}(\mathbf{Z}_2) \cap GL_{n-2}(\mathbf{Q}_2)$. *Assume that* $m \geq i(B_3) + 3$, *and* $-u_1u_2 \equiv 1 \bmod 4$. *Then we have*

$$\alpha_2(2^m u_1 \perp 2^m u_2 \perp B_3, H_k) = 2^{-2k+n+1}\alpha_2(2^m K \perp B_3, H_k)$$
$$+(1-2^{-k})(1+2^{1-k})4^{-1}\sum_{u=1,3,5,7}\alpha_2(2^m u \perp B_3, H_{k-1}),$$

where K *is* H *or* Y *according as* $-u_1u_2 \equiv 1 \bmod 8$ *or not.*

Remark. The theorem can also be proved by using [KH], Theorem 2.6. For further information on local densities, see [Kat1], [Kat2], [Kat4],[Kat5], and [Kat6].

Next we give a functional equation of $F(B; X)$. To do this, let $f(B; X)$ be a polynomial in X such that $f(B; p^{-s}) = b_p(B; s)$. Then the following proposition is derived from a functional equation of (global) Siegel series by Mizumoto [Mi].

Proposition 4. *Let* C *and* $\hat{C}$ *be non-degenerate symmetric matrices of degree* n *with entries in* $\mathbf{Z}_p$. *Assume that there exists a matrix* V *of degree* n *with entries in* $\mathbf{Z}_p$ *such that* $C = \hat{C}[V]$ *and* $f(\hat{C}, X)$ *is not identically zero. Put*

$$G(C, \hat{C}; X) = \frac{f(C; X)}{X^{\mathrm{ord}_p(\det V)} f(\hat{C}, X)}$$

Then we have

$$G(C, \hat{C}; p^{-n-1}X^{-1}) = G(C, \hat{C}; X).$$

Remark. To prove the above proposition, we used a global functional equation. But as T. Watanabe pointed to the author, the above functional equation can also be proved by rewriting the functional equation in [Kar], 4.8.

Now by the above proposition and a classification theorem of quadratic forms over local fields, we give the following functional equation of $F(B; X)$, whicn is one of key ingredients to prove Theorem 1.

Theorem 5. *For a non-degenerate half-integral matrix* B *of degree* n *over* $\mathbf{Z}_p$ *put* $D(B) = 2^{2[n/2]} \det B$ *and* $d(B) = \mathrm{ord}_p(D(B))$. *Further put*

$$\delta(B) = \begin{cases} 2[(d(B) + 1 - \delta_{2p})/2] & \text{if } n \text{ is even} \\ d(B) & \text{if } n \text{ is odd} \end{cases}$$

with δ_{2p} Kronecker's delta, and

$$e(B) = \begin{cases} \delta(B) - 2 + 2\xi(B)^2 & \text{if } n \text{ is even} \\ \delta(B) & \text{if } n \text{ is odd} \end{cases}.$$

Then we have

$$F(B; p^{-n-1}X^{-1}) = \zeta(B)(p^{(n+1)/2}X)^{-e(B)}F(B; X),$$

where $\zeta(B) = \eta(B)$ or 1 according as n is odd or even.

Now to prove Theorem 1, we give two types of induction formulae for $F(B; X)$. First let p be any prime number. For non-degenerate half-integral matrices B and B_2 of degree n and $n-1$, respectively over $\mathbf{Z}_p$, put $\delta = \delta(B)$ and $\tilde{\delta} = \delta(B_2)$. Further put $\xi = \xi(B), \xi' = \xi'(B)$, and $\tilde{\eta} = \eta(B_2)$ if n is even, and $\tilde{\xi} = \xi(B_2), \tilde{\xi}' = \xi'(B_2)$, and $\eta = \eta(B)$ if n is odd. Here we make the convention that B_2 is the empty matrix and that we have $\tilde{\xi} = \tilde{\xi}' = 1$ and $\tilde{\delta} = 0$ if $n = 1$. We note that $\delta = \delta_1(B)$ and $\tilde{\delta} = \delta_2(B)$. Define rational functions $C(B, B_2; X)^{(1)}$ and $C(B, B_2; X)^{(0)}$ in X by

