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Two-dimensional Toda cellular automaton

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Introduction

The direct connections between soliton equations and soliton cellular automata(SCA's) was found recently [1, 2, 3]. This finding resulted from investigation of the Toda cellular automaton [4, 5] and will offer new fields both in the study of cellular automata (CA's) and in that of integrable systems with infinite degrees of freedom. The SCA's are integrable in the sense that they have N-soliton solutions (N>2) and infinitely many conserved quantities. In order to obtain SCA's from integrable partial differntial equations, we need two steps of discretization: discretization of independent variables and that of dependent variables (ultra-discretization). As for the former step, Hirota proposed a unified difference equation named Discrete Analog of Generalized Toda Equation (DAGTE), from which a number of important integrable difference and/or differential equations are obtained by reduction and variable transformations [6, 7]. The relation of DAGTE to the τ function of KP hierarchy was clarified by Miwa, and an algebraic approach to integrable partial diffrence equations with fermion field operators was established by Date, Jimbo and Miwa [8, 9]. In the present paper, we show a general formalism to construct SCA's based on this algebraic approach. As a consequence, a 2+1 dimensional SCA (2D Toda cellular automaton) is obtained from ultara-discretization of DAGTE [10]. We also show the SCA analog of Bäcklund transformation and Lax representations [7, 11].

SCA construction in terms of fermion operators

In early 80's, Sato established a unified theory of solitons [12]. He showed that any integrable differential equation can be regarded as a dynamical system on a universal Grassmann manifold (UGM). A solution to the nonlinear equation corresponds to a point of UGM. It is called the τ function. Using the Plücker relation of UGM, we obtain Hirota's bilinear identity for the τ function. Date, Jimbo, Kashiwara and Miwa developed the Sato theory giving its link with infinite dimensional Lie algebras by the method of field theory and vertex operators [13]. Then the τ function is expressed as a vacuum expectation value of a fermion field operator.

In terms of usual fermion creation and annihilation operators, which satisfy

$$[\psi_i,\psi_j^*]_+\equiv\psi_i\psi_j^*+\psi_j^*\psi_i=\delta_{i,-j}$$

and

$$[\psi_i, \psi_j]_+ = [\psi_i^*, \psi_j^*]_+ = 0, \quad (i, j \in \mathbf{Z} + 1/2),$$

the N-soliton solution is expressed as [9]

$$\tau(\mathbf{t}) = \langle vac| \prod_{i=1}^{N} (1 + c_i \psi(p_i, \mathbf{t}) \psi^*(q_i, \mathbf{t})) |vac\rangle$$

where

$$\begin{aligned} \psi(p,t) &\equiv e^{\xi(\mathbf{t},p)} \psi(p), \quad \psi^*(q,t) \equiv e^{-\xi(\mathbf{t},q)} \psi^*(q), \\ \psi(p) &\equiv \sum_{j \in \mathbf{Z}+1/2} \psi_j p^{-j-1/2}, \quad \psi^*(q) \equiv \sum_{j \in \mathbf{Z}+1/2} \psi_j^* q^{-j-1/2} \end{aligned}$$

The operator product is a radically ordered product as usual, c_i is an arbitrary constant, $\mathbf{t}$ denotes the time variables and a function $\xi(\mathbf{t}, p)$ is arbitrary in principle, though we need a careful choice in order to get a significant differential or difference equation. Then the Hirota's bilinear identity is a consequnce of an elementary complex analysis:

$$\sum_{all} Res \left[\langle vac|\psi_{1/2}^* \psi(z, \mathbf{t}) g_N(\mathbf{t}) |vac\rangle \langle vac| \psi_{-1/2} \psi^*(z, \mathbf{t}') g_N(\mathbf{t}') |vac\rangle \right] = 0, \qquad (1)$$

where

$$g_N(\mathbf{t}) = \prod_{i=1}^{N} (1 + c_i \psi(p_i, \mathbf{t}) \psi^*(q_i, \mathbf{t})) .$$

