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The log utility and the paradox of Petersburg

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Summary. There is an objective meaning of the log utility if we consider that one continues betting a constant per cent of one's money. We point this out in a general case, and consider mathematically and numerically what happens if one continues betting in the paradox of Petersburg.

1. Introduction

A meaning to use the log utility of the amount of money is usually explained by subjective satisfaction. Today this is often explained in textbooks on decision theory and Bayesian statistics. Bernoulli [1] proposed the log utility to solve the problem by Montmort [2] called the paradox of Petersburg today. It is often believed that Bernoulli [1] is the original of this paradox, but Bernoulli [1] quotes Montmort [2], though the author has not got the original of Montmort [2]. There is, however, an objective meaning of the log utility. This is an easy fact, but the author has not found it in literature.

In Section 2, we shall make a setup and point out this fact in a general case when we continue betting, also note its limitations. In Section 3, we shall consider mathematically what happens if we continue betting in the paradox of Petersburg. In Section 4, we shall consider it numerically by giving graphs. In Section 5, we shall give some remarks.

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2. The log utility in a general case when we continue betting

We shall make a following setup and point out an objective meaning of the log utility in a general case when we continue betting.

Assume that Peter has y ducats ($y > 0$) first, though the value of y is not essential as will be seen. He agrees to Paul that when he pays Paul b ducats, Paul will give him bX ducats, where X is an unknown nonnegative random variable. Let $0 \le p \le 1$ and assume that $b = py$, that is, Paul uses $100p$ per cent of his money to bet. The meaning of $p \le 1$ is that he keeps out of debt to bet. On speculation in stocks, it essentially means that he does not make credit transaction. After this bet, he has $y - py + pyX = y(1 - p + pX)$ ducats. Then the increment of his log utility in this bet is given by $U := \log y(1 - p + pX) - \log y = \log(1 - p + pX)$, which is independent of y, where we define $\log 0 = -\infty$. A radix of log, say $c(> 1)$, is not essential. We assume that log means natural logarithm (i.e., $c = e$) for convenience of a mathematical approach. Its merit *in practice* is that $U \approx -p + pX$ holds when $X \approx 1$. If we change c, then the new U is a constant and positive multiple of the old U. When X is very large, there is a merit to choose $c = 10$ in practice because if we do so, he has $10^U y$ ducats after this bet. Let μ be the increment of his mean utility (moral expectation) of this bet, that is, $\mu := E[U]$, assuming its existence (possibly $\pm\infty$). Those who agree to the log utility consider that this bet is favorable if $\mu > 0$ and unfavorable if $\mu < 0$. If he continues betting $100p$ per cent of his money, where p is a constant, it is really so. If we explain this fact precisely, it is as follows:

Let $X_1, X_2, X_3, \ldots$ be independent random variables with the same distribution of X. First, Peter has y ducats. He pays Paul py ducats and Paul gives him pyX_1 ducats. Then he has $Y_1 := y(1 - p + pX_1)$ ducats.

Second, he pays Paul pY_1 ducats and Paul gives him pY_1X_2 ducats. Then he has $Y_2 := Y_1(1-p+pX_2) = y(1-p+pX_1)(1-p+pX_2)$ ducats, and so on. After betting n times, he has $Y_n := y(1-p+pX_1)(1-p+pX_2)\cdots(1-p+pX_n)$ ducats. Since $\log Y_n = \log y + \log(1-p+pX_1) + \log(1-p+pX_2) + \cdots + \log(1-p+pX_n)$, applying the strong law of large numbers, we get the following results. If $\mu > 0$, then $\lim_{n\to\infty} Y_n = \infty$ with probability 1. If $\mu < 0$, then $\lim_{n\to\infty} Y_n = 0$ with probability 1. Moreover, assume that $\sigma^2 = \mathrm{Var}[U]$ exists and $0 < \sigma < \infty$. For $t > 0$, by Chebyshev's inequality, we get

$$P\left[\exp(\mu n - t\sigma\sqrt{n}) < \frac{Y_n}{y} < \exp(\mu n + t\sigma\sqrt{n})\right] \geq 1 - \frac{1}{t^2} \quad \text{for any given } n.$$

Its right-hand side is, for example, 0.96 for $t = 5$. In addition, for any t, by the central limit theorem, we get

$$P\left[\frac{Y_n}{y} > \exp(\mu n - t\sigma\sqrt{n})\right] \approx 1 - \frac{1}{\sqrt{2\pi}} \int_t^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \quad \text{for a sufficiently large } n,$$

where we can get the value of its right-hand side by a table of the normal distribution. For example, it is approximately 0.98 for $t = 2$.

We should also recognize limitations to use the log utility. When n is given, rather than to consider μ, it is better to consider $\nu := \mu n - t\sigma\sqrt{n}$ or $\lambda := \max\{\nu, n\xi\}$, where ξ is the maximum value satisfying $\log(1-p+pX) \geq \xi$ with probability 1, and $t > 0$ is taken appropriately to consider safety.

3. Paradox of Petersburg—a mathematical approach

We shall consider mathematically what happens if Peter continues betting in the paradox of Petersburg.

Assume that $J = j$ with probability 2^{-j} for $j = 1, 2, 3, \ldots$, and $X = k2^J$, where k is a positive constant. (Originally, Montmort [2] and Bernoulli [1] consider the case that Paul gives Peter 2^{j-1} ducats with probability 2^{-j}.) It is well known that $E[X] = \infty$. However,

$$\mu = E[U] = \sum_{j=1}^{\infty} 2^{-j} \log(1 - p + kp2^j),$$

and we see that this is finite, but generally difficult to calculate its exact value. On the following theorems, see Appendix A for proofs.

Theorem 1. *For $p = 1$, the following assertions hold.*
(i) $\mu = \log 4k$.
(ii) *If $k > 1/4$, continuing this bet, he increases his money to infinity with probability* 1.
(iii) *If $k < 1/4$, continuing this bet, he decreases his money to zero with probability* 1.

Note that Theorem 1 (i) is essentially obtained by Bernoulli [1]. Next, denote $q := 1 - p$, and for $p \in [0, 1)$, let $r := kp/q$ and $\eta := E[\log(1 + r2^J)]$. We use η to evaluate not only μ but also σ. To evaluate μ, we get the following theorem.

Theorem 2. *For $p \in [0, 1)$, the following assertions hold.*

$$\mu = \eta + \log q,$$
$$\eta = \log 4 + \sum_{j=1}^{j_0} 2^{-j} \log(r + 2^{-j}) + \rho_{j_0} \geq 0,$$

where

$$2^{-j_0} \max\{\log r, -(j_0 + 2) \log 2\} \leq \rho_{j_0} < 2^{-j_0} \log(r + 2^{-j_0 - 1}).$$

Next, we shall consider maximizing $\mu = \mu(p, k)$ by moving p.

Theorem 3. *The following assertions hold.*

(i) *There exists* $\mu_1(p, k) := (\partial/\partial p)\mu(p, k)$ *for* $p \in (0, 1]$ *and it strictly decreases with respect to* $p \in (0, 1]$.

(ii) $\mu_1(0+, k) := \lim_{p \downarrow 0} \mu_1(p, k) = \infty$.

(iii) *The function* $\mu(p, k)$ *is continuous and strictly concave with respect to* $p \in [0, 1]$.

(iv) *For each* $k \in (0, \infty)$, *there exists a unique* $p = p_0 = p_0(k)$ *that maximizes* $\mu(p, k)$.

(v) $p_0(k) = 1$ *for* $k \in [1/3, \infty)$.

(vi) $0 < p_0(k) < \dfrac{1}{3(1 - 2k)} < 1$ *for* $k \in (0, 1/3)$.

(vii) *The function* $p_0(k)$ *is continuous and strictly increases with respect to* $k \in (0, 1/3]$.

(viii) $p_0(0+) := \lim_{k \downarrow 0} p_0(k) = 0$.

Next we shall examine σ^2.

Theorem 4. *The following assertions hold.*

(i) *If* $p = 1$, *then* $\sigma^2 = 2\log^2 2$, *which is independent of* k.

(ii) *If* $p \in [0, 1)$, *then* $\sigma^2 = \sigma^2(r) = \mathrm{Var}[\log(1 + r2^J)]$, *which is a function of* $r = kp/q$.

(iii) $\sigma^2(\infty) := \lim_{r \to \infty} \sigma^2(r) = 2\log^2 2$, *which coincides with the value in (i).*

(iv) *There exists* $\sigma^{2\prime}(r)$ *for* $r \in (0, \infty)$ *and it is positive.*

(v) *The function* $\sigma^2(r)$ *is continuous and strictly increases with respect to* $r \in [0, \infty)$.

(vi) *The function* $\sigma^2(kp/q)$ *is continuous and strictly increases with respect to* $p \in [0, 1]$.

