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Classification of involutions of lattices with conditions and real algebraic curves on a hyperboloid *

(条件付き対合付き格子の分類と hyperboloid 上の実代数曲線)

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§1. Introduction

Real algebraic curves on a hyperboloid (i.e., $\boldsymbol{R}P^1 \times \boldsymbol{R}P^1$) or an ellipsoid have been studied by several people, D. A. Gudkov ([5]), V. I. Zvonilov ([23],[24],[25],[26]), P. Gilmer ([4]), G. Mikhalkin ([14],[16],[15]), the author ([12],[11],[10],[13],[21]) and others. The author has been studying especially curves of bidegree (4,4) on a hyperboloid. The classification of "real schemes" (i.e., isotopic classification on $\boldsymbol{R}P^1 \times \boldsymbol{R}P^1$) of nonsingular real algebraic curves of bidegree (4,4) on a hyperboloid was completed by Zvonilov ([25]) and the author ([13]) independently.

In the same paper [25], Zvonilov also judged the "dividingness" (see §6) of each real scheme and the "complex orientation" of each dividing curve. He did this work by using "Rokhlin type formula" obtained by himself ([23]) and Gilmer's results on the rotation numbers of dividing curves ([4]).

In the meanwhile, after her work of the isotopic classification, the author started to apply Nikulin's theory of "involutions of lattices with conditions" (see [19]) to curves of bidegree (4,4) on a hyperboloid. I. Itenberg ([6],[8],[7],[9]) and A. Degtyarev ([2],[3]) also have done similar approaches for singular curves of degree 6 in $\boldsymbol{R}P^2$ or singular surfaces of degree 4 in $\boldsymbol{R}P^3$. In 1995, the author finished enumerating up all the "genera" of our "involutions of lattices with our condition", i.e., the 2-dimensional cohomology groups of the double coverings of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ branched along nonsingular real algebraic curves of bidegree (4,4). The result of that work was first appeared in [21]. But "the table of all the genera" in [21] has some mistypes, duplications and a wrong topological interpretation. So the author distributed a revised table to some people. (The present article also includes the revised table in §5.)

Anyway, since then, the author has been investigating the topological properties of curves which realize each genus, where 'topological properties' mean real schemes, dividingness, complex orientations, e.t.c. In this article, the author will collect and arrange the processes and

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results of her investigation stated above, and prove some known or unknown facts by using 'the table of genera'. Finally, she will indicate some summarized questions.

Acknowledgment

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§2. Our situation (I)

Let A be a nonsingular **real** algebraic curve of bidegree(4,4) in $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ and Y be the double covering of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ branched along A. Then the complex conjugation of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ is lifted into two anti-holomorphic involutions of Y, which are denoted by T^+ and T^-. (For the details, see [12],[11],[13].)

We set $L = H^2(Y; \boldsymbol{Z})$. Since the bidegree is (4,4), Y is a $K3$ surface. And so L is an even unimodular lattice of signature (3,19). We set $e_1 = \pi^*([\infty \times \boldsymbol{P}^1])$ and $e_2 = \pi^*([\boldsymbol{P}^1 \times \infty])$, where $\pi : Y \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ is the covering map. Then we see $e_1 \cdot e_1 = e_2 \cdot e_2 = 0$ and $e_1 \cdot e_2 = 2$. Let T be T^+ or T^-. Then we see $T^*(e_i) = -e_i \quad (i = 1,\ 2)$. Let S be the subgroup of L generated by e_1 and e_2. Then S is a primitive subgroup of L. We set $\varphi = T^*$ and $\theta = \varphi|_S$.

We now obtain two "lattices with involutions" (L, φ) and (S, θ). Let i denote the inclusion map : $S \to L$, and we set $G = \{\mathrm{id}_S\}$. Then

$$(L, \varphi, i)$$

is an *involution of a lattice with condition* (S, θ, G) in the sense of Nikulin [19]. We will give precise definitions in the next section.

§3. Definitions

By a *lattice* we mean a nondegenerate symmetric bilinear form over $\boldsymbol{Z}$. By a *homomorphism of lattices* we mean a group homomorphism preserving the bilinear form.

By a *condition* (on an involution of a lattice) we mean a triple (S, θ, G), where S is a nondegenerate lattice, θ is an involution of S, and G is a distinguished subgroup of $O(S, \theta)$, where we set $O(S, \theta) = \{f :$ automorphism of$S | f \circ \theta = \theta \circ f\}$. In [19] S is assumed to be possibly degenerate, but in this article we assume that it is nondegenerate.

By an *involution (of a lattice) with condition* (S, θ, G) we mean a triple (L, φ, i), L is a lattice, φ is an involution of L and $i : S \subset L$ is a primitive embedding of lattices which satisfies $\varphi \circ i = i \circ \theta$. Two involutions (L, φ, i) and (L', φ', i') with condition (S, θ, G) are called *isomorphic* if there is an isomorphism $u : L \to L'$ of lattices with involutions (that is, $\varphi' \circ u = u \circ \varphi$) such that u preserves the condition (S, θ, G) (that is, $u \circ i = i' \circ g$ for some $g \in G$). Moreover, we introduce a weaker equivalence relation. We say two involutions (L, φ, i) and (L', φ', i') with condition (S, θ, G) belong to a same *genus* if for every prime p ($= 2, 3, 5, 7, \cdots$, and ∞), there exists an Z_p-isomorphism $u : L \otimes_Z Z_p \to L' \otimes_Z Z_p$ of induced lattices with induced involutions (that is, $\overline{\varphi'} \circ u = u \circ \overline{\varphi}$) such that u preserves the condition (S, θ, G) (that is, $u \circ i = i' \circ g$ for some $g \in G$). (We are refered to, for example, p.43 of [17] for the definition of 'genus'. The author could not find the clear definition of the genus of an involution of a lattice with a condition in [19].)

In this article, as in [19], we treat only even lattices. If M is a (nondegenarate) lattice, we set $A_M = M^*/M$, which is called the ***discriminant group*** of M, and q_M denotes the ***discriminant (quadratic) form*** of M. (For the details, see p.109 of [18].)

For an involution of a lattice (L, φ, i) with condition (S, θ, G) stated above, we consider the restricted lattices:

$$L_\pm = \{x \in L | \varphi(x) = \pm x\}$$

and

$$S_\pm = \{x \in S | \theta(x) = \pm x\}.$$

Since we see that the discriminant group $A_{L_+} = L_+^*/L_+$ is isomorphic to the direct sum of some $Z/2$'s. Let a denote the number of those $Z/2$'s. And let $(t_{(+)}, t_{(-)})$ denotes the signature of L_+.