In order to construct integrable CA's, we should put $\mathbf{t} = \{t, j_1, j_2, \cdots, j_m\}$ and impose the condition: $e^{\xi(\mathbf{t},p) - \xi(\mathbf{t},q)} = e^{(-\omega t + k_1 j_1 + \cdots + k_m j_m)/\epsilon}$, where two of the ω and k_j's are arbitrary integers. This condition gives the dispersion relation $\omega = \omega(k_1, \cdots, k_m : \epsilon)$ for we have only two free parameters p and q. At the same time, p and q have asymptotic forms: $p = e^{P/\epsilon} + \cdots$ and $q = e^{Q/\epsilon} + \cdots$. Thus putting $\rho(\mathbf{t}) \equiv \lim_{\epsilon \to +0} \epsilon \log[\tau(\mathbf{t})]$ with some careful choice of coefficients of the bilinear identity, $\rho(\mathbf{t})$ satisfies an equation similar to Hirota's bilinear identity, from which we obtain an m dimensional CA and its N-soliton solutions. Since the CA is thus constructed, it naturally inherits the geometrical and algebraic nature of the τ function.

Bäcklund transformation of difference-difference equations and its CA analog

Bäcklund transformations are classified into two types: transformation of an N-soliton solution to an N-soliton solution with different phases and that to an (N+1)-soliton solution. We define $g_N(\mathbf{t})$, $f_N(\mathbf{t})$ and $g_{N+1}(\mathbf{t})$ as

$$\begin{aligned} g_N(\mathbf{t}) &= \prod_{i=1}^{N} (1 + c_i \psi(p_i, \mathbf{t}) \psi^*(q_i, \mathbf{t})) \\ f_N(\mathbf{t}) &= \psi(p_{N+1}, \mathbf{t}) g_N(\mathbf{t}) \\ g_{N+1}(\mathbf{t}) &= (1 + \psi(p_{N+1}, \mathbf{t}) \psi(q_{N+1}, \mathbf{t})) g_N(\mathbf{t}). \end{aligned}$$

The N-soliton solution, N-soliton solution with different phase and (N+1)-soliton solutions are respectively given as

$$\begin{aligned}\tau_N(\mathbf{t}) &= \langle vac|g_N(\mathbf{t})|vac\rangle,\\ \sigma_N(\mathbf{t}) &= \langle vac|f_N(\mathbf{t})|vac\rangle,\\ \tau_{N+1}(\mathbf{t}) &= \langle vac|g_{N+1}(\mathbf{t})|vac\rangle.\end{aligned}$$

Both types of Bäcklund transformations can be obtained by the bilinear identity

$$\sum_{all} Res\left[\langle vac|\psi^*_{1/2}\psi(z,\mathbf{t})g_M(\mathbf{t})|vac\rangle\langle vac|\psi^*(z,\mathbf{t}')f_M(\mathbf{t}')|vac\rangle\right] = 0, \tag{2}$$

and the identity $[1+\psi(p_{N+1},\mathbf{t})\psi^*(q_{N+1},\mathbf{t})]\,f_N(\mathbf{t}) = f_N(\mathbf{t})$. It should be noted that the Bäcklund transformation is essentially equal to the Lax representation of the difference-difference equation. Extension to multi-components systems can be done by fermion operators with *colors*. The CA analog of the Bäcklund transformation is straightforwardly obtained by ultra-discretization in the same manner as described in the previous section.

2D Toda cellular automaton

Let $\mathbf{t} = (l,m,n)$ and $e^{-\xi(\mathbf{t},k)} = k^l(1-ak)^m(1-bk^{-1})^n$. Then the bilinear identity (1) turns into

$$\begin{aligned}\tau(l,m,n+1)\tau(l,m+1,n) \quad &-ab\tau(l-1,m+1,n)\tau(l+1,m,n+1)\\ &-(1-ab)\tau(l,m+1,n+1)\tau(l,m,n) = 0.\end{aligned} \tag{3}$$

By variable transformation
$\chi(t,x,y) = \chi(l+m-n, l, m+n-1) = \tau(l,m,n)$ and $\delta^2 \equiv ab$, we have

$$\begin{aligned}\chi(t-1,x,y)\chi(t+1,x,y) \quad &-\delta^2\chi(t,x-1,y)\chi(t,x+1,y)\\ &-(1-\delta^2)\chi(t,x,y+1)\chi(t,x,y-1) = 0.\end{aligned} \tag{4}$$