At the last part of Section 2, we noted limitations to use the log utility. Here, $\xi = \xi(p) = \log(q + 2kp)$ holds. We shall consider maximizing $\nu = \nu(p, k, n, t) = \mu(p)n - t\sigma(kp/q)\sqrt{n}$ and $\lambda = \lambda(p, k, n, t) = \max\{\nu(p, k, n, t), n\log(q + 2kp)\}$. On this point, we obtain the following theorem. For further details, we shall consider numerically in the next section.

Theorem 5. *For any fixed $k > 0$, $n = 1, 2, 3, \ldots$, and $t > 0$, the function $\nu(p, k, n, t)$ with respect to $p \in [0, 1]$ takes its maximum value at $p = p_1 = p_1(k, n, t)$ (say), and it satisfies $p_1 \leq p_0$. In particular, if $p_0 < 1$, then the strict inequality $p_1 < p_0$ holds. If $k < 1/2$, then the function $\lambda(p, k, n, t)$ with respect to $p \in [0, 1]$ takes its maximum value at the same point $p = p_1$.*

To evaluate σ^2, we get the following theorem.

Theorem 6. *For $r \in [0, \infty)$, the following assertion holds.*

$$\sigma^2 = \zeta - \eta^2,$$

where $\eta = \eta(r) := E[\log(1 + r2^J)]$ is evaluated in Theorem 2 and

$$\begin{aligned} \zeta = \zeta(r) &:= E[\log^2(1 + r2^J)] \\ &= 6\log^2 2 + \sum_{j=1}^{j_0} 2^{-j}\{j\log 4 + \log(r + 2^{-j})\}\log(r + 2^{-j}) + \tilde{\rho}_{j_0} \geq 0, \end{aligned}$$

$$2^{-j_0}\max\{(j_0 + 2)\log r, -(j_0^2 + 4j_0 + 6)\log 2\}\log 4 \leq \tilde{\rho}_{j_0}$$

$$< \begin{cases} 2^{-j_0}[(j_0 + 2)\log 4\log(r + 2^{-j_0-1}) + \min\{\log^2 r, (j_0^2 + 4j_0 + 6)\log^2 2\}] \\ \hfill \text{if } r \leq 1 - 2^{-j_0-1}, \\ 2^{-j_0}[(j_0 + 2)\log 4\log(r + 2^{-j_0-1}) + \max\{\log^2 r, \log^2(r + 2^{-j_0-1})\}] \\ \hfill \text{if } r > 1 - 2^{-j_0-1}. \end{cases}$$

4. Paradox of Petersburg—a numerical approach

We shall give numerical results. Figures 1 to 14 are log-linear plots of $\varphi = \varphi(p, k, n, t) := \exp \lambda(p, k, n, t)$ with respect to $p \in [0, 1]$. The axes origin is $(0, 1)$ in each figure because it is important whether $\varphi(p, k, n, t) > 1$ or not. We denote $10 \wedge m := 10^m$ in figures. For each $t = 0, 1, 2, 3, 4, 5$, the curve of $\varphi(p, k, n, t)$ is the $(t + 1)$th highest. For example, in Figure 2, there are only three curves because the cases $t = 2, 3, 4, 5$ coincide in this figure. If we do not truncate the curves under $\varphi(p, k, n, t) = 0.5$, then they do not coincide. Note that $k = 1/3$ is the case that k is the smallest value that satisfies $p_0(k) = 1$, and $k = 1/4$ is the case $\mu(1, k) = 0$. There is not a special meaning for $k = 1/8$. See Appendix B for the way to obtain the figures. We see that, to maximize $\varphi(p, k, n, t)$ (or $\lambda(p, k, n, t)$) with respect to $p \in [0, 1]$, for $t = 1, 2, 3, 4, 5$, we should take much smaller p than p_0, in particular, if n is not so large. It is danger to bet in the paradox of Petersburg not so large times. For safety, Peter has to continue betting hundreds or thousands of times. We should, however, recognize that, if he really does so, then $\varphi(p, k, n, 0) = \exp \mu n$ for an appropriate p is extremely large. If he really owns such a huge amount of money like $y \exp \mu n$ ducats, it worries him about the great confusion of economy and that even he cannot live on. This is also a limitation of the log utility. In practice, however, Paul will go bankrupt before Peter owns such a huge amount of money. Peter will have $y + M$ ducats with probability 1 where M is the largest amount of money that Paul can pay, if he continues betting $100p$ per cent of his money satisfying $\mu(p, k) > 0$.