We define the invariants δ_φ and $\delta_{\varphi S}$ as follows.

$$\delta_\varphi = \begin{cases} 0 & \text{if } x \cdot \varphi(x) \equiv 0 \text{ (mod2) } \forall x \in L \\ 1 & \text{otherwise} \end{cases}$$

$$\delta_{\varphi S} = \begin{cases} 0 & \text{if } x \cdot \varphi(x) \equiv x \cdot s_\varphi \text{ (mod2) } \forall x \in L \\ & \text{for some } s_\varphi \text{ in } S \\ 1 & \text{otherwise} \end{cases}$$

Then (L, φ, i) is of one of the following 3 types:

Type 0: $\delta_\varphi = 0$ (then, $\delta_{\varphi S} = 0$)
Type Ia: $\delta_\varphi = 1$ and $\delta_{\varphi S} = 0$
Type Ib: $\delta_{\varphi S} = 1$

For the elements $x_\pm \in S_\pm$, we define the invariant

$$\delta_{x_\pm} = \begin{cases} 0 & \text{if } x_\pm \cdot L_\pm \equiv 0 \text{ (mod2)} \\ 1 & \text{otherwise} \end{cases}$$

Then we get two functions $\delta_\pm : x_\pm \mapsto \delta_{x_\pm}$, and we define

$$H_\pm = \frac{1}{2}\delta_\pm^{-1}(0)/S_\pm.$$

We see they are contained in $(\frac{1}{2}S_{\pm} \cap S_{\pm}^{*})/S_{\pm}$. An another equivalent definition of $H_{\pm}$ is given in p.105 of [19]. We use the above definition because of the importance of topological interpretations (see for example, [10] and Lemma 4 in §6) of the invariants $\delta_{x_{\pm}}$.

Finally, we define the group $H_{+} \oplus_{\gamma} H_{-}$ and the embedding $\gamma_r : H_{+} \oplus_{\gamma} H_{-} \to A_{L_{+}}$ as in p.105 of [19]. And we set $q_r = \gamma_r^{*} q_{L_{+}}$, where $q_{L_{+}}$ is the discriminant form of L_{+}. Then q_r is a 'finite quadratic form' (see p.108 of [18] for the definition). And note that the form q_r is **possibly degenerate**. See also p.108 of [18] for the definition of degeneracy of finite quadratic forms.

We put $q = q_{L_{+}}$ and let v_q $(\in A_{L_{+}})$ denote *the characteristic element*(see p.108 of [19]) of q. We see the following:

$\delta_{\varphi} = 0$ if and only if $v_q = 0$

$\delta_{\varphi S} = 0$ if and only if v_q is contained in $\gamma_r(H_{+} \oplus_{\gamma} H_{-})$

Thus, for Type Ia, we denote by v the element of $H_{+} \oplus_{\gamma} H_{-}$ such that $\gamma_r(v) = v_q$. We call it *the characteristic element of the embedding* γ_r.

§4. Our situation (II)

We return to our situation stated in §2. We first remark that $t_{(+)} = 1$ in our case. (For the reason, see p.156 of [18].) Next, it is obvious that $S_{+} = \{0\}$ and $S_{-} = S$ because $\theta = -1$. We see the discriminant group (recall §3) $A_{S_{-}} = S_{-}^{*}/S_{-} = S^{*}/S$ is generated by $[e_1^{*}]$ $(= [\frac{1}{2}e_2])$ and $[e_2^{*}]$ $(= [\frac{1}{2}e_1])$, and hence it is isomorphic to $\boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2$, where e_i^{*} $(i = 1, 2)$ is the dual element of e_i. While $A_{S_{+}} = H_{+} = \{0\}$, H_{-} is a subgroup of $(S_{-}^{*} \cap (\frac{1}{2}S_{-}))/S_{-} = A_{S_{-}}$, namely, one of the following 5 subgroups:

$$\{0\},\ <[\tfrac{1}{2}e_1]>,\ <[\tfrac{1}{2}e_2]>,\ <[\tfrac{1}{2}h]>\ \text{and}\ A_{S_{-}},$$

where we set $h = e_1 + e_2$.

§5. Applications of Nikulin's results to our situation

We now fix our condition (S, θ, G), namely, S is the lattice represented by $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, $\theta = -1$ and $G = \{\mathrm{id}_S\}$. And we restrict ourselves to involutions of lattices (L, φ, i) with the condition (S, θ, G), where L is the even unimodular lattice of signature (3,19) (so-called the $K3$ lattice) and $t_{(+)} = 1$.

In our case, since $H_{+} = 0$, we have $q_r = (-q_{S_{-}})|H_{-}$ (recall §3), and hence, it is just determined by H_{-}. For the embedding $\gamma_r : q_r \to q$, we have $\gamma_r = \gamma_{H_{-}} = \gamma_{L_{+}S_{-}}$ (see p.105 of [19]). And, in the case of TypeIa, the characteristic element v of the embedding γ_r (recall §3) is contained in H_{-}.

We now apply the results of Theorem 1.6.3 and Theorem 1.8.3 of [19] to our situation. We get the following conclusions:

(1) The genus of (L, φ, i) is uniquely determined by the 'type' (Type 0, Type Ia or Type Ib), the invariants a, $t_{(-)}$, H_-, and in the case of Type Ia, the characteristic element v ($\in H_-$) of the embedding γ_r.
(2) Two lists H_- and H'_- (with identical 'type' and invariants a, $t_{(-)}$), and in the case of Type Ia, v ($\in H_-$) and v' ($\in H'_-$) give identical genera if and only if $H_- = H'_-$, and $v = v'$ for Type Ia.
(3) There exists an involution of a lattice (L, φ, i) with the condition (S, θ, G) (fixed as above) with L even unimodular of signature (3,19), $t_{(+)} = 1$, an designated 'type' (Type 0, Type Ia or Type Ib), invariants a, $t_{(-)}$, H_-, and, for Type Ia, the characteristic element of the embedding γ_r being v ($\in H_-$) if and only if the 'type' and these invariants a, $t_{(-)}$, H_-, and v ($\in H_-$) (for Type Ia) satisfy the **Conditions 1.8.1** and **1.8.2** of [19].

Then let us enumerate up all the data of the invariants:

$$\text{'type' (Type 0, Type Ia or Type Ib)}, \ a, \ t_{(-)}, \ H_-,$$

$$\text{(and in the case of Type Ia, the characteristic element } v \ (\in H_-))$$

which satisfy the Conditions 1.8.1 and 1.8.2 of [19]. Actually, this is a hard and tedious task. The results are written in Tables 1–3 below.

Notation: In Tables 1–3, the symbols a, t(-) and H- mean a, $t_{(-)}$ and H_- respectively. And the symbols 0, e1, e2, h and S- stand for the data of H_-, namely, the subgroups $\{0\}$, $< [\frac{1}{2}e_1] >$, $< [\frac{1}{2}e_2] >$, $< [\frac{1}{2}h] >$ and A_{S_-} (recall §4) respectively.

We remark that in our case, every H_- in Type Ia is generated by a unique nonzero element, and hence it is nothing but the characteristic element. Hence, we don't need to designate the characteristic elements in Type Ia, either.