$$C(B, B_2; X)^{(1)} = \begin{cases} \dfrac{1 - p^{n/2}\xi X}{1 - p^{n+1}X^2} & \text{if } n \text{ is even} \\ \dfrac{1}{1 - p^{(n+1)/2}\tilde{\xi}X} & \text{if } n \text{ is odd,} \end{cases}$$

and

$$C(B, B_2; X)^{(0)} = \begin{cases} \dfrac{(-1)^{\xi+1}\xi'\tilde{\eta}(1 - p^{n/2+1}X\xi)}{1 - p^{n+1}X^2} & \\ \times (p^{n/2}X)^{\delta-\tilde{\delta}+\xi^2}p^{\delta/2} & \text{if } n \text{ is even} \\ \dfrac{(-1)^{\tilde{\xi}}\tilde{\xi}'\eta}{1 - p^{(n+1)/2}X\tilde{\xi}} & \\ \times (p^{(n-1)/2}X)^{\delta-\tilde{\delta}+2-\tilde{\xi}^2}p^{(2\delta-\tilde{\delta}+2)/2} & \text{if } n \text{ is odd.} \end{cases}$$

We note that by definition we have

$$C(B, B_2; X)^{(1)} = \frac{1}{1 - pX} \text{ and } C(B, B_2; X)^{(0)} = \frac{-(pX)^{\delta+1}}{1 - pX}$$

if $n = 1$.

Then by Theorem 3 (1) and Theorem 5 we have.

Theorem 6. *Let $B_1 = (b_1)$ and B_2 be non-degenerate half-integral matrices of degree 1 and $n-1$, respectively over $\mathbf{Z}_p$, and put $B = B_1 \perp B_2$. Assume that $\mathrm{ord}_p(b_1) \geq i(B_2) - 1 + 2\delta_{2p}$. Then we have*

$$F(B; X) = C(B, B_2; X)^{(1)}F(B_2; pX) + C(B, B_2; X)^{(0)}F(B_2; X).$$

Next we must consider a more complicate case for $p = 2$; let B_2 be a non-degenerate half-integral matrix of degree $n-2$ over $\mathbf{Z}_2$. Let

$$B_1 = 2^m K \text{ with } K = H \text{ or } Y,$$

or

$$B_1 = 2^m u_1 \perp 2^m u_2 \text{ with } u_1, u_2 \in \mathbf{Z}_2^*.$$

Put $B = B_1 \perp B_2$. Put $\delta = \delta(B), \tilde{\delta} = \delta(2^m \perp B_2)$, and $\hat{\delta} = \delta(B_2)$. We note that $\delta = \delta_1(B), \tilde{\delta} = \delta_2(B)$, and $\hat{\delta} = \delta_3(B)$. Further put

$$\sigma = \begin{cases} (2\tilde{\delta} - \delta - \hat{\delta} + 2)/2 & \text{if } n \text{ is even, } B_1 = 2^m u_1 \perp 2^m u_2, \text{ and } d(B) \text{ is odd} \\ & \text{or if } n \text{ is even, } B_1 = 2^m K, \text{ and } \xi(B_2) = 0 \\ 2 & \text{if } n \text{ is odd, } B_1 = 2^m K, \text{ and } d_2(B) \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Further we define other quantities as follows; let n be even. Then put $\xi = \xi(B), \xi' = 1 + \xi - \xi^2, \hat{\xi} = \xi(B_2), \hat{\xi}' = 1 + \hat{\xi} - \hat{\xi}^2$, and

$$\tilde{\eta} = \begin{cases} \eta(2^m u_2 \perp B_2) & \text{if } B_1 = 2^m u_1 \perp 2^m u_2, \\ & \text{and } d(B_2) \text{ is even} \\ (-1)^{((n-1)^2-1)/8} h(B_2)(2^m, (-1)^{(n-2)/2} \det B_2)_2 & \text{if } B_1 = 2^m K, \text{ and } \xi(B_2) \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Let n be odd. Then put $\eta = \eta(B), \hat{\eta} = \eta(B_2), \tilde{\xi}' = 1$ and $\tilde{\xi} = 1$ or 0 according as $B_1 = 2^m K$ and $d_2(B)$ is even or not. Now for $i = 1, 2$ and $j = 0, 1$ define a rational function $C(B, B_2; X)^{(ij)}$ in X by