Figures 1-2

Figure 1: k=1/3, n=1

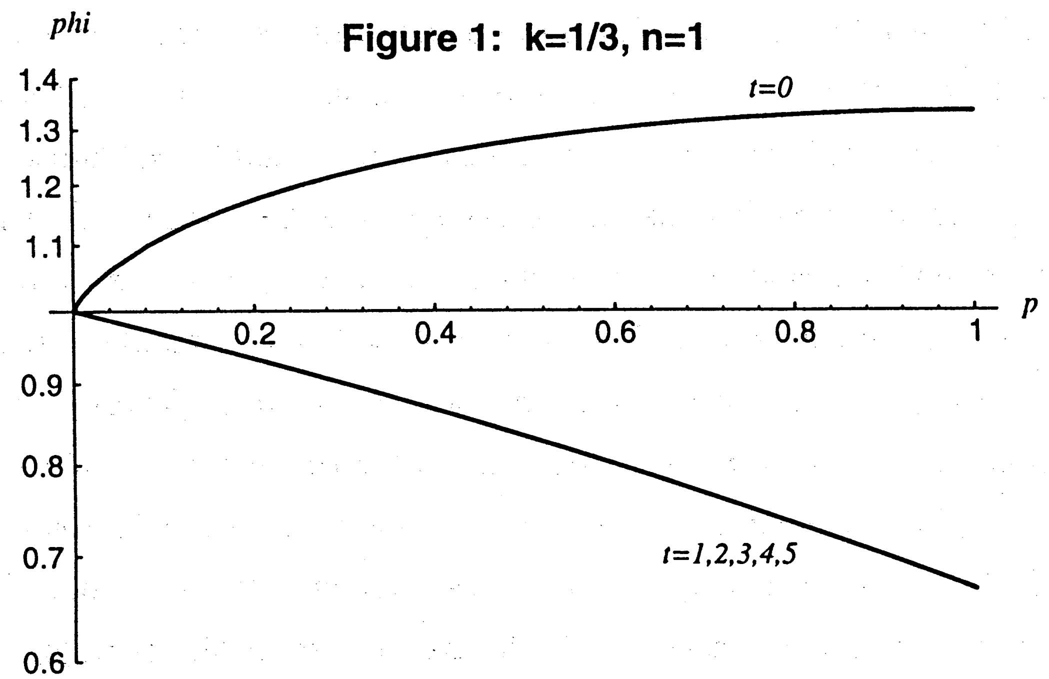
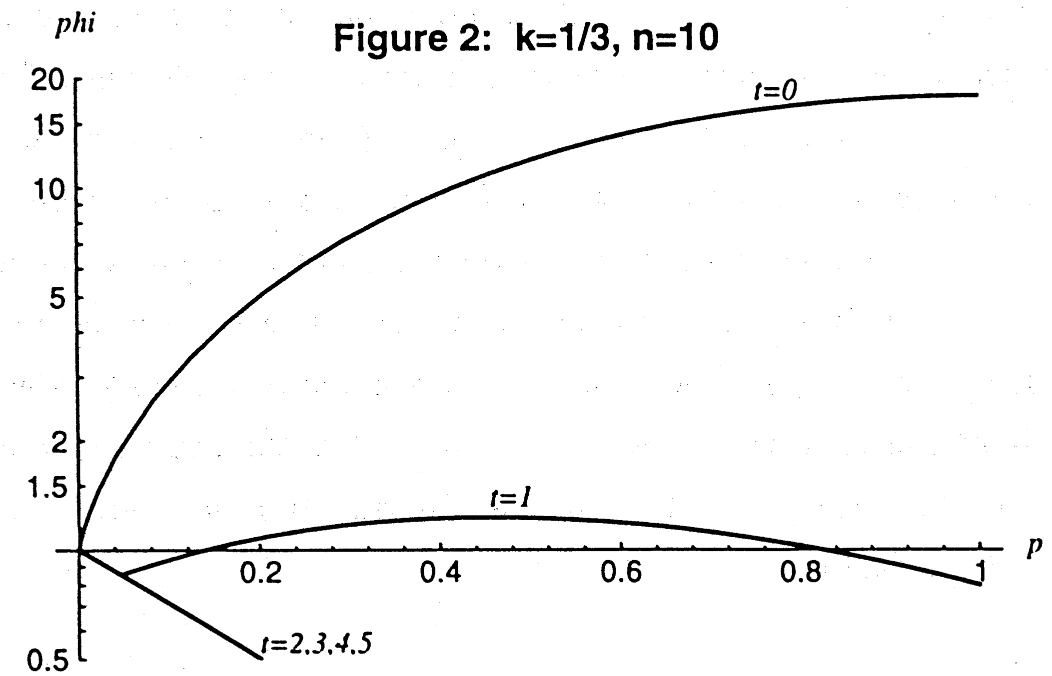
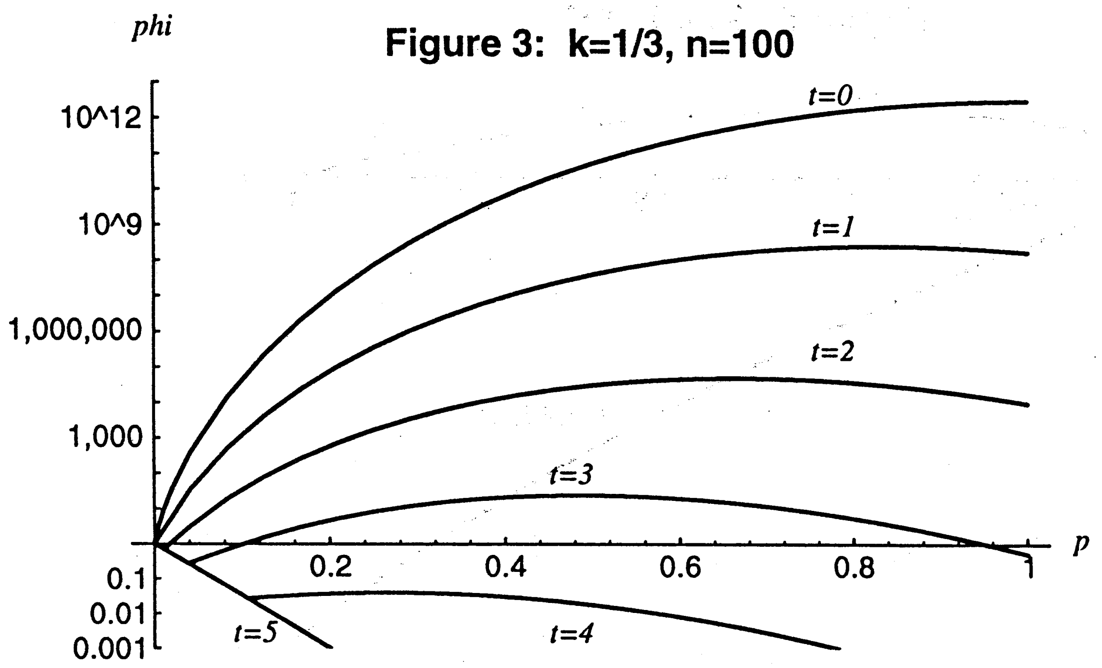
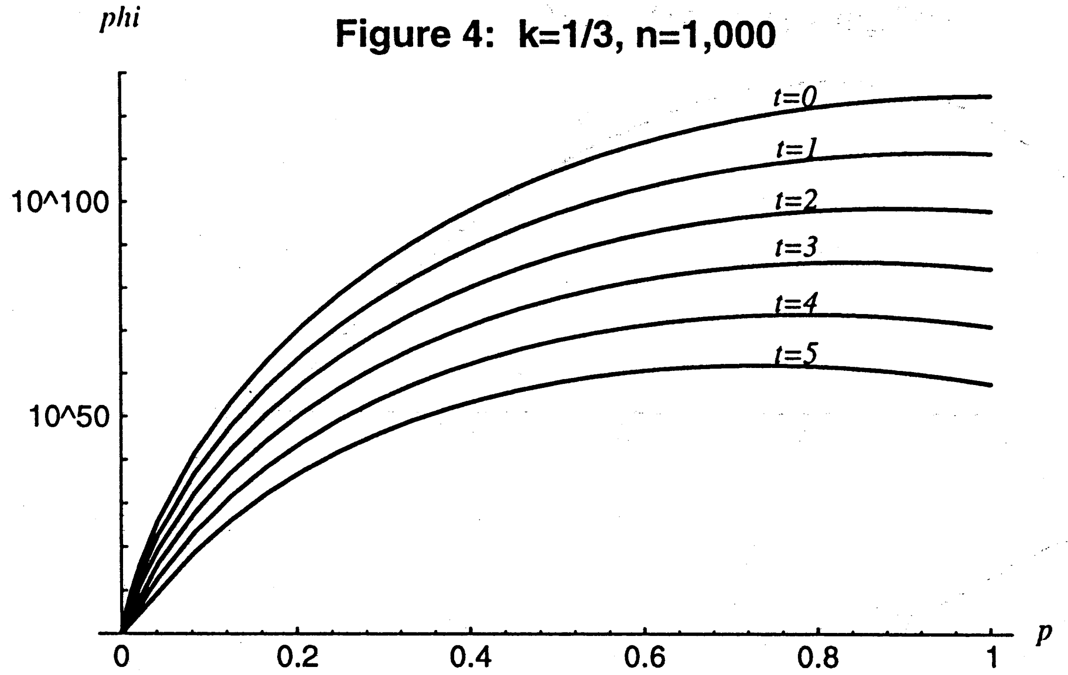


Figure 2: k=1/3, n=10



Figures 3-4


phi
Figure 3: k=1/3, n=100
t=0
t=1
t=2
t=3
t=4
t=5
10^12
10^9
1,000,000
1,000
0.1
0.01
0.001
0.2
0.4
0.6
0.8
1
p


phi
Figure 4: k=1/3, n=1,000
t=0
t=1
t=2
t=3
t=4
t=5
10^100
10^50
0
0.2
0.4
0.6
0.8
1
p

Figures 5-6

Figure 5: k=1/4, n=1

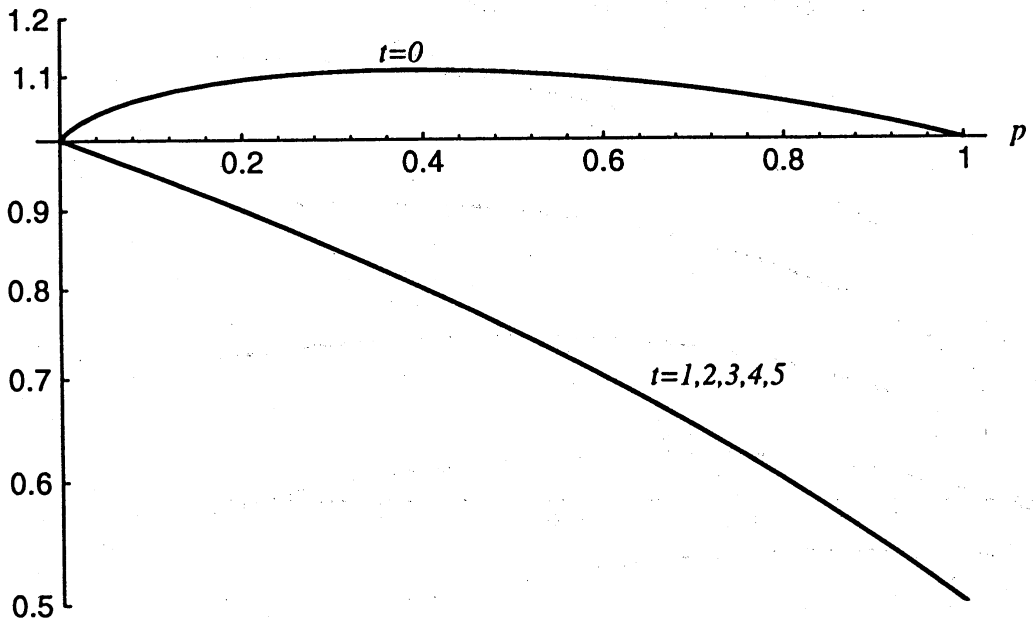
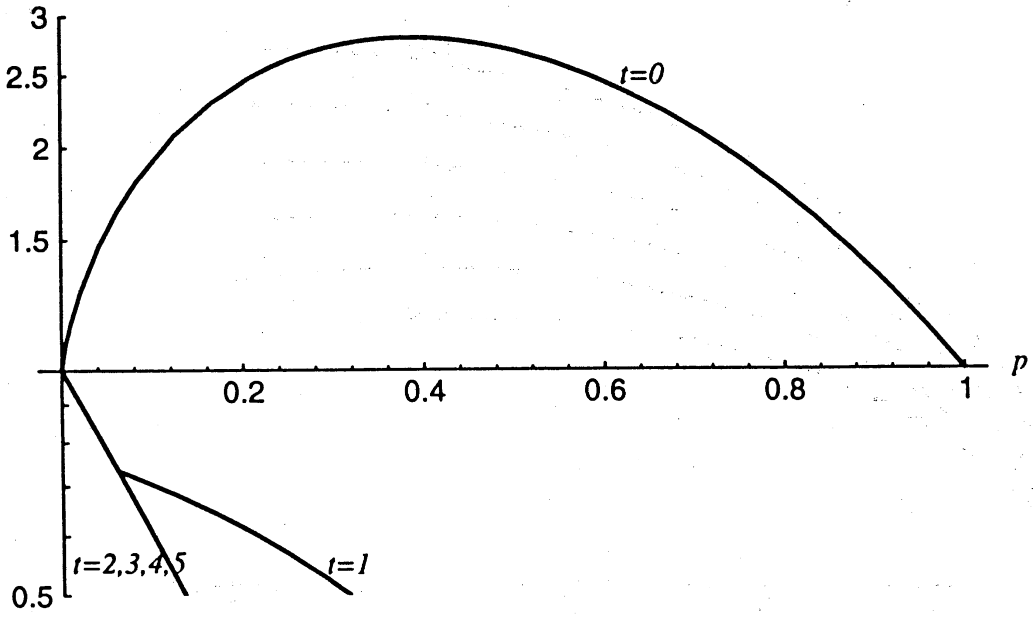