In each 'type' (Type 0, Type Ia or Type Ib), the data (a, $t_{(-)}$, H_-) are in bijective correspondence with the genera because of the conclusions (1),(2) above and the fact that a and $t_{(-)}$ are genus invariants (see p.137 of [18]). Thus we see that there are 51 genera of Type 0, 34 genera of Type Ia and 174 genera of Type Ib in our situation.

Type0

a	t(-)	H-
0	1	0
0	9	0
0	17	0
2	1	0
2	1	e1
2	1	e2
2	1	h
2	1	S-
2	5	0
2	5	h
2	9	0
2	9	e1
2	9	e2
2	9	h
2	9	S-
2	13	0
2	13	h
2	17	e1
2	17	e2
2	17	h
2	17	S-
4	5	0
4	5	e1
4	5	e2
4	5	h
4	5	S-
4	9	0
4	9	e1
4	9	e2
4	9	h
4	9	S-
4	13	e1
4	13	e2
4	13	h
4	13	S-
4	17	S-
6	9	0
6	9	e1
6	9	e2
6	9	h
6	9	S-
6	13	S-
8	9	0
8	9	e1
8	9	e2
8	9	h
8	9	S-
10	9	e1
10	9	e2
10	9	h
10	9	S-

Table 1: Type0

TypeIa

a	t(-)	H-
2	1	e1
2	1	e2
2	3	h
2	7	h
2	9	e1
2	9	e2
2	11	h
2	15	h
2	17	e1
2	17	e2
4	3	h
4	5	e1
4	5	e2
4	7	h
4	9	e1
4	9	e2
4	11	h
4	13	e1
4	13	e2
4	15	h
6	5	e1
6	5	e2
6	7	h
6	9	e1
6	9	e2
6	11	h
6	13	e1
6	13	e2
8	7	h
8	9	e1
8	9	e2
8	11	h
10	9	e1
10	9	e2

Table 2: TypeIa

TypeIb

a	t(-)	H-
1	0	0
1	2	0
1	8	0
1	10	0
1	16	0
2	1	0
2	3	0
2	7	0
2	9	0
2	11	0
2	15	0
3	2	0
3	2	e1
3	2	e2
3	2	h
3	2	S-
3	4	0
3	4	h
3	6	0
3	6	h
3	8	0
3	8	e1
3	8	e2
3	8	h
3	8	S-
3	10	0
3	10	e1
3	10	e2
3	10	h
3	10	S-
3	12	0
3	12	h
3	14	0
3	14	h
3	16	e1
3	16	e2
3	16	h
3	16	S-
3	18	S-
4	3	0
4	3	e1
4	3	e2
4	3	h
4	3	S-
4	5	0
4	5	h
4	7	0
4	7	e1
4	7	e2
4	7	h
4	7	S-
4	9	0
4	9	e1
4	9	e2
4	9	h
4	9	S-
4	11	0
4	11	e1
4	11	e2
4	11	h
4	11	S-
4	13	0
4	13	h
4	15	e1
4	15	e2
4	15	h
4	15	S-
4	17	S-
5	4	0
5	4	e1
5	4	e2
5	4	h
5	4	S-
5	6	0
5	6	e1
5	6	e2
5	6	h
5	6	S-
5	8	0
5	8	e1
5	8	e2
5	8	h
5	8	S-
5	10	0
5	10	e1
5	10	e2
5	10	h
5	10	S-
5	12	0
5	12	e1
5	12	e2
5	12	h
5	12	S-
5	14	e1
5	14	e2
5	14	h
5	14	S-
5	16	S-
6	5	0
6	5	e1
6	5	e2
6	5	h
6	5	S-
6	7	0
6	7	e1
6	7	e2
6	7	h
6	7	S-
6	9	0
6	9	e1
6	9	e2
6	9	h
6	9	S-
6	11	0
6	11	e1
6	11	e2
6	11	h
6	11	S-
6	13	e1
6	13	e2
6	13	h
6	13	S-
6	15	S-
7	6	0
7	6	e1
7	6	e2
7	6	h
7	6	S-
7	8	0
7	8	e1
7	8	e2
7	8	h
7	8	S-
7	10	0
7	10	e1
7	10	e2
7	10	h
7	10	S-
7	12	e1
7	12	e2
7	12	h
7	12	S-
7	14	S-
8	7	0
8	7	e1
8	7	e2
8	7	h
8	7	S-
8	9	0
8	9	e1
8	9	e2
8	9	h
8	9	S-
8	11	e1
8	11	e2
8	11	h
8	11	S-
8	13	S-
9	8	0
9	8	e1
9	8	e2
9	8	h
9	8	S-
9	10	e1
9	10	e2
9	10	h
9	10	S-
9	12	S-
10	9	e1
10	9	e2
10	9	h
10	9	S-
10	11	S-
11	10	S-

Table 3: TypeIb

§6. Topological interpretations of each genus

In this section, for each our genus, we investigate the topological properties of nonsingular real algebraic curves of bidegree (4,4) on $\boldsymbol{R}P^1 \times \boldsymbol{R}P^1$ which realize that genus. We recall our situation stated in §2.

Let $\boldsymbol{R}A$ be the real part of A, i.e., $A \cap \boldsymbol{R}P^1 \times \boldsymbol{R}P^1$. See the section 2 of [13] for the definitions of the following notions concerning $\boldsymbol{R}A$:

the notion of $(M-i)$-*curves of bidegree* (4,4),

the *torsion* (s,t) $(\in \boldsymbol{Z} \times \boldsymbol{Z})$ of each connected component of $\boldsymbol{R}A$,

oval, non-oval, odd branch, even branch

We can set B^+ $(B^-) = \{F \geq 0\}$ $(\{F \leq 0\})$ $(\subset \boldsymbol{R}P^1 \times \boldsymbol{R}P^1)$, where we fix a defining (real) polynomial F of A. We recall the two anti-holomorphic involutions T^+ and T^- of Y, and let $\boldsymbol{R}Y^{\pm}$ denote the fixed point sets of $T^{\pm}$. Then, since our bidegree is (4,4), we can regard $\boldsymbol{R}Y^{\pm}$ as the doubles of $B^{\pm}$ respectively (see Remark 3.2 of [12] for the reason) replacing F by $-F$ if necessary.

We call $\boldsymbol{R}A$ a *dividing* curve (or curve *of type* I ([25])) if $A \setminus \boldsymbol{R}A$ is disconnected, and *non-dividing* curve (or curve *of type* II) if otherwise. Moreover, following [20], we call a real scheme is *of type* I if all the curves with this scheme are of type I, *of type* II if they all are of type II, and *of indeterminate type* if some are of type I and others are of type II.