$$C(B, B_2; X)^{(11)} = \begin{cases} \dfrac{1 - 2^{n/2}\xi X}{1 - 2^{n+1}X^2} & \text{if } n \text{ is even} \\ \dfrac{1}{1 - 2^{(n+1)/2}\tilde{\xi} X} & \text{if } n \text{ is odd,} \end{cases}$$

$$C(B, B_2; X)^{(10)} = \begin{cases} \dfrac{(-1)^{\xi+1}\xi'\tilde{\eta}(1 - 2^{n/2+1}X\xi)}{1 - 2^{n+1}X^2} & \\ \times (2^{n/2}X)^{\delta - \tilde{\delta} + \xi^2 + \sigma} 2^{\delta/2} & \text{if } n \text{ is even} \\ \dfrac{(-1)^{\tilde{\xi}}\eta}{1 - 2^{(n+1)/2}X\tilde{\xi}} & \\ \times (2^{(n-1)/2}X)^{\delta - \tilde{\delta} + 2 - \tilde{\xi}^2 + \sigma} 2^{(2\delta - \tilde{\delta} + 2 + \sigma)/2} & \text{if } n \text{ is odd,} \end{cases}$$

$$C(B, B_2; X)^{(21)} = \begin{cases} \dfrac{1}{1 - 2^{n/2}\hat{\xi}X} & \text{if } n \text{ is even} \\ \dfrac{1 - 2^{(n-1)/2}\tilde{\xi}X}{1 - 2^n X^2} & \text{if } n \text{ is odd,} \end{cases}$$

and

$$C(B,B_2;X)^{(20)}=\begin{cases}\dfrac{(-1)^{\hat{\xi}}\hat{\xi}'\tilde{\eta}}{1-2^{n/2}X^{\hat{\xi}}} & \\ \times(2^{(n-2)/2}X)^{\tilde{\delta}-\hat{\delta}+2-\hat{\xi}^2-\sigma}2^{(2\tilde{\delta}-\hat{\delta}+2-2\sigma)/2} & \text{if } n \text{ is even}\\ \dfrac{(-1)^{\tilde{\xi}+1}\hat{\eta}(1-2^{(n+1)/2}X^{\tilde{\xi}})}{1-2^nX^2} & \\ \times(2^{(n-1)/2}X)^{\tilde{\delta}-\hat{\delta}+\tilde{\xi}^2-\sigma}2^{(\tilde{\delta}-\sigma)/2} & \text{if } n \text{ is odd.}\end{cases}$$

Here we make the convention that B_2 is the empty matrix and that we have $\tilde{\eta}=\hat{\xi}=\hat{\xi}'=1,\tilde{\delta}=m$ and $\hat{\delta}=0$ if $n=2$. Thus we have

$$C(B,B_2;X)^{(11)}=\frac{1-2^2\xi X}{1-2^3X^2},$$

$$C(B,B_2;X)^{(10)}=\frac{(-1)^{\xi+1}\xi'(1-2^2\xi X)}{1-2^3X^2}(2X)^{\delta-m+\xi^2}2^{\delta/2},$$

$$C(B,B_2;X)^{(21)}=\frac{1}{1-2X},$$

and

$$C(B,B_2;X)^{(20)}=-\frac{(2X)^m}{1-2X}$$

if $B=2^mK$ or $2^mu_1\perp 2^mu_2$. Then by Theorem 3, (2),(3),(4) and Theorem 5 we have

Theorem 7. *Let $B_1=2^mu_1\perp 2^mu_2$ with $u_1,u_2\in\mathbf{Z}_2^*$ or $B_1=2^mK$ with $K=H$ or Y. Let $B_2\in\mathcal{H}_{n-2}(\mathbf{Z}_2)$ and put $B=B_1\perp B_2$. Assume that $m\geq i(B_2)+1$. Then we have*

$$F(B;X)=C(B,B_2;X)^{(11)}C(B,B_2;2X)^{(21)}F(B_2;4X)$$
$$+\{C(B,B_2;X)^{(11)}C(B,B_2;2X)^{(20)}+C(B,B_2;X)^{(10)}C(B,B_2;X)^{(21)}\}F(B_2;2X)$$
$$+C(B,B_2;X)^{(10)}C(B,B_2;X)^{(20)}F(B_2;X).$$

Proof of Theorem 1. The theorem can be proved by using Theorems 6 and 7 repeatedly.