Figure 6: k=1/4, n=10



Figures 7-8

Figure 7: k=1/4, n=100

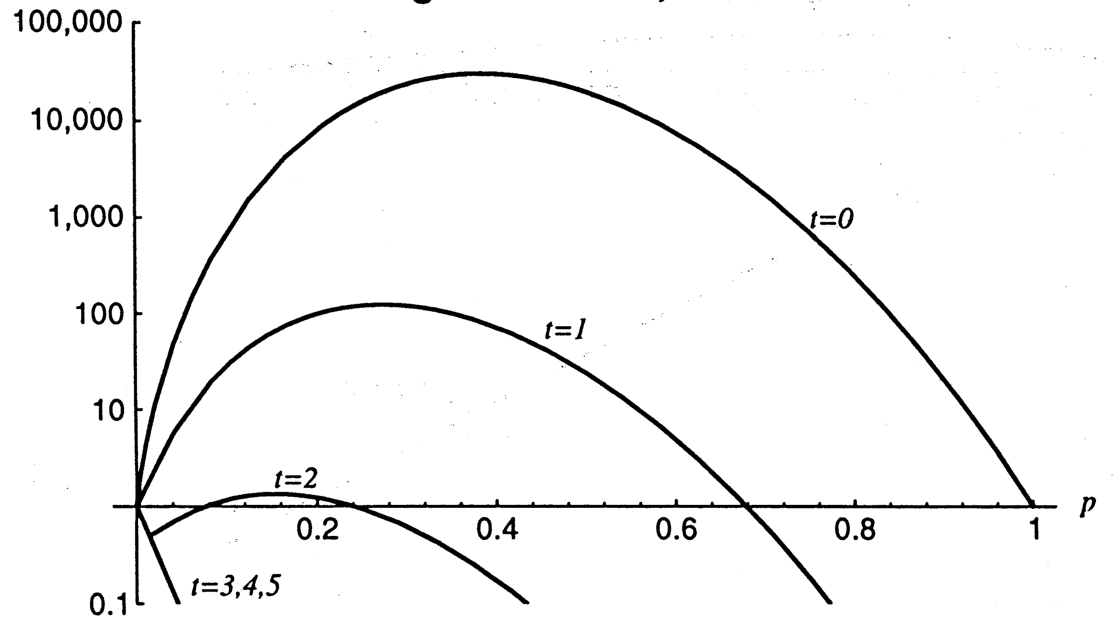
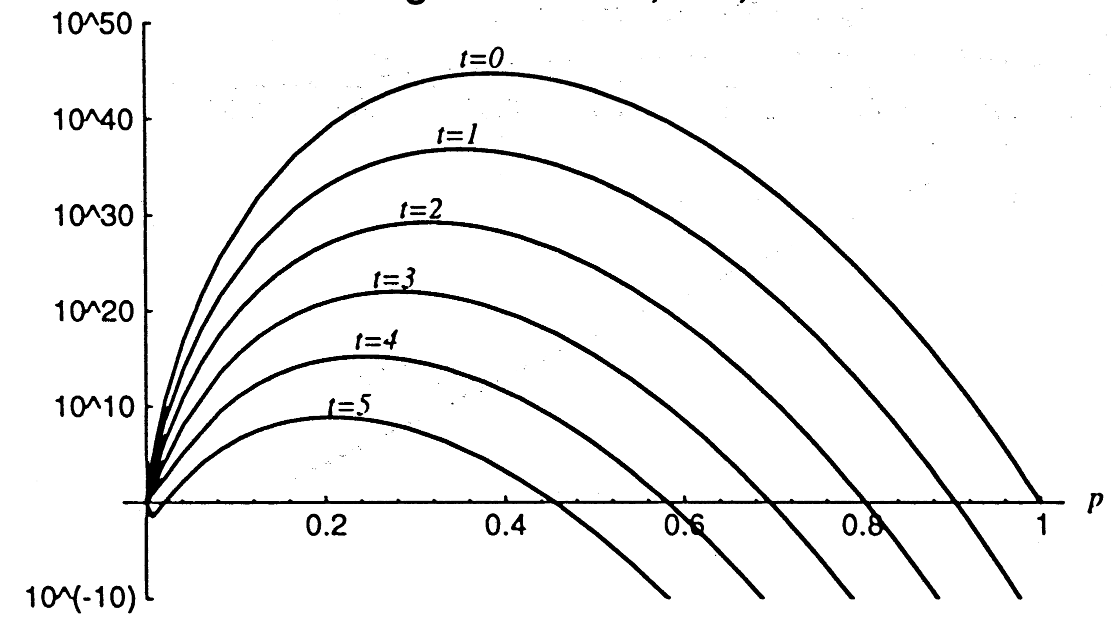
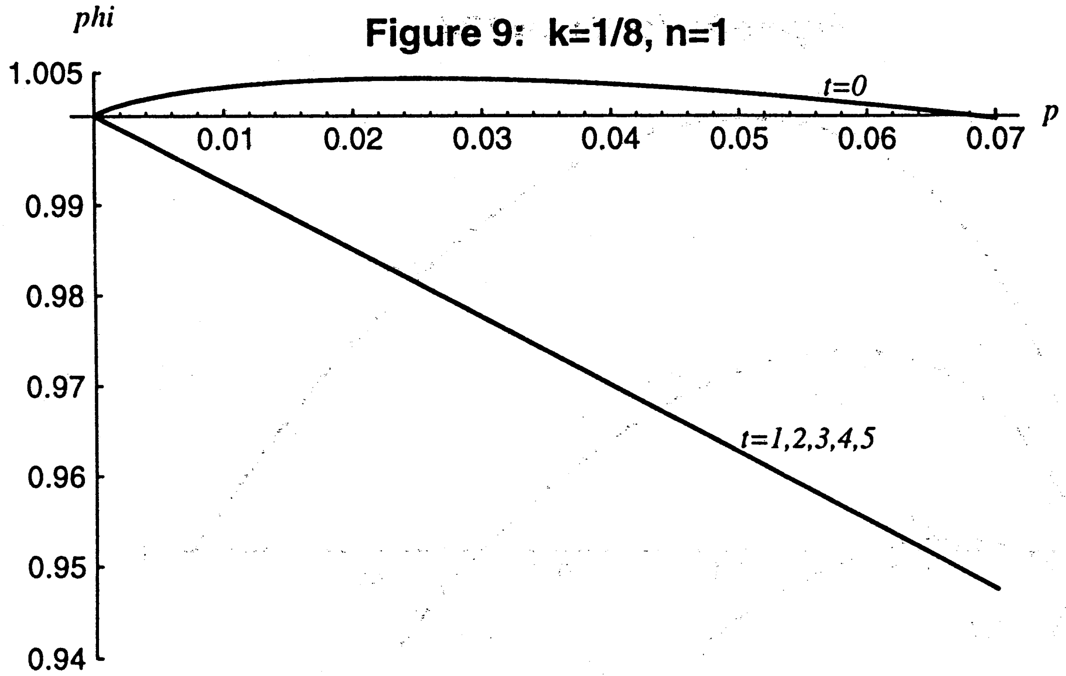
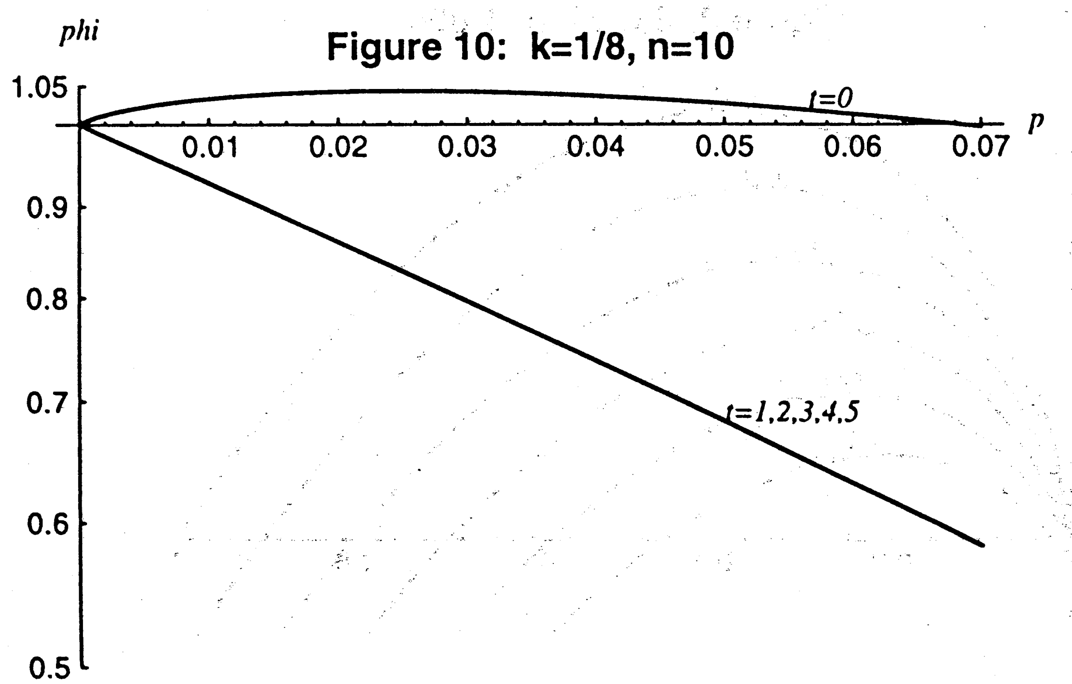