Lemma 1 ([12]) *For a nonsingular real algebraic curve $\boldsymbol{R}A$ of bidegree $(4,4)$ on $\boldsymbol{R}P^1 \times \boldsymbol{R}P^1$, we have the following:*
(1)$[\boldsymbol{R}Y^+] = [\boldsymbol{R}Y^-]$ *in* $H_2(Y;\boldsymbol{Z}/2)$
(2)*If $\boldsymbol{R}A$ is dividing, then*

$$[\boldsymbol{R}Y^{\pm}] = \begin{cases} 0 & (\textit{if } \boldsymbol{R}A \textit{ has only ovals}) \\ (\hat{l}\, e_1)_{\mathrm{mod}\, 2} & (\textit{if } \boldsymbol{R}A \textit{ has odd branches with odd } s) \\ (\hat{l}\, e_2)_{\mathrm{mod}\, 2} & (\textit{if } \boldsymbol{R}A \textit{ has odd branches with odd } t) \\ (\hat{l}\, h)_{\mathrm{mod}\, 2} & (\textit{if } \boldsymbol{R}A \textit{ has even branches with } (|s|,|t|) = (1,1)) \end{cases}$$

in $H_2(Y;\boldsymbol{Z}/2)$, where $\hat{l}$ is the integer defined in [12], *and we use the same notations for the Poincaré duals of the cohomology classes e_i $(i=1,2)$ defined in §2.*

We next quote the following collection of useful results. See [18] for the terminology.

Theorem 2 **([18], Theorems 3.10.5 and 3.10.6)** *If Y belongs to a coarse projective equivalence class of real $K3$ surfaces corresponding to an isomorphism class of polarlized integral involutions (L,φ,h) of the even unimodular lattice of signature $(3,19)$ with $h^2 = n$(: a designated even positive integer), $t_{(+)} = 1$, and the invariants $t_{(-)}$, a, δ_h, δ_φ and $\delta_{\varphi,h}$, then we have the following:*

(1) *The real part $\boldsymbol{R}Y$ of Y is an orientable closed surface which is homeomorphic to*

$$\begin{cases} \emptyset & \textit{if } \delta_\varphi = 0,\ (a, t_{(-)}) = (10, 9) \\ T^2 \amalg T^2 & \textit{if } \delta_\varphi = 0,\ (a, t_{(-)}) = (8, 9) \\ \Sigma_g \amalg k(S^2) & \textit{in the remaining cases,} \end{cases}$$

where we set $g = \frac{21-a-t_{(-)}}{2}$ *and* $k = \frac{1-a+t_{(-)}}{2}$, Σ_g *denotes the orientable closed surface of genus* g, *and* $k(S^2)$ *means the disjoint union of* k *copies of* S^2.

(2) *When* $\boldsymbol{R}Y \neq \emptyset$,

$\delta_h = 0 \iff$ *the linear system* $|h|_{\boldsymbol{R}}$ *cuts out on* $\boldsymbol{R}Y$ *a cycle homologous to 0 in* $H_1(\boldsymbol{R}Y; \boldsymbol{Z}/2)$.

(3) $\delta_\varphi = 0 \iff [\boldsymbol{R}Y] = 0$ *in* $H_2(Y; \boldsymbol{Z}/2)$.

(4) $\delta_{\varphi,h} = 0 \iff [\boldsymbol{R}Y] = h_{\mathrm{mod}\,2}$ *in* $H_2(Y; \boldsymbol{Z}/2)$.

Let us return to the situation in §2 again. We set $T = T^+$ or T^- and $\varphi = T^*$. Let $\boldsymbol{R}Y$ be the fixed point set of T. We set $h = e_1 + e_2$ in §4. Then (L, φ, h) is a 'polarized integral involution' ([18]) with $h^2 = 4$. Hence, by Lemma 1 and Theorem 2, we have the following:

Lemma 3 *Let $\boldsymbol{R}A$ be a nonsingular real algebraic curve of bidegree* $(4,4)$ *on* $\boldsymbol{R}P^1 \times \boldsymbol{R}P^1$. *Then we have the following:*

(1) $\delta_\varphi = 0 \iff [\boldsymbol{R}Y] = 0$ *in* $H_2(Y; \boldsymbol{Z}/2)$.

(2) $\delta_{\varphi,h} = 0 \iff [\boldsymbol{R}Y] = h_{\mathrm{mod}\,2}$ *in* $H_2(Y; \boldsymbol{Z}/2)$.

Moreover, suppose that $\boldsymbol{R}A$ is dividing. Then we have the following:

(3) $[\boldsymbol{R}Y] = 0$ *in* $H_2(Y; \boldsymbol{Z}/2) \iff \boldsymbol{R}A$ *has only ovals, or it has non-ovals with* $\hat{l}$ *even.*

(4) $[\boldsymbol{R}Y] = h_{\mathrm{mod}\,2}$ *in* $H_2(Y; \boldsymbol{Z}/2) \iff \boldsymbol{R}A$ *has non-ovals with* $(|s|, |t|) = (1, 1)$ *and* $\hat{l}$ *odd.*

When $\boldsymbol{R}A$ has only ovals, B^+ or B^- contains 'the outermost component' (cf. [12]). As stated above, $\boldsymbol{R}Y^\pm$ are the doubles of $B^\pm$ respectively. Thus we can divide the situations of (Y, T) into the following 4 cases:

A: $\boldsymbol{R}A$ has only ovals and $\boldsymbol{R}Y$ contains the double of the outermost component.

A': $\boldsymbol{R}A$ has only ovals and $\boldsymbol{R}Y$ does not contain the double of the outermost component.

B: $\boldsymbol{R}A$ has odd branches.

C: $\boldsymbol{R}A$ has even branches.

We now recall all the real schemes (i.e., isotopy types) of curves of bidegree (4,4) on a hyperboloid, which are given in §3.11 of [25] or at the end of [13]. We also give the correspondence between the notations for real schemes used in [25] and [13]. See the following table:

Notation in [25] (also in [22])	**Notation in [13]**
$<\lambda_1 \amalg 1 < \lambda_2 >>$ (where $(\lambda_1, \lambda_2) = (0,9),\ (4,5),\ (8,1),\ (0,8),\ (3,5),\ (4,4),\ (7,1)$)	$\frac{\lambda_2}{1}\lambda_1$
$<\lambda_1 \amalg 1 < 0 >>$ (where $0 \leq \lambda_1 \leq 8$)	$\lambda_1 + 1$
$<\lambda_1 \amalg 1 < \lambda_2 >>$ (where $\lambda_1 \geq 0,\ \lambda_2 \geq 1$ and $\lambda_1 + \lambda_2 \leq 7$)	$\frac{\lambda_2}{1}\lambda_1$
$<2<1>>$	$\frac{1\,1}{1\,1}$
$<0>$	$\emptyset$
$<e_1,\ \lambda_1,\ e_1, \lambda_2>$ (where $(\lambda_1, \lambda_2) = (8,0),\ (4,4),\ (7,0),\ (4,3)$; or $\lambda_1 \geq \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 \leq 6$)	$\vert\lambda_1\vert\lambda_2$ $\left(\text{or } \frac{\lambda_1}{\lambda_2}\right)$
$<2(e_1 + 2e_2)>$	$2(1,2)$
$<4(e_1)>$	$4(1,0)$
$<e_1 + e_2,\ \lambda_1,\ e_1 + e_2, \lambda_2>$ (where $\lambda_1 \geq \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 \leq 8$)	$/\lambda_1/\lambda_2$
$<4(e_1 + e_2)>$	$4(1,1)$