4 Examples

In this section we give some examples.

(1). Let $n=1$. Then for any $B\in\mathcal{H}_1(\mathbf{Z}_p)\cap GL_1(\mathbf{Q}_p)$ such that $\mathrm{ord}_p(B)=m$ we have

$$F(B;X)=\frac{1-(pX)^{m+1}}{1-pX}=\sum_{i=0}^{m}(pX)^i.$$

This is well known.

(2). Let $n = 2$ and $B = (b_{ij})_{1 \le ij \le 2}$. Let $\delta = 2[(d_1(B) + 1 - \delta_{2p})/2]$ and $\tilde{\delta} = d_2(B)$. Further put $\xi = \xi(B), \xi' = \xi'(B)$. Then by Theorems 4.1 and 4.2 we have

$$F(B;X) = \frac{(1-p\xi X)(1-(p^2X)^{\tilde{\delta}+1})}{(1-p^3X^2)(1-p^2X)}$$

$$+\frac{(-1)^{\xi+1}\xi'(pX)^{\delta-\tilde{\delta}+\xi^2}p^{\delta/2}(1-p^2\xi X)(1-(pX)^{\tilde{\delta}+1})}{(1-p^3X^2)(1-pX)}.$$

Rewriting this, we have

$$F(B;X) = \xi(pX)^{\delta-\tilde{\delta}+1}p^{\delta/2}\sum_{i=0}^{\tilde{\delta}}(pX)^i$$

$$+(1-p\xi X)\sum_{i=0}^{\tilde{\delta}}(p^2X)^i\sum_{j=0}^{\delta/2+\xi^2-i-1}(p^3X^2)^j.$$

This coincides with [Kau]. See also [Ma2] and [Ki2].

(3). Let $n = 3$. Let $B = (b_{ij})_{1 \le i,j \le 3}$. Put $\eta = (-1)^{\delta_{2p}h(B)}, \delta = d_1(B), \tilde{\delta} = 2[(d_2(B) + 1 - \delta_{2p})/2]$, and $\hat{\delta} = d_3(B)$. To give an explicit form of $F(B;X)$ we define another invariant $\tilde{\xi}$ as follows; let $p \neq 2$. Let

(3.1) $B \cong p^{r_1}u_1 \perp p^{r_2}u_2 \perp p^{r_3}u_3$

be the Jordan decomposition of B as in Section 4, where $r_1 \ge r_2 \ge r_3 \ge 0$ and $u_1, u_2, u_3 \in \mathbf{Z}_p^*$. Then put

$$\tilde{\xi} = \chi(-p^{r_2}u_2p^{r_3}u_3).$$

Next let $p = 2$. Then B is equivalent to one of the following canonical forms:

(3.2) $B \cong 2^{r_1}u_1 \perp 2^{r_2}u_2 \perp 2^{r_3}u_3$ with $r_1 \ge r_2 \ge r_3 \ge 0, u_1, u_2, u_3 \in \mathbf{Z}_2^*$

(3.3) $B \cong 2^{r_1}K \perp 2^{r_3}u_3$ with $r_1 \ge r_3 + 1 \ge 0, u_3 \in \mathbf{Z}_2^*$

(3.4) $B \cong 2^{r_1}u_1 \perp 2^{r_2}K$ with $r_1 \ge r_2 \ge 0, u_1 \in \mathbf{Z}_2^*$.