Figure 8: k=1/4, n=1,000



Figures 9-10


phi
Figure 9: k=1/8, n=1
1.005
0.99
0.98
0.97
0.96
0.95
0.94
0.01
0.02
0.03
0.04
0.05
0.06
0.07
p
t=0
t=1,2,3,4,5


phi
Figure 10: k=1/8, n=10
1.05
0.9
0.8
0.7
0.6
0.5
0.01
0.02
0.03
0.04
0.05
0.06
0.07
p
t=0
t=1,2,3,4,5

Figures 11-12

Figure 11: k=1/8, n=100

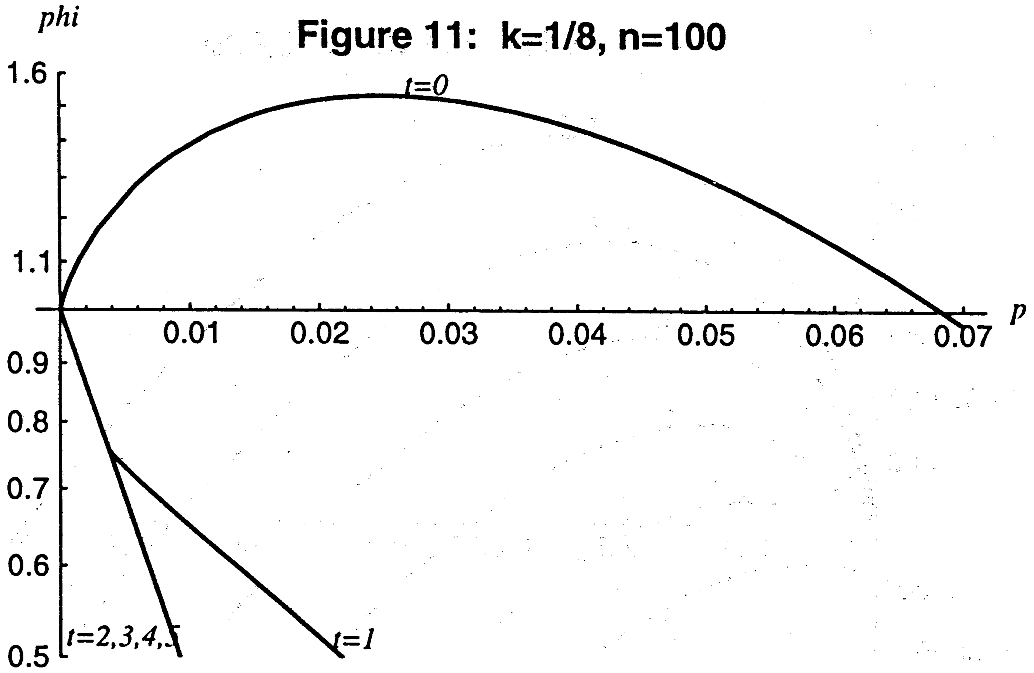
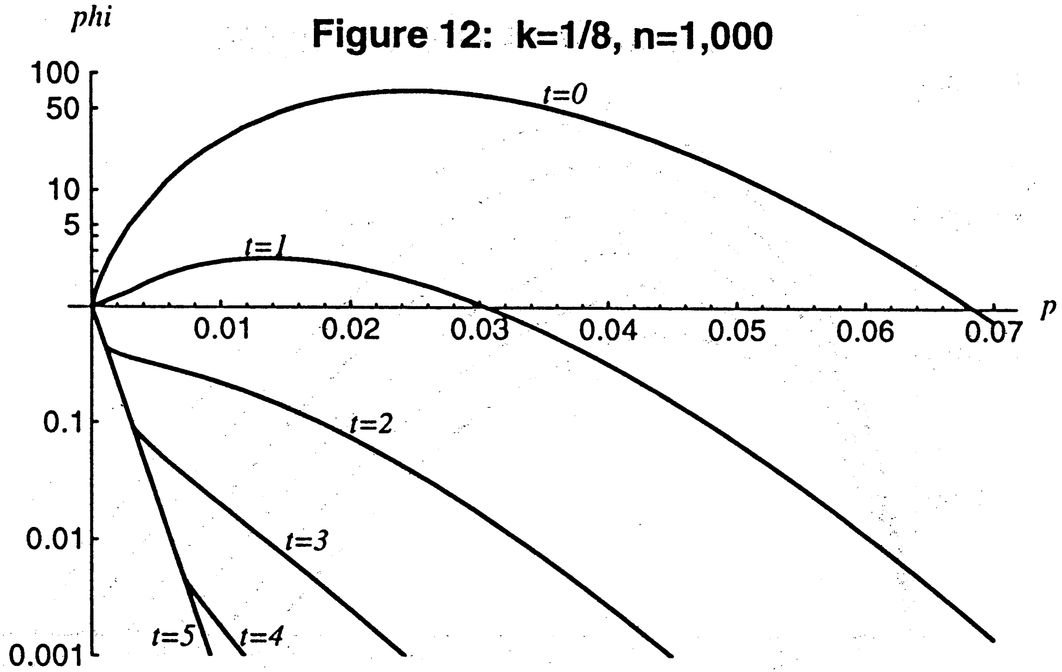
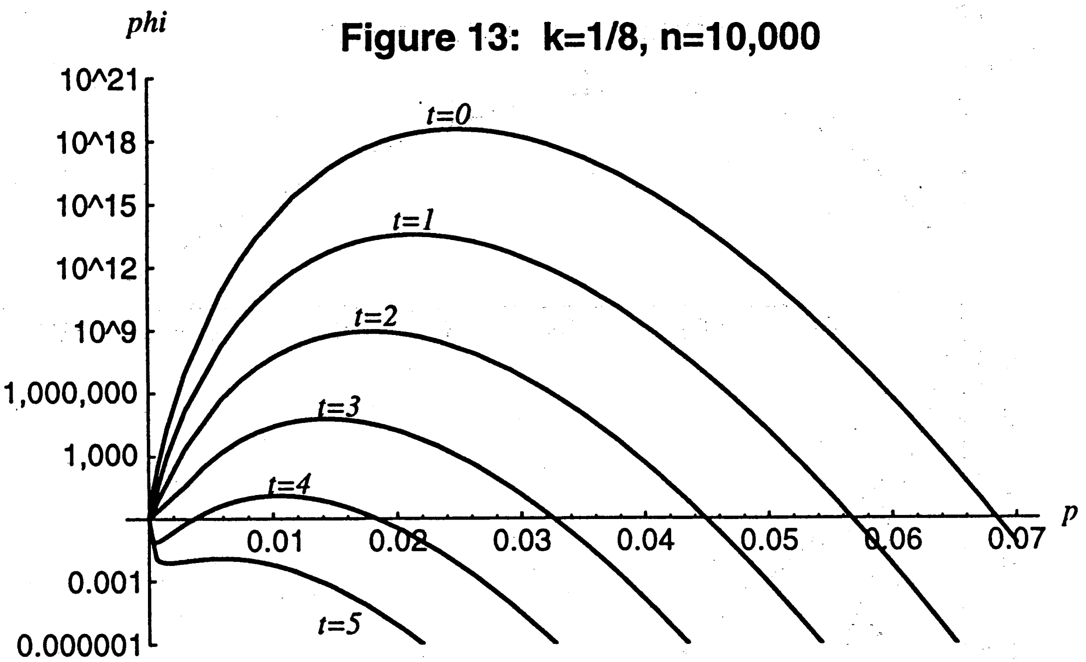
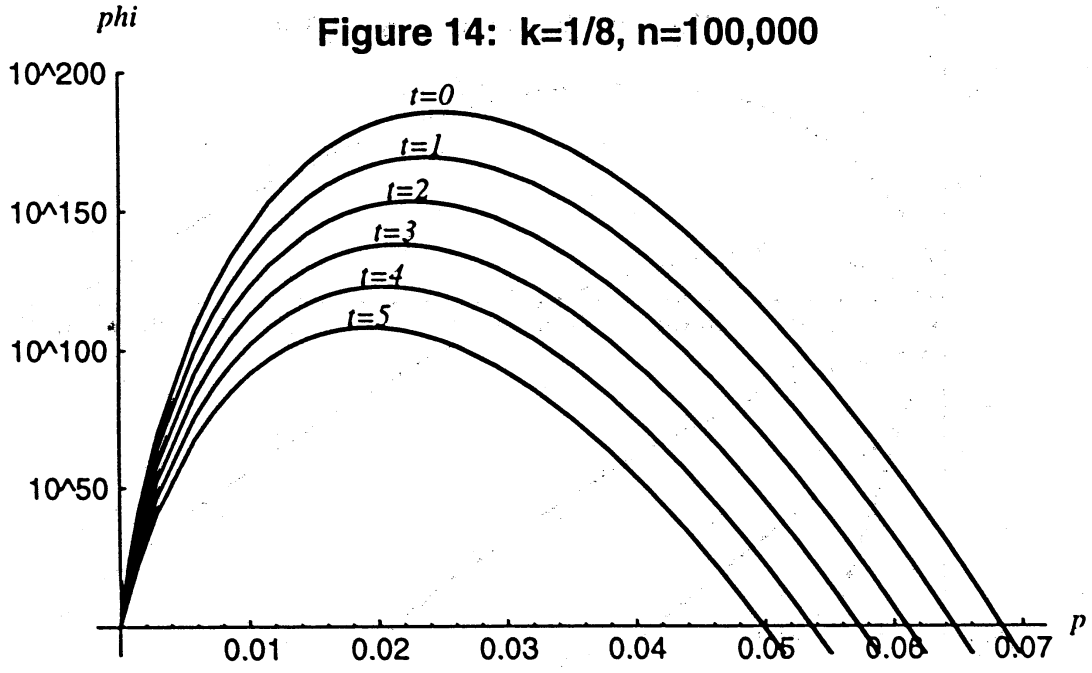