It is easily seen that the real scheme of a curve determines the topological types of $\boldsymbol{R}Y^{\pm}$ as in the following table:

Real scheme	$\Longrightarrow$ **The topological types of** $\boldsymbol{R}Y^{\pm}$
$\frac{\lambda_2}{1}\lambda_1$	$\Sigma_{\lambda_1+2} \amalg \lambda_2 S^2$ (A case) and $\Sigma_{\lambda_2} \amalg \lambda_1 S^2$ (A' case)
$\lambda_1 + 1$	Σ_{λ_1+2} (A case) and $(\lambda_1 + 1) S^2$ (A' case)
$\frac{1\,1}{1\,1}$	$\Sigma_3 \amalg 2 S^2$ (A case) and $T^2 \amalg T^2$ (A' case)
$\emptyset$	$T^2 \amalg T^2$ (A case) and $\emptyset$ (A' case)
$\vert\lambda_1\vert\lambda_2$	$\Sigma_{\lambda_1+1} \amalg \lambda_2 S^2$ and $\Sigma_{\lambda_2+1} \amalg \lambda_1 S^2$
$2(1,2)$	T^2 (both)
$4(1,0)$	$T^2 \amalg T^2$ (both)
$/\lambda_1/\lambda_2$	$\Sigma_{\lambda_1+1} \amalg \lambda_2 S^2$ and $\Sigma_{\lambda_2+1} \amalg \lambda_1 S^2$
$4(1,1)$	$T^2 \amalg T^2$ (both)

Now the subgroup H_- is defined by the invariants δ_{e_1}, δ_{e_2} and δ_h. Recall the definitions of $\delta_{x_\pm}$ in §3.

Lemma 4 ([13], Lemma 2) *Let $\boldsymbol{R}A$ be a nonsingular real algebraic curve of bidegree* $(4,4)$ *on $\boldsymbol{R}P^1 \times \boldsymbol{R}P^1$. If $\boldsymbol{R}A$ has odd branches with odd s (resp. t), then we have $\delta_{e_1} = 0$ (resp. $\delta_{e_2} = 0$).*

The following lemma can be proved in the similar way to Lemma 4 above:

Lemma 5 *Let $\boldsymbol{R}A$ be a nonsingular real algebraic curve of bidegree* $(4,4)$ *on $\boldsymbol{R}P^1 \times \boldsymbol{R}P^1$. Then we have the following:*
(1) If (Y,T) is in A' case and $\boldsymbol{R}Y \neq \emptyset$, then $\delta_{e_1} = \delta_{e_2} = \delta_h = 0$, namely, $H_- = A_{S_-}$.
(2) If we are in C case, then $\delta_h = 0$.

For only h, we can prove "the inverse assertion" by Theorem 2, (2) above, and we get the following:

Lemma 6 *Let $\boldsymbol{R}A$ be a nonsingular real algebraic curve of bidegree* $(4,4)$ *on $\boldsymbol{R}P^1 \times \boldsymbol{R}P^1$. If $\boldsymbol{R}Y \neq \emptyset$ and $\delta_h = 0$ for (Y,T), then we are in A' case or C case.*

Lemma 7 *Let $\boldsymbol{R}A$ be a nonsingular real algebraic curve of bidegree* $(4,4)$ *on $\boldsymbol{R}P^1 \times \boldsymbol{R}P^1$. Then, for (Y,T), we have*

$$x \cdot T_*(x) \equiv x \cdot [\boldsymbol{R}Y] \pmod 2 \quad \forall x \in H_2(Y;\boldsymbol{Z})$$

.

Proof. $T : Y \to Y$ is an orientation preserving involution, and its fixed point set $\boldsymbol{R}Y$ is an orientable closed surface (Theorem 2, (1)). Hence, by Lemma 3 of [1], we get the required results.□

Remark 8 *By the above lemma, we see*

$$v_q = [\frac{1}{2}[\boldsymbol{R}Y]] \quad \in L_+^*/L_+ = A_{L_+},$$

where v_q is the characteristic element (recall §3) of q.

Proposition 9 *Let $\boldsymbol{R}A$ be a nonsingular real algebraic curve of bidegree* $(4,4)$ *on $\boldsymbol{R}P^1 \times \boldsymbol{R}P^1$. If $\boldsymbol{R}A$ is dividing, then (L,φ,i) is of Type 0 or Type Ia.*

Proof. If $\boldsymbol{R}A$ is dividing, by (2) of Lemma 1, we have $[\boldsymbol{R}Y] \equiv s_\varphi \pmod{2L}$ for some $s_\varphi \in S$. By Lemma 7, we have $\delta_{\varphi S} = 0$.□

Our aim is to restrict the real schemes of the curves which realize each genus enumerated in Tables 1–3.

We first present 'candidates' of the real schemes of the curves which realize each genus by using the above results. See **Tables 4–6** below.

Then we get some further results from Tables 4–6:

Proposition 10 *In Type Ia, A cases are impossible.*
(Namely, every real scheme with the superscript $^{1)}$ *in Table 5 can be removed.)*

Proof. In Table 5 (i.e., Type Ia), the real schemes 8, $\frac{4}{1}3$, $\frac{1}{1}4$, $\frac{3}{1}2$, $\frac{5}{1}$, 4, $\frac{2}{1}1$, $\frac{1}{1}$ are presented as candidates in the columun A. Suppose that there exists a curve $\boldsymbol{R}A$ such that its real scheme is 8, (Y, T^-) is in A case, (L, φ, i) is of Type Ia, and $H_- =< [\frac{1}{2}e_1] >$. Since (L, φ, i) is of Type Ia and $H_- =< [\frac{1}{2}e_1] >$, we see $v = [\frac{1}{2}e_1]$. By Remark 8, we have $v_q = [\frac{1}{2}[\boldsymbol{R}Y^-]] \in A_{L_+}$. Hence, $\gamma_r([\frac{1}{2}e_1]) = [\frac{1}{2}[\boldsymbol{R}Y^-]]$, where $\gamma_r = \gamma_{L_+S_-}$ (recall §5). This means $\frac{1}{2}e_1 + \frac{1}{2}[\boldsymbol{R}Y^-] \in L$. In the meanwhile, by Lemma 1, $[\boldsymbol{R}Y^+] \equiv [\boldsymbol{R}Y^-] \pmod{2L}$. Hence, we also get $e_1 \equiv [\boldsymbol{R}Y^+] \pmod{2L}$. Hence, for the same curve $\boldsymbol{R}A$ with the different involution T^+, the associated involution of our lattice (L, φ', i') with our condition is of Type Ia, too. It is obvious that (Y, T^+) is in A' case. So $\boldsymbol{R}Y^+$ is homeomorphic to $8S^2$. Then, by Theorem 2 (1), we see $(a, t_{(-)}) = (4, 17)$. But this pair of $(a, t_{(-)})$ does not appear in Type Ia. This is a contradiction. For the remaining real schemes, we can also prove the same assertion in the same way.□