This is the DAGTE proposed by Hirota. The equation (3) or (4) reduces to a discrete analogue of the 2 dimensional Toda equation [14],

$$\begin{aligned}V(l,m,n+1) - V(l,m,n) &= I(l,m+1,n)V(l,m,n+1) - I(l+1,m,n)V(l,m,n),\\ I(l,m+1,n) - I(l,m+1,n) &= V(l-1,m,n+1) - V(l,m,n).\end{aligned}$$

where

$$V(l,m,n) = \frac{\tau(l+1,m,n+1)\tau(l-1,m+1,n)}{\tau(l,m+1,n)\tau(l,m,n)}$$

$$I(l,m,n) = \frac{1}{\delta}\left\{(1-\delta^2)\frac{\tau(l,m,n+1)\tau(l-1,m,n)}{\tau(l,m,n)\tau(l-1,m,n+1)} - 1\right\}.$$

Let us derive an ultra-discrete version of eq. Eq. (4). The dependent variable transformation,

$$\chi(t,x,y)=\exp[S(t,x,y)] \tag{5}$$

yields

$$\exp[\Delta_t^2 S(t,x,y)]-\delta^2\exp[\Delta_x^2 S(t,x,y)]-(1-\delta^2)\exp[\Delta_y^2 S(t,x,y)]=0, \tag{6}$$

or equivalently,

$$\exp[(\Delta_t^2-\Delta_y^2)S(t,x,y)]=(1-\delta^2)\left(1+\frac{\delta^2}{1-\delta^2}\exp[(\Delta_x^2-\Delta_y^2)S(t,x,y)]\right). \tag{7}$$

Each operator Δ_t, Δ_x and Δ_y represents central difference operator defined, for example, by

$$\Delta_t^2 S(t,x,y)=S(t+1,x,y)-2S(t,x,y)+S(t-1,x,y).$$

Taking a logarithm of Eq. (7) and operating $(\Delta_x^2-\Delta_y^2)$, we have

$$(\Delta_t^2-\Delta_y^2)u(t,x,y)=(\Delta_x^2-\Delta_y^2)\log\left(1+\frac{\delta^2}{1-\delta^2}\exp[u(t,x,y)]\right), \tag{8}$$

where

$$u(t,x,y)=(\Delta_x^2-\Delta_y^2)S(t,x,y). \tag{9}$$

We finally take an ultra-discrete limit of eq. Eq. (8). Putting

$$u(t,x,y)=\frac{v_\varepsilon(t,x,y)}{\varepsilon},\qquad \frac{\delta^2}{1-\delta^2}=\mathrm{e}^{-\frac{\theta_0}{\varepsilon}}, \tag{10}$$

and taking the small limit of ε, we obtain the following equation,

$$(\Delta_t^2-\Delta_y^2)v(t,x,y) = (\Delta_x^2-\Delta_y^2)F(v(t,x,y)-\theta_0), \tag{11}$$

$$F(X) = \max[0,X]. \tag{12}$$

We have rewritten $\lim_{\varepsilon\to+0} v_\varepsilon(t,x,y)$ as $v(t,x,y)$ in Eq. (11). We call the ultra-discrete system satisfying the above Eq. (11) the *2D Toda cellular automaton.*

It should be noted that the N-soliton solution can also be found through the same limiting procedure. This is given by

$$v(t,x,y)=$$
$$\max_{\mu=0,1}\left[\max_{i=1,2,\cdots,N}[\mu_i(K_i+P_i)],\max_{i<j}[\mu_i\mu_j\Theta_{ij}]\right]$$

$$
\begin{aligned}
&+\max_{\mu=0,1}\left[\max_{i=1,2,\cdots,N}[\mu_i(K_i-P_i)],\max_{i<j}[\mu_i\mu_j\Theta_{ij}]\right]\\
&-\max_{\mu=0,1}\left[\max_{i=1,2,\cdots,N}[\mu_i(K_i+Q_i)],\max_{i<j}[\mu_i\mu_j\Theta_{ij}]\right]\\
&-\max_{\mu=0,1}\left[\max_{i=1,2,\cdots,N}[\mu_i(K_i-Q_i)],\max_{i<j}[\mu_i\mu_j\Theta_{ij}]\right],\\
&\quad K_i = P_i x + Q_i y + \Omega_i t,\\
&\quad |\Omega_i| = \max[|P_i|, |Q_i| + \theta_0] - \max[0, \theta_0].
\end{aligned}
$$