Figure 12: k=1/8, n=1,000



Figures 13-14





5. Some remarks

We shall give some remarks. When Peter continues betting a constant *amount* of money, he increases his money to infinity with probability 1 *if he can borrow any large amount of money.* If he cannot, however, he may go bankrupt before increasing his money. When Peter continues betting a constant *per cent* of his money, there is no possibility of Peter's bankruptcy. There is, however, a problem that how they manage a smaller amount than the smallest unit of money. If they manage each time of their bet, Peter may decrease his money and $100p$ per cent of his money may become smaller than the smallest unit of money. In particular, if k is small, then he should take a small p, so this problem is important. To avoid this problem, they should manage as follows: He continues this bet for a long time without paying or receiving money in practice. After stopping it, he pays or receives money in practice, with managing only at last a smaller amount than the smallest unit of money. Then there is no problem.

Next, assume that $X = k2^{2^J}$ instead of $X = k2^J$. Then $\mu = E[U] = \infty$ for $p \in (0, 1]$. Therefore, continuing this bet, he increases his money to infinity with probability 1, and we cannot determine p by the log utility.

Appendix A

Proof of Theorem 1. Since $U = \log k + J \log 2$, we have $E[J] = \sum_{j=1}^{\infty} 2^{-j} j = 2$, hence (i) holds, so (ii) and (iii) follow. □

Proof of Theorem 2. Clearly $\eta \geq 0$ by definition. We have

$$\begin{aligned}\mu =& E[\log q(1 + r2^J)] \\ =& \log q + E[\log(1 + r2^J)] \\ =& \eta + \log q\end{aligned}$$

$$\begin{aligned}
\eta &= \sum_{j=1}^{\infty} 2^{-j} \log(1 + r2^j) \\
&= \sum_{j=1}^{\infty} 2^{-j} \log 2^j (r + 2^{-j}) \\
&= \log 4 + \sum_{j=1}^{\infty} 2^{-j} \log(r + 2^{-j}) \\
&= \log 4 + \sum_{j=1}^{j_0} 2^{-j} \log(r + 2^{-j}) + \rho_{j_0} \quad \text{(say)}.
\end{aligned}$$

We can evaluate ρ_{j_0} as follows:

$$\rho_{j_0} = 2^{-j_0} \sum_{j=1}^{\infty} 2^{-j} \log(r + 2^{-j_0-j}) = 2^{-j_0} \log(r + \theta 2^{-j_0-1})$$

where $0 < \theta < 1$,

hence

$$2^{-j_0} \log r < \rho_{j_0} < 2^{-j_0} \log(r + 2^{-j_0-1}),$$

and

$$\begin{aligned}
\rho_{j_0} &\geq 2^{-j_0} \sum_{j=1}^{\infty} 2^{-j} \log 2^{-j_0-j} \\
&= -\,2^{-j_0} (\log 2) \sum_{j=1}^{\infty} 2^{-j} (j_0 + j) \\
&= -\,2^{-j_0} (\log 2) \left(j_0 \sum_{j=1}^{\infty} 2^{-j} + \sum_{j=1}^{\infty} 2^{-j} j \right) \\
&= -\,2^{-j_0} (j_0 + 2) \log 2.
\end{aligned}$$

From the two inequalities above, we have

$$2^{-j_0} \max\{\log r, -(j_0 + 2) \log 2\} \leq \rho_{j_0} < 2^{-j_0} \log(r + 2^{-j_0-1}). \quad \square$$

Proof of Theorem 3. We have

$$\begin{aligned}\mu_1(p,k) =& \sum_{j=1}^{\infty} 2^{-j} \frac{k2^j - 1}{1+(k2^j-1)p} \\ =& \frac{1}{p}\sum_{j=1}^{\infty} 2^{-j}\left\{1 - \frac{1}{1+(k2^j-1)p}\right\} \\ =& \frac{1}{p}\left(1 - \sum_{j=1}^{\infty} \frac{1}{kp4^j + q2^j}\right)\end{aligned}$$

for $p \in (0,1]$, where the first line of the equation above can be justified by its locally uniform convergence. We get (i) from the first line. Regarding the summation as the integration by the counting measure and using the monotone convergence theorem, we get (ii). We shall show (iii). To prove the continuity, it is enough to show that $\mu(p,k)$ is continuous at $p = 0$. We get this by Lebesgue's dominant conversion theorem, because if $k2^j - 1 > 0$, then $2^{-j}\log(1 - p + kp2^j)$ is positive and increases with respect to p. The strict concaveness follows from (ii) and the continuity. We get (iv) from (iii). By calculation, we get $\mu_1(1,k) = 1 - 1/3k$, so (v) follows. Assume that $k \in (0,1/3)$. We have $p_0 > 0$ by (ii) and (iii). We get $\mu_1(1,k) < 0$ by calculation, so $p_0 < 1$. Hence p_0 satisfies $\mu_1(p_0,k) = 0$ by (i), and

$$1 = \sum_{j=1}^{\infty} \frac{1}{kp_0 4^j + q_0 2^j} > \sum_{j=1}^{\infty} \frac{1}{(kp_0 + q_0/2)4^j} = \frac{1}{3(kp_0 + q_0/2)}$$

where $q_0 := 1 - p_0$. Solving this inequality with respect to p_0, we get $p_0(k) < 1/\{3(1-2k)\}$, and $1/\{3(1-2k)\} < 1$ is straightforwardly shown. Hence we have (vi). We shall show (vii). If $k < k' \leq 1/3$, then $\mu_1(p_0(k), k') > \mu_1(p_0(k), k) = 0$, so we get $p_0(k') > p_0(k)$ by (i), hence $p_0(k)$ strictly increases with respect to $k \in (0,1/3]$. We shall show its continuity. For any sequence $\{k_m\}$ in $(0,1/3]$ that converges to $k \in (0,1/3]$, we have