Proposition 11 *In Type 0, the real schemes* $\frac{5}{1}$, $\frac{2}{1}1$ *and* $/4/0$ *are impossible.*
(Namely, every real scheme with the superscript $^{2)}$ *in Table 4 can be removed.)*

Proof. We consider a curve $\boldsymbol{R}A$ such that its real scheme is $\frac{5}{1}$ and (Y, T^-) is in A case. Then, for the same curve $\boldsymbol{R}A$ with the different involution T^+, (Y, T^+) is in A' case, and $\boldsymbol{R}Y^+$ is homeomorphic to Σ_5. By Theorem 2 (1), we see $(a, t_{(-)}) = (6, 5)$. By Lemma 5, we have $H_- = A_{S_-}$. Since $(a, t_{(-)}, H_-) = (6, 5, A_{S_-})$ appears only in Type Ib, we see $\delta_\varphi = 1$. Hence, $[\boldsymbol{R}Y^+] \neq 0$ $(\in H_2(Y; \boldsymbol{Z}/2))$ because of Remark 8 and the end of §3, or Theorem 2 (3). Then we also get $[\boldsymbol{R}Y^-] \neq 0$ $(\in H_2(Y; \boldsymbol{Z}/2))$. Hence we have $\delta_\varphi = 1$ also for T^-. This means $\frac{5}{1}$ does not appear in Type 0.

We next consider a curve $\boldsymbol{R}A$ such that its real scheme is $\frac{2}{1}1$ and (Y, T^-) is in A case. Then, for the same curve $\boldsymbol{R}A$ with the different involution T^+, $\boldsymbol{R}Y^+$ is homeomorphic to $\Sigma_2 \amalg 1S^2$. By Theorem 2 (1), we see $(a, t_{(-)}) = (8, 9)$. If moreover $\delta_\varphi = 0$, then the real part is homeomorphic to $T^2 \amalg T^2$ by the same theorem. Hence we have $\delta_\varphi = 1$. Then we can prove that $\frac{2}{1}1$ does not appear in Type 0 in the same way as $\frac{5}{1}$.

We last consider a curve $\boldsymbol{R}A$ such that its real scheme is $/4/0$. Then $\boldsymbol{R}Y^+$ or $\boldsymbol{R}Y^-$ is homeomorphic to Σ_5. Hence, we have $(a, t_{(-)}) = (6, 5)$, and $[\frac{1}{2}h] \in H_-$ by Lemma 5. Since such genera appear only in Type Ib, we get $\delta_\varphi = 1$. Thus we can prove that $/4/0$ does not appear in Type 0 in the same way as above.□

Type 0 $(\delta_\varphi = 0)$						
a	$t_{(-)}$	H_-	A	A'	B	C
0	1	$\{0\}$	$\frac{1}{1}8$			
0	9	$\{0\}$	$\frac{5}{1}4$			
0	17	$\{0\}$	$\frac{9}{1}$			
2	1	$\{0\}$	8			
2	1	$< [\frac{1}{2}e_1] >$	8 [4)]		$\|0\|8$	
2	1	$< [\frac{1}{2}e_2] >$	8 [4)]		$\overline{\frac{0}{8}}$	
2	1	$< [\frac{1}{2}h] >$				/0/8
2	1	A_{S_-}		$\frac{9}{1}$		/0/8 [4)]
2	5	$\{0\}$	$\frac{2}{1}5$			
2	5	$< [\frac{1}{2}h] >$				/2/6
2	9	$\{0\}$	$\frac{4}{1}3$			
2	9	$< [\frac{1}{2}e_1] >$	$\frac{4}{1}3$ [4)]		$\|4\|4$	
2	9	$< [\frac{1}{2}e_2] >$	$\frac{4}{1}3$ [4)]		$\overline{\frac{4}{4}}$	
2	9	$< [\frac{1}{2}h] >$				/4/4
2	9	A_{S_-}		$\frac{5}{1}4$		/4/4 [4)]
2	13	$\{0\}$	$\frac{6}{1}1$			
2	13	$< [\frac{1}{2}h] >$				/6/2
2	17	$< [\frac{1}{2}e_1] >$			$\|8\|0$	
2	17	$< [\frac{1}{2}e_2] >$			$\overline{\frac{8}{0}}$	
2	17	$< [\frac{1}{2}h] >$				/8/0
2	17	A_{S_-}		$\frac{1}{1}8$		/8/0 [4)]
4	5	$\{0\}$	$\frac{1}{1}4$			
4	5	$< [\frac{1}{2}e_1] >$	$\frac{1}{1}4$ [4)]		$\|1\|5$	
4	5	$< [\frac{1}{2}e_2] >$	$\frac{1}{1}4$ [4)]		$\overline{\frac{1}{5}}$	
4	5	$< [\frac{1}{2}h] >$				/1/5
4	5	A_{S_-}		$\frac{6}{1}1$		/1/5 [4)]
4	9	$\{0\}$	$\frac{3}{1}2$			
4	9	$< [\frac{1}{2}e_1] >$	$\frac{3}{1}2$ [4)]		$\|3\|3$	
4	9	$< [\frac{1}{2}e_2] >$	$\frac{3}{1}2$ [4)]		$\overline{\frac{3}{3}}$	
4	9	$< [\frac{1}{2}h] >$				/3/3
4	9	A_{S_-}		$\frac{4}{1}3$		/3/3 [4)]
4	13	$< [\frac{1}{2}e_1] >$	$\frac{5}{1}$ [2)]		$\|5\|1$	
4	13	$< [\frac{1}{2}e_2] >$	$\frac{5}{1}$ [2)]		$\overline{\frac{5}{1}}$	
4	13	$< [\frac{1}{2}h] >$				/5/1
4	13	A_{S_-}		$\frac{2}{1}5$		/5/1 [4)]

4	17	A_{S_-}		8		
6	9	$\{0\}$	$\frac{2}{1}1^{\ 2)},\ \frac{1\,1}{1\,1}$			
6	9	$<[\frac{1}{2}e_1]>$	$\frac{2}{1}1^{\ 2)},\ \frac{1\,1}{1\,1}^{\ 4)}$		$\|2\|2$	
6	9	$<[\frac{1}{2}e_2]>$	$\frac{2}{1}1^{\ 2)},\ \frac{1\,1}{1\,1}^{\ 4)}$		$\overline{\frac{2}{2}}$	
6	9	$<[\frac{1}{2}h]>$				$/2/2$
6	9	A_{S_-}		$\frac{3}{1}2$		$/2/2^{\ 4)}$
6	13	A_{S_-}		$\frac{1}{1}4$		$/4/0^{\ 2)}$
8	9	$\{0\}$	$\emptyset$			
8	9	$<[\frac{1}{2}e_1]>$	$\emptyset$		$4(1,0)$	
8	9	$<[\frac{1}{2}e_2]>$	$\emptyset$		$4(0,1)$	
8	9	$<[\frac{1}{2}h]>$				$4(1,1)$
8	9	A_{S_-}		$\frac{1\,1}{1\,1}$		$4(1,1)^{\ 4)}$
10	9	$<[\frac{1}{2}e_1]>$		$\emptyset$		
10	9	$<[\frac{1}{2}e_2]>$		$\emptyset$		
10	9	$<[\frac{1}{2}h]>$		$\emptyset$		
10	9	A_{S_-}		$\emptyset$		