Then define $\tilde{\xi}$ by

$$\tilde{\xi} = \begin{cases} \chi(-2^{r_2}u_2 2^{r_3}u_3) & \text{if } B \text{ is of type (3.2) and } r_1 > r_2, \\ 1 & \text{if } B \text{ is of type (3.3) and } d_2(B) \text{ is even} \\ 0 & \text{if } B \text{ is of type (3.3) and } d_2(B) \text{ is odd} \\ & \text{or if } B \text{ is of type (3.2) and } r_1 = r_2, \\ \chi(-\det K) & \text{if } B \text{ is of type (3.4)} \end{cases}.$$

Now let p be any prime number. Then we define $\sigma = 2$, or 0 according as B is of type (3.3) and $d_2(B)$ is even, or not, and put $\tilde{\delta}' = \tilde{\delta} - \sigma$. Further put $\tilde{\xi}' = 1 + \tilde{\xi} - \tilde{\xi}^2$. Then we have

$F(B;X)$

$$= \frac{1}{1-p^5X^2}\frac{1-(p^3X)^{\hat{\delta}+1}}{1-p^3X} - \eta(p^3X^2)^{(\delta-\hat{\delta}+2)/2}p^{(\delta+\hat{\delta})/2}\frac{1-(pX)^{\hat{\delta}+1}}{(1-p^3X^2)(1-pX)}$$

$$+\{\frac{(-1)^{\tilde{\xi}+1}\tilde{\xi}'(p^2X)^{\tilde{\delta}'-\hat{\delta}+\tilde{\xi}^2}p^{\tilde{\delta}'/2}(1-p^3X\tilde{\xi})}{(1-p^2X\tilde{\xi})(1-p^5X^2)}$$

$$-\frac{(-1)^{\tilde{\xi}+1}\eta\tilde{\xi}'(pX)^{\delta-\tilde{\delta}'+2-\tilde{\xi}^2}p^{(2\delta-\tilde{\delta}'+2)/2}(1-pX\tilde{\xi})}{(1-p^2X\tilde{\xi})(1-p^3X^2)}\}\frac{1-(p^2X)^{\hat{\delta}+1}}{1-p^2X}.$$

Rewriting this, we have

$$F(B;X)=\sum_{i=0}^{\hat{\delta}}(p^3X)^i\sum_{j=0}^{\tilde{\delta}'/2-i-1}(p^5X^2)^j$$

$$+\eta(p^3X^2)^{(2\delta-\tilde{\delta}'+2-2\tilde{\xi}^2)/2}(pX)^{-\delta}\sum_{i=0}^{\hat{\delta}}(p^2X)^i\sum_{j=0}^{\tilde{\delta}'/2+\tilde{\xi}^2-i-1}(p^3X^2)^j$$

$$+\tilde{\xi}^2(p^5X^2)^{\tilde{\delta}'/2}(p^2X)^{-\hat{\delta}}\sum_{i=0}^{\hat{\delta}}(p^2X)^i\sum_{j=0}^{\delta-2\tilde{\delta}'+\hat{\delta}-1}(p^2\tilde{\xi}X)^i.$$

This coincides with [Kat2], Theorem 1.1, though it is slightly different from the latter in formulation. See also [Ki3].

5 Comments

(1) Our method of the proof of Theorem 1 seems applicable to other types of Eisensten series.

(2) As is well known $E_{n,k}(Z,0)$ has the following Fourier expansion

$$E_{n,k}(Z,0)=\sum_B c_{n,k}(B)e(BZ),$$

where B runs over all semi-positive definite half-integral matrices of degree n over $\mathbf{Z}$. Then we define Koecher-Maass Dirichlet series $M_{n,k}(s)$ associated with $E_{n,k}(Z,0)$ by

$$M_{n,k}(s)=\sum_B\frac{c_{n,k}(B)}{\epsilon(B)(\det B)^s}$$

where B runs over a complete set of representatives of $GL_n(\mathbf{Z})$-equivalence classes of positive definite half-integral matrices of degree n over $\mathbf{Z}$, and $\epsilon(B)=\#\{X\in M_n(\mathbf{Z});{}^tXBX=B\}$. Ibukiyama and the author has given an explicit form of $M_{n,k}(s)$ without using Theorem 1 (cf. [IK]). It is interesting problem to get the same result directly from Theorem 1.

(3) We have proved Theorems 6 and 7 assuming the functional equation for the global Siegel series. But conversely by these theorems we can prove Theorem 5, and therfore get the functional equation for the global Siegel series. Thus it seems very interesting problem to prove Theorems 6 and 7 directly from the local theory of quadratic forms.

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