$\sum_{j=1}^{\infty} 1/\{k_m p_0(k_m)4^j + q_0(k_m)2^j\} = 1$, and there exists a subsequence $\{k_{i_m}\}$ such that $\{p_0(k_{i_m})\}$ converges (to p, $q := 1-p$, say). If $p = 0$, then, since $p_0(k)$ is positive and strictly increases with respect to k, we get $k = 0$, which is a contradiction. Hence $p \neq 0$, and we may assume that $p_0(k_{i_m}) > p/2$ and $k_{i_m} > k/2$. Therefore, $k_{i_m} p_0(k_{i_m})4^j + q_0(k_{i_m})2^j\} > (k/2)(p/2)4^j$. Hence we can use Lebesgue's dominant conversion theorem and get $\sum_{j=1}^{\infty} 1/\{kp4^j + q2^j\} = 1$, so $p = p_0(k)$, that is, $\lim_{m\to\infty} p_0(k_{i_m}) = p_0(k)$. Hence we have (vii). We shall show (viii). Assume that $1/3 > k_1 > k_2 > \cdots$ and $\lim_{m\to\infty} k_m = 0$. Then, $\sum_{j=1}^{\infty} 1/\{k_m p_0(k_m)4^j + q_0(k_m)2^j\} = 1$, and $\{p_0(k_m)\}$ decreases with respect to n, so it converges (to p, $q := 1-p$, say). Since $k_m p_0(k_m)4^j + q_0(k_m)2^j \geq q_0(k_m)2^j \geq q_0(k_1)2^j$, by Lebesgue's dominant conversion theorem, we get $\sum_{j=1}^{\infty} 1/q2^j = 1$, so $q = 1$ and $p = 0$. Hence we have (viii). □

Proof of Theorem 4. If $p = 1$, then $U = \log k + J \log 2$, $E[J^2] = \sum_{j=1}^{\infty} 2^{-j} j^2 = 6$, so $\sigma^2 = (6 - 2^2)\log^2 2 = 2\log^2 2$, and we get (i). If $0 \leq p < 1$, then $U = \log q + \log(1 + r2^J)$, so we get (ii). We have $\sigma^2(\infty) := \lim_{r\to\infty} \sigma^2(r) = \lim_{p\uparrow 1}\{E[\log^2\{1 + p(2^J - 1)\}] - \mu^2(p,1)\} = E[\log^2 2^J] - \mu^2(1,1) = 2\log^2 2$, where the third equality is justified by the monotone convergence theorem, so we have (iii). Denoting $\eta := E[\log(1 + r2^J)]$, we have $\sigma^2(r) = \sum_{j=1}^{\infty} 2^{-j}\log^2(1 + r2^j) - \eta^2(r)$, hence

$$
\begin{aligned}
\frac{\sigma^{2\prime}(r)}{2} &= \sum_{j=1}^{\infty} 2^{-j}\frac{1}{r + 2^{-j}}\log(1 + r2^j) - \eta(r)\sum_{j=1}^{\infty} 2^{-j}\frac{1}{r + 2^{-j}} \\
&= \sum_{j=1}^{\infty} 2^{-j}\frac{1}{r + 2^{-j}}\{\log(1 + r2^j) - \eta(r)\}
\end{aligned}
$$

for $r \in (0, \infty)$, where the first line of the equation above can be justified by locally uniform convergence of the sums. Since we can take $j_1 = j_1(r) \in$

$[1,\infty)$ satisfying $\eta(r) = \log(1 + r2^{j_1})$, we get

$$\frac{\sigma^{2\prime}(r)}{2} = \sum_{j=1}^{\infty} 2^{-j} \left(\frac{1}{r + 2^{-j}} - \frac{1}{r + 2^{-j_1}} \right) \{\log(1 + r2^j) - \eta(r)\}$$
$$+ \frac{1}{r + 2^{-j_1}} \sum_{j=1}^{\infty} 2^{-j} \{\log(1 + r2^j) - \eta(r)\}.$$

The second sum is 0 by the definition of η. In the first sum, we have

$$\left(\frac{1}{r + 2^{-j}} - \frac{1}{r + 2^{-j_1}} \right) \{\log(1 + r2^j) - \eta(r)\} \geq 0,$$

where the equality holds if and only if $j = j_1$. Hence $\sigma^{2\prime}(r) > 0$ holds for $r \in (0, \infty)$, so we have (iv). It is easy to show that $\sigma^2(r)$ is continuous with respect to r by Lebesgue's dominant conversion theorem and the continuity of $\eta(r)$. From this and (iv), we get (v). By (v) and using (iii) at $p = 1$, we have (vi). □

Proof of Theorem 5. By Theorem 3 (iii) and Theorem 4 (vi), the function $\nu(p, k, n, t)$ is continuous with respect to p on the compact set $[0, 1]$, so it takes its maximum value. We see $p_1 \leq p_0$ by Theorem 4 (vi). We shall show that the strict inequality holds if $p_0 < 1$. By Theorem 3 (v) and (vi), $0 < p_0 < 1$ holds. We may assume $p_1 > 0$. Then, denoting $\nu_1(p, k, n, t) := (\partial/\partial p)\nu(p, k, n, t)$, we get $\nu_1(p_1, k, n, t) = 0$. If $p_0 = p_1$, then, denoting $r_0 = kp_0/q_0$, we have $0 = \nu_1(p_1, k, n, t) = \nu_1(p_0, k, n, t) = \mu_1(p_0, k)n - kt\sigma'(r_0)\sqrt{n}/q_0^2 = -kt\sigma'(r_0)\sqrt{n}/q^2$, so $\sigma'(r_0) = 0$ and $\sigma^{2\prime}(r_0) = 2\sigma(r_0)\sigma'(r_0) = 0$, which contradicts Theorem 4 (iv). We shall show the last part. It is enough to prove $\lambda(p_1, k, n, t) \geq \lambda(p, k, n, t)$ for $p \in [0, 1]$. If $\lambda(p_1, k, n, t) \neq \nu(p_1, k, n, t)$, then, we get $\nu(p_1, k, n, t) < \lambda(p_1, k, n, t) = n \log(q_1 + 2kp_1) < 0 = \lambda(0, k, n, t) = \nu(0, k, n, t)$, where the first inequality follows by the assumption and the definition of ν and the second one follows by $k < 1/2$. This is a contradiction because $\lambda(p,k,n,t)$ takes its maximum value at $p = p_1$. Hence $\lambda(p_1,k,n,t) = \nu(p_1,k,n,t)$ holds. If $\lambda(p,k,n,t) = \nu(p,k,n,t)$, then $\lambda(p_1,k,n,t) = \nu(p_1,k,n,t) \geq \nu(p,k,n,t) = \lambda(p,k,n,t)$. If $\lambda(p,k,n,t) \neq \nu(p,k,n,t)$, then $\lambda(p_1,k,n,t) = \nu(p_1,k,n,t) \geq \nu(0,k,n,t) = \lambda(0,k,n,t) = 0 > n\log(q+2kp) = \lambda(p,k,n,t)$. Hence $\lambda(p_1,k,n,t) \geq \lambda(p,k,n,t)$ holds anyhow. Therefore, we have completed the proof. □

Proof of Theorem 6. By Theorem 4 (ii), we get $\sigma^2 = \zeta - \eta^2$. Clearly $\zeta \geq 0$ by definition. We have

$$\begin{aligned}
\zeta =& \sum_{j=1}^{\infty} 2^{-j} \log^2(1 + r2^j) \\
=& \sum_{j=1}^{\infty} 2^{-j} \{j \log 2 + \log(r + 2^{-j})\}^2 \\
=& 6\log^2 2 + (\log 4) \sum_{j=1}^{\infty} 2^{-j} j \log(r + 2^{-j}) + \sum_{j=1}^{\infty} 2^{-j} \log^2(r + 2^{-j}) \\
=& 6\log^2 2 + (\log 4) \left\{ \sum_{j=1}^{j_0} 2^{-j} j \log(r + 2^{-j}) + \rho_{j_0}^{(1)} \right\} \\
&+ \left\{ \sum_{j=1}^{j_0} 2^{-j} \log^2(r + 2^{-j}) + \rho_{j_0}^{(2)} \right\} \quad \text{(say)} \\
=& 6\log^2 2 + \sum_{j=1}^{j_0} 2^{-j} \{j \log 4 + \log(r + 2^{-j})\} \log(r + 2^{-j}) + \tilde{\rho}_{j_0} \quad \text{(say)}.
\end{aligned}$$