Table 4

Type Ia ($\delta_\varphi = 1$ and $delta_{\varphi S} = 0$)						
a	$t_{(-)}$	H_-	A	A'	B	C
2	1	$<[\frac{1}{2}e_1]>$	$8^{\ 1)}$		$\|0\|8$	
2	1	$<[\frac{1}{2}e_2]>$	$8^{\ 1)}$		$\overline{\frac{0}{8}}$	
2	3	$<[\frac{1}{2}h]>$				$/1/7$
2	7	$<[\frac{1}{2}h]>$				$/3/5$
2	9	$<[\frac{1}{2}e_1]>$	$\frac{4}{1}3^{\ 1)}$		$\|4\|4$	
2	9	$<[\frac{1}{2}e_2]>$	$\frac{4}{1}3^{\ 1)}$		$\overline{\frac{4}{4}}$	
2	11	$<[\frac{1}{2}h]>$				$/5/3$
2	15	$<[\frac{1}{2}h]>$				$/7/1$
2	17	$<[\frac{1}{2}e_1]>$			$\|8\|0$	
2	17	$<[\frac{1}{2}e_2]>$			$\overline{\frac{8}{0}}$	
4	3	$<[\frac{1}{2}h]>$				$/0/6$
4	5	$<[\frac{1}{2}e_1]>$	$\frac{1}{1}4^{\ 1)}$		$\|1\|5$	
4	5	$<[\frac{1}{2}e_2]>$	$\frac{1}{1}4^{\ 1)}$		$\overline{\frac{1}{5}}$	
4	7	$<[\frac{1}{2}h]>$				$/2/4$
4	9	$<[\frac{1}{2}e_1]>$	$\frac{3}{1}2^{\ 1)}$		$\|3\|3$	
4	9	$<[\frac{1}{2}e_2]>$	$\frac{3}{1}2^{\ 1)}$		$\overline{\frac{3}{3}}$	

4	11	$< [\frac{1}{2}h] >$				/4/2
4	13	$< [\frac{1}{2}e_1] >$	$\frac{5}{1}$ [1]		$\|5\|1$	
4	13	$< [\frac{1}{2}e_2] >$	$\frac{5}{1}$ [1]		$\frac{\overline{5}}{1}$	
4	15	$< [\frac{1}{2}h] >$				/6/0
6	5	$< [\frac{1}{2}e_1] >$	4 [1]		$\|0\|4$	
6	5	$< [\frac{1}{2}e_2] >$	4 [1]		$\frac{\overline{0}}{4}$	
6	7	$< [\frac{1}{2}h] >$				/1/3
6	9	$< [\frac{1}{2}e_1] >$	$\frac{2}{1}1$ [1]		$\|2\|2$	
6	9	$< [\frac{1}{2}e_2] >$	$\frac{2}{1}1$ [1]		$\frac{\overline{2}}{2}$	
6	11	$< [\frac{1}{2}h] >$				/3/1
6	13	$< [\frac{1}{2}e_1] >$			$\|4\|0$	
6	13	$< [\frac{1}{2}e_2] >$			$\frac{\overline{4}}{0}$	
8	7	$< [\frac{1}{2}h] >$				/0/2
8	9	$< [\frac{1}{2}e_1] >$	$\frac{1}{1}$ [1]		$\|1\|1$	
8	9	$< [\frac{1}{2}e_2] >$	$\frac{1}{1}$ [1]		$\frac{\overline{1}}{1}$	
8	11	$< [\frac{1}{2}h] >$				/2/0
10	9	$< [\frac{1}{2}e_1] >$			$\|0\|0,\ 2(1,2)$	
10	9	$< [\frac{1}{2}e_2] >$			$\frac{\overline{0}}{0},\ 2(2,1)$	

Table 5

Type Ib ($\delta_\varphi = 1$ and $\delta_{\varphi S} = 1$)						
a	$t_{(-)}$	H_-	A	A'	B	C
1	0	$\{0\}$	9			
1	2	$\{0\}$	$\frac{1}{1}7$			
1	8	$\{0\}$	$\frac{4}{1}4$			
1	10	$\{0\}$	$\frac{5}{1}3$			
1	16	$\{0\}$	$\frac{8}{1}$			
2	1	$\{0\}$	8			
2	3	$\{0\}$	$\frac{1}{1}6$			
2	7	$\{0\}$	$\frac{3}{1}4$			
2	9	$\{0\}$	$\frac{4}{1}3$			
2	11	$\{0\}$	$\frac{5}{1}2$			
2	15	$\{0\}$	$\frac{7}{1}$			
3	2	$\{0\}$	7			
3	2	$< [\frac{1}{2}e_1] >$	7 [4]		$\|0\|7$	
3	2	$< [\frac{1}{2}e_2] >$	7 [4]		$\frac{\overline{0}}{7}$	

3	2	$<[\frac{1}{2}h]>$				/0/7
3	2	A_{S_-}		$\frac{8}{1}$		/0/7 [4)]
3	4	$\{0\}$	$\frac{1}{1}5$			
3	4	$<[\frac{1}{2}h]>$				/1/6
3	6	$\{0\}$	$\frac{2}{1}4$			
3	6	$<[\frac{1}{2}h]>$				/2/5
3	8	$\{0\}$	$\frac{3}{1}3$			
3	8	$<[\frac{1}{2}e_1]>$	$\frac{3}{1}3$ [4)]		$\|3\|4$	
3	8	$<[\frac{1}{2}e_2]>$	$\frac{3}{1}3$ [4)]		$\frac{\overline{3}}{4}$	
3	8	$<[\frac{1}{2}h]>$				/3/4
3	8	A_{S_-}		$\frac{5}{1}3$		/3/4 [4)]
3	10	$\{0\}$	$\frac{4}{1}2$			
3	10	$<[\frac{1}{2}e_1]>$	$\frac{4}{1}2$ [4)]		$\|4\|3$	
3	10	$<[\frac{1}{2}e_2]>$	$\frac{4}{1}2$ [4)]		$\frac{\overline{4}}{3}$	
3	10	$<[\frac{1}{2}h]>$				/4/3
3	10	A_{S_-}		$\frac{4}{1}4$		/4/3 [4)]
omitted (similar to the above)						
4	13	$\{0\}$	$\frac{5}{1}$			
4	13	$<[\frac{1}{2}h]>$				/5/1
omitted (similar to the above)						
6	5	$\{0\}$	4			
6	5	$<[\frac{1}{2}e_1]>$	4 [4)]		$\|0\|4$	
6	5	$<[\frac{1}{2}e_2]>$	4 [4)]		$\frac{\overline{0}}{4}$	
6	5	$<[\frac{1}{2}h]>$				/0/4
6	5	A_{S_-}		$\frac{5}{1}$		/0/4 [4)]
omitted (similar to the above)						
6	9	$\{0\}$	$\frac{2}{1}1$			
6	9	$<[\frac{1}{2}e_1]>$	$\frac{2}{1}1$ [4)]		$\|2\|2$	
6	9	$<[\frac{1}{2}e_2]>$	$\frac{2}{1}1$ [4)]		$\frac{\overline{2}}{2}$	
6	9	$<[\frac{1}{2}h]>$				/2/2
6	9	A_{S_-}		$\frac{3}{1}2$		/2/2 [4)]