Each phase shift term $\Theta_{ij}(1 \le i < j \le N)$ satisfies the relation,

$$
\begin{aligned}
&\max\left[\Theta_{ij} + \max[0,\theta_0] + |\Omega_i + \Omega_j|, \max[0,\theta_0] + |\Omega_i - \Omega_j|\right]\\
&\quad = \max\left[\Theta_{ij} + |P_i + P_j|, \Theta_{ij} + \theta_0 + |Q_i + Q_j|, |P_i - P_j|, \theta_0 + |Q_i - Q_j|\right].
\end{aligned}
$$

Finally we briefly show an SCA analog of Bäcklund transformation for 2D Toda cellular automaton. Putting $\mathbf{t} = (l+1, m+1, n)$ and $\mathbf{t}' = (l, m, n)$ in Eq (2), we obtain the Bäcklund transformation for Eq. (3) as

$$
\begin{aligned}
&\tau_M(l,m,n)\sigma_N(l+1,m+1,n+1) = \tau_M(l,m+1,n)\sigma_N(l+1,m,n)\\
&\quad + a\tau_M(l+1,m,n)\sigma_N(l,m+1,n), \qquad \text{where } M = N \text{ or } N+1,
\end{aligned}
$$

or by the variable transformation

$$
f(t-1,x,y)f'(t+1,x,y+1) = f(t,x,y+1)f'(t,x+1,y) + af(t,x+1,y)f'(t,x,y+1).
$$

The ultra-discrete limit of the above bilinear identity is an SCA analog of Bäcklund transformation and written as

$$
\begin{aligned}
&\rho(t-1,x,y) + \rho'(t+1,x,y+1) =\\
&\quad \max[\rho(t,x,y+1) + \rho'(t,x+1,y), \rho(t,x+1,y) + \rho'(t,x+1,y) - \theta_0].
\end{aligned}
$$

If we put $\mathbf{t} = (l, m+1, n+1)$ or $(l+1, m, n+1)$, another Bäcklund transformation is obtained. We can deduce Lax representation of DAGTE and its conserved quantities from the set of bilinear identities. Ultra-discrete limit of them are an SCA analog of Lax represetation and conserved quantities of 2D Toda cellular automaton.

References

[1] D. Takahashi and J. Satsuma, J. Phys. Soc. Jpn. **59** (1990) 3514.

[2] T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma, Phys. Rev. Lett. 76 (1996) 3247.

[3] J. Matsukidaira, J. Satsuma, D. Takahashi, T. Tokihiro and M. Torii, to be published.

[4] D. Takahashi, *in the present volume.*

[5] D. Takahashi and J. Matsukidaira, Phys. Lett. A **209** (1995) 184.

[6] R. Hirota, J. Phys. Soc. Jpn. **45** (1981) 3785.

[7] R. Hirota, *Direct Method in Soliton Theory* (Iwanami, Tokyo, 1992) [in Japanese].

[8] T. Miwa, Proc. Japan Acad. **58A** (1982) 9.

[9] E. Date, M. Jimbo and T. Miwa, J. Phys. Soc. Jpn. **51** (1982) 4125 .

[10] S. Moriwaki, A. Nagai, J.Satsuma, T. Tokihiro, M. Torii, J. Matsukidaira and D. Takahashi, unpublished.

[11] T. TOkihiro and J. Satsuma, unpublished.

[12] M. Sato, RIMS Kokyuroku **439** (1981) 30 .

[13] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Physica **4D** (1982) 343, and references cited there.

[14] R. Hirota, S. Tsujimoto and T. Imai, RIMS Kokyuroku **822** (1992) 144.