We can evaluate $\rho_{j_0}^{(1)}$ as follows:

$$\begin{aligned}
\rho_{j_0}^{(1)} =& 2^{-j_0} \sum_{j=1}^{\infty} 2^{-j} (j_0 + j) \log(r + 2^{-j_0 - j}) \\
=& j_0 \rho_{j_0} + 2^{-j_0} \sum_{j=1}^{\infty} 2^{-j} j \log(r + 2^{-j_0 - j})
\end{aligned}$$

$$=j_0\rho_{j_0} + 2^{-j_0}\cdot 2\log(r+\theta^{(1)}2^{-j_0-1}) \qquad \text{where } 0<\theta^{(1)}<1,$$

hence

$$2^{-j_0}(j_0+2)\log r < \rho^{(1)}_{j_0} < 2^{-j_0}(j_0+2)\log(r+2^{-j_0-1}),$$

and

$$\begin{aligned}\rho^{(1)}_{j_0} \geq & j_0\rho_{j_0} + 2^{-j_0}\sum_{j=1}^{\infty} 2^{-j} j\log 2^{-j_0-j}\\ =& j_0\rho_{j_0} - 2^{-j_0}(\log 2)\sum_{j=1}^{\infty}2^{-j}j(j_0+j)\\ =& j_0\rho_{j_0} - 2^{-j_0}(\log 2)\left(j_0\sum_{j=1}^{\infty}2^{-j}j + \sum_{j=1}^{\infty}2^{-j}j^2\right)\\ \geq & -2^{-j_0}j_0(j_0+2)\log 2 - 2^{-j_0}(2j_0+6)\log 2\\ =& -2^{-j_0}(j_0^2+4j_0+6)\log 2.\end{aligned}$$

From the two inequalities above, we have

$$\begin{aligned}2^{-j_0}\max\{(j_0+2)\log r, -(j_0^2+4j_0+6)\log 2\}&\\ \leq \rho^{(1)}_{j_0} < 2^{-j_0}(j_0+2)\log(r+2^{-j_0-1}).&\end{aligned}$$

We can evaluate $\rho^{(2)}_{j_0}$ as follows:

$$\rho^{(2)}_{j_0} = 2^{-j_0}\sum_{j=1}^{\infty}2^{-j}\log^2(r+2^{-j_0-j}) = 2^{-j_0}\log^2(r+\theta^{(2)}2^{-j_0-1})$$

$$\text{where } 0<\theta^{(2)}<1,$$

hence

$$0\leq \rho^{(2)}_{j_0} < 2^{-j_0}\max\{\log^2 r, \log^2(r+2^{-j_0-1})\},$$

and if $r + 2^{-j_0-1} \le 1$, we get

$$\begin{aligned}
\rho_{j_0}^{(2)} \le& 2^{-j_0} \sum_{j=1}^{\infty} 2^{-j} \log^2 2^{-j_0-j} \\
=& 2^{-j_0} (\log^2 2) \sum_{j=1}^{\infty} 2^{-j} (j_0 + j)^2 \\
=& 2^{-j_0} (\log^2 2) \left(j_0^2 \sum_{j=1}^{\infty} 2^{-j} + 2j_0 \sum_{j=1}^{\infty} 2^{-j} j + \sum_{j=1}^{\infty} 2^{-j} j^2 \right) \\
=& 2^{-j_0} (j_0^2 + 4j_0 + 6) \log^2 2.
\end{aligned}$$

From the two inequalities above, we have

$$0 \le \rho_{j_0}^{(2)} \le \begin{cases} 2^{-j_0} \min\{\log^2 r, (j_0^2 + 4j_0 + 6) \log^2 2\} & \text{if } r \le 1 - 2^{-j_0-1}, \\ 2^{-j_0} \max\{\log^2 r, \log^2 (r + 2^{-j_0-1})\} & \text{if } r > 1 - 2^{-j_0-1}. \end{cases}$$

Since $\tilde{\rho}_{j_0} = \rho_{j_0}^{(1)} \log 4 + \rho_{j_0}^{(2)}$, we can get the inequality on $\tilde{\rho}_{j_0}$, so we have completed the proof. □

Appendix B

We shall explain the way to obtain the figures. The author has used *Mathematica* for Macintosh. Let

$$\begin{aligned}
q :=& 1 - p, \\
r :=& \frac{kp}{q} \quad \text{if } p \ne 1, \\
\overline{\eta} :=& \log 4 + \sum_{j=1}^{j_0} 2^{-j} \log(r + 2^{-j}) + 2^{-j_0} \log(r + 2^{-j_0-1}), \\
\underline{\eta} :=& \max\{0, \\
& \log 4 + \sum_{j=1}^{j_0} 2^{-j} \log(r + 2^{-j}) + 2^{-j_0} \max\{\log r, -(j_0 + 2) \log 2\}\},
\end{aligned}$$

$$\overline{\mu} := \begin{cases} \log 4k & \text{if } p = 1, \\ \overline{\eta} + \log q & \text{if } p \neq 1, \end{cases}$$

$$\underline{\mu} := \begin{cases} \log 4k & \text{if } p = 1, \\ \underline{\eta} + \log q & \text{if } p \neq 1, \end{cases}$$

$$\begin{aligned} \overline{\zeta} :=& 6 \log^2 2 + \sum_{j=1}^{j_0} 2^{-j} \{ j \log 4 + \log(r + 2^{-j}) \} \log(r + 2^{-j}) \\ &+ \begin{cases} 2^{-j_0} [(j_0 + 2) \log 4 \log(r + 2^{-j_0 - 1}) + \min\{\log^2 r, (j_0^2 + 4j_0 + 6) \log^2 2\}] \\ \qquad\qquad\qquad\qquad \text{if } r \leq 1 - 2^{-j_0 - 1}, \\ 2^{-j_0} [(j_0 + 2) \log 4 \log(r + 2^{-j_0 - 1}) + \max\{\log^2 r, \log^2(r + 2^{-j_0 - 1})\}] \\ \qquad\qquad\qquad\qquad \text{if } r > 1 - 2^{-j_0 - 1}, \end{cases} \end{aligned}$$

$$\begin{aligned} \underline{\zeta} :=& \max\{0, 6 \log^2 2 + \sum_{j=1}^{j_0} 2^{-j} \{ j \log 4 + \log(r + 2^{-j}) \} \log(r + 2^{-j}) \\ &+ 2^{-j_0} \max\{(j_0 + 2) \log r, -(j_0^2 + 4j_0 + 6) \log 2\} \log 4\}, \end{aligned}$$

$$\overline{\sigma} := \begin{cases} \sqrt{2} \log 2 & \text{if } p = 1, \\ \sqrt{\overline{\zeta} - \underline{\eta}^2} & \text{if } p \neq 1, \end{cases}$$

$$\underline{\sigma} := \begin{cases} \sqrt{2} \log 2 & \text{if } p = 1, \\ \sqrt{\max\{0, \underline{\zeta} - \overline{\eta}^2\}} & \text{if } p \neq 1, \end{cases}$$

$$\overline{\lambda} := \max\{\overline{\mu} n - t \underline{\sigma} \sqrt{n}, n \log(q + 2kp)\},$$

$$\underline{\lambda} := \max\{\underline{\mu} n - t \overline{\sigma} \sqrt{n}, n \log(q + 2kp)\}.$$

Then $\underline{\eta} \leq \eta \leq \overline{\eta}$, $\underline{\mu} \leq \mu \leq \overline{\mu}$, $\underline{\zeta} \leq \zeta \leq \overline{\zeta}$, $\underline{\sigma} \leq \sigma \leq \overline{\sigma}$, and $\underline{\lambda} \leq \lambda \leq \overline{\lambda}$ follow by Section 3. The author has made *Mathematica* draw curves of $\overline{\lambda}$'s and $\underline{\lambda}$'s with respect to p, letting $j_0 = 20$ in Figures 1 to 10, and $j_0 = 30$ in Figures 11 to 14. Then, for each k, n, and t, the curves of them look coincident, so we can regard them as the curve of λ. *Mathematica* can compute infinite sums numerically but the author has avoided it because Wolfram [3] (p. 832, see also pp. 689–690) notes, "You should realize that with sufficiently pathological summands, the algorithms used by `NSum` (a numerical sum) can give wrong answers."

In this way, we get graphs of λ's. By making adequate ticks, they become log-linear plots of φ's. For this purpose, the author has selected adequate values for the vertical coordinate, not *Mathematica* automatically selected. For each of them, say $(\varphi =)\varphi_0$, the author has made a tick of g to the place of $(\lambda =) \log \varphi_0$ on the vertical coordinate. In some figures, curves are truncated. For example, in Figure 2, the curves under $\varphi(p, k, n, t) = 0.5$ are truncated. This is also done by the author, not automatically. The author has made ticks and truncation considering that the reader can easily see important parts of graphs, in particular, whether $\varphi > 1$ or not, and avoiding misunderstanding.

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