omitted (similar to the above)						
8	9	$\{0\}$	$\frac{1}{1}$			
8	9	$<[\frac{1}{2}e_1]>$	$\frac{1}{1}$ 4)		$\|1\|1$	
8	9	$<[\frac{1}{2}e_2]>$	$\frac{1}{1}$ 4)		$\overline{\frac{1}{1}}$	
8	9	$<[\frac{1}{2}h]>$				$/1/1$
8	9	A_{S_-}		$\frac{2}{1}1$		$/1/1$ 4)
8	11	$<[\frac{1}{2}e_1]>$			$\|2\|0$	
8	11	$<[\frac{1}{2}e_2]>$			$\overline{\frac{2}{0}}$	
8	11	$<[\frac{1}{2}h]>$				$/2/0$
8	11	A_{S_-}		$\frac{1}{1}2$		$/2/0$ 4)
8	13	A_{S_-}		4		
omitted (similar to the above)						
9	10	$<[\frac{1}{2}e_1]>$			$\|1\|0$	
9	10	$<[\frac{1}{2}e_2]>$			$\overline{\frac{1}{0}}$	
9	10	$<[\frac{1}{2}h]>$				$/1/0$
9	10	A_{S_-}		$\frac{1}{1}1$		$/1/0$ 4)
9	12	A_{S_-}		3		
10	9	$<[\frac{1}{2}e_1]>$			$\|0\|0,\ 2(1,2)$ 3)	
10	9	$<[\frac{1}{2}e_2]>$			$\overline{\frac{0}{0}},\ 2(2,1)$ 3)	
10	9	$<[\frac{1}{2}h]>$				$/0/0$
10	9	A_{S_-}		$\frac{1}{1}$		$/0/0$ 4)
10	11	A_{S_-}		2		
11	10	A_{S_-}		1		

Table 6

We next consider the dividingness of nonsingular real algebraic curves of bidegree $(4, 4)$ on a hyperboloid. We first quote the following known result:

Proposition 12 ([12]) *For the dividingness of nonsingular real algebraic curves* $\boldsymbol{R}A$ *of bidegree* $(4,4)$ *on a hyperboloid, we have the following:*
(1)*M-curves are dividing.*
(2)*The number of the connected components of a dividing curve* $\boldsymbol{R}A$ *is even.*

(3) *The real schemes* $\frac{2}{1}5$ *and* $\frac{6}{1}1$ *are of* **type I**.
(4) *The real schemes* $\frac{1}{1}6$, $\frac{3}{1}4$, $\frac{5}{1}2$, $\frac{7}{1}$, 6, $\frac{2}{1}3$, $\frac{4}{1}1$, $\frac{1}{1}2$, $\frac{3}{1}$ *and* 2 *are of* **type II** *(by the Arnol'd's type congruence.)*
(5) *The real schemes* $|\lambda_1|\lambda_2$ *or* $/\lambda_1/\lambda_2$ *with* $\lambda_1-\lambda_2$ *odd are of* **type II**.
(6) *The real schemes* $\frac{1}{1}\frac{1}{1}$, $4(1,0)$ *and* $4(1,1)$ *are of* **type I**.

By Proposition 9, we immediately get the following:

Proposition 13 (1) *Curves in Type Ib are not dividing.* (2) *The real schemes which appear only in Type Ib are of* **type II**.

By (2) above, we get different proofs of the following results:

Corollary 14 (Zvonilov [25],3.11) *We have the following:*
(1) *The real schemes* $\frac{5}{1}$, 4, $\frac{2}{1}1$ *and* $\frac{1}{1}$ *are of* **type II**.
(2) *The real schemes* $|6|0$, $|4|2$, $|3|1$ *and* $|2|0$ *are of* **type II**.
(3) *The real schemes* $/4/0$, $/1/1$ *and* $/0/0$ *are of* **type II**.

Remark:
(1) Zvonilov proved the above assertions using his results in [23].
(2) Gilmer's result ([4]) on the rotation numbers of dividing curves can also contribute to the above assertion.
(3) We can prove the non-dividingness of 4 by Gilmer's Theorem 2 (b) in [4].

However, at present, it seems that we cannot prove the following assertions by means of our Tables 4–6.

Proposition 15 (Zvonilov [25],3.11) *We have the following:*
(1) *The real scheme* $|0|0$ *is of* **type II**.
(2) *The real schemes* $|5|1$ *and* $2(1,2)$ *are of* **type I**.

Remark:
By the above result, we can remove $2(1,2)$ from Table 6 (i.e.,Type Ib).

§7. Some questions

In §6, we tried to restrict the real schemes of the curves which realize each genus enumerated in Tables 1–3. In this section, we give some questions.

Question 1 *In the situation of §2, we set* $K=\boldsymbol{R}Y\cap\pi^{-1}(\infty\times\boldsymbol{P}^1)$ *(resp.* $\boldsymbol{R}Y\cap\pi^{-1}(\boldsymbol{P}^1\times\infty)$*), where* $\boldsymbol{R}Y$ *denotes the fixed point set of* T. *We suppose that* $\boldsymbol{R}Y\neq\emptyset$. *Then, is the following assertion true? "If* δ_{e_1} *(resp.* δ_{e_2}*)* $=0$*, then* $[K]=0$ $(\in H_1(\boldsymbol{R}Y;\boldsymbol{Z}/2))$*."*

If the above assertion is true, then we can remove the real schemes with the superscript [4)] from Tables 4–6.

Question 2 *In the case of Type Ia, is* $(a, t_{\underline{(-)}}, H_-) = (10, 9, e_1)$ *(resp.* $(10, 9, e_2)$*) realized by both a curve with its real scheme* $|0|0$ *(resp.* $\frac{\overline{0}}{0}$*) and a curve with its real scheme* $2(1, 2)$ *(resp.* $2(2, 1)$*)?*

Question 3 *In each case of Type 0 and Type Ia, is it possible that some dividing curves and some non-dividing curves realze an identical value of* $(a, t_{(-)}, H_-)$ *(i.e., a genus)?*

Question 4 *In the case of Type 0, the 4 genera with* $(a, t_{(-)}) = (10, 9)$ *are all realized by any curves (with their real parts empty)?*

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