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Global existence of solutions to the parabolic systems of chemotaxis

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1. Introduction

We consider time-global existence of solutions of some parabolic systems related to chemotaxis. We consider the following system which is called Keller-Segel model.

$$
\begin{cases}
u_t = \nabla \cdot (\nabla u - \chi u \nabla \phi(v)), & x \in \Omega,\ t > 0, \\
\varepsilon v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\
\dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\
u(\cdot, 0) = u_0, v(\cdot, 0) = v_0 & x \in \Omega,
\end{cases}
$$

where χ and ε are postive constants, Ω is a bounded and connected domain of $\mathbf{R}^2$ with smooth boundary $\partial\Omega$, ϕ is a smooth function on $(0, \infty)$ with $\phi' > 0$, u_0 and v_0 are smooth, nonnegative and non-trivial on $\bar{\Omega}$.

Keller-Segel model was introduced by Keller and Segel [6] to describe the initiation of chemotactic aggregation of cellular slime molds. $u(x, t)$ represents the cell density at place x and time t. $v(x, t)$ represents the concentration of chemical substance at place x and time t.

Let me explain Keller-Segel model.

The first equation means change of cell density. The term $(-\nabla u + \chi \nabla \phi(v))$ means the flow of cells. The term $-\nabla u$ means the flow due to diffusion. As $\nabla \phi(v) = \phi' \nabla v$, then the term $\chi u \phi' \nabla v$ means the chemotactic flow due to response to chemical attractant. Namely, cells sense the gradient of chemical concentration. This phenomenon is called chemotaxis. And chemical substance is an attractant, then the positivity of ϕ' is neccessary. Then the function ϕ means the relation between the intensity of chemotactic flux and v, ∇v. ϕ is called sensitivity function. Cells measure the gradient of $\phi(v)$. Several forms of ϕ are suggested in biology.

The second equation means change of concentration of chemical substance. The term $(-1/\varepsilon)\nabla v$ means the flow due to diffusion. The term of v/ε means the degradation by reactions. The term u/ε means the production by cells. Then the degradation and the production are proportional to chemical concentration and cell density, respectively.

Those phnomenon suggests the posibility of aggregation. Namely, first, cells move toward higher concentration. Then cells aggregate at the place and product much attractant. Then cell and chemical substance aggregate at the place.

Then we consider the following problem :

Investigate whether solutions can exist globally in time or not for several forms of the sensitivity function ϕ.

In particular, ϕ is specified as the following two cases :

(A1) $\phi(v) = v$,

(A2) $\phi(v) = \log v$.

First, we describe a result in one dimensional case. In the following theorem, ϕ is a smooth function with $\phi' > 0$.

Theorem 1 *Assume that $\Omega = (0, L)$, u_0 is a nonnegative smooth function on $[0, L]$ and v_0 is a positive smooth function on $[0, L]$. Then the solution is globally bounded in time. Namely, $T_{max} = \infty$ and*

$$\sup_{t\geq 0}(\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty}) < \infty,$$

where T_{max} is the maximal time of existence.

Then, in two dimensional case, we expect one dimensional blow-up can not occur.

Theorem 2 *Assume $\phi(v) = v$.*

(i) If $\|u_0\|_{L^1} < 4\pi/\chi$, then the solution is globally bounded in time.

(ii) Let $\Omega = \{x \in \mathrm{R}^2; |x| < L\}$ and (u_0, v_0) be radial in x. If $\|u_0\|_{L^1} < 8\pi/\chi$, then the solution is globally bounded in time.

We expect that the restriction of $L^1 - norm$ is necessary. Because, there are the following conjecture and resuluts.

Childress [2] and Childress and Percus [3] have given a conjecture such that if $\int_\Omega u_0(x)dx < 8\pi/\chi$ then the solution exists globally in time, and if $\int_\Omega u_0(x)dx > 8\pi/\chi$ then the solution can blow up in finite time, in the case of $\phi(v) = v$ and radial initial functions (u_0, v_0) on $\Omega = \{x \in \mathbf{R}^2; |x| < L\}$.

T. Nagai [7] deal with the limiting system as $\varepsilon \to 0$. He has given a result such that if $\int_\Omega u_0(x)dx < 8\pi/\chi$ then the solution is globally bounded in time, and if $\int_\Omega u_0(x)dx > 8\pi/\chi$ and $\int_\Omega u_0(x)|x|^2dx \ll 1$, then the solution blows up in finite time, in the case of $\phi(v) = v$ and radial initial functions (u_0, v_0) on $\Omega = \{x \in \mathbf{R}^2; |x| < L\}$.

Theorem 3 *Assume $\phi(v) = \log v$ and v_0 is positive in $\bar{\Omega}$.*

(i) If $\chi < 1$, then the solution globally exists in time. Namely, $T_{max} = \infty$ and

$$\sup_{0 \le t \le T} (\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty}) = C_T < \infty$$

for $T > 0$.

(ii) Let $\Omega = \{x \in \mathrm{R}^2; |x| < L\}$ and (u_0, v_0) be radial in x. If $\chi < 5/2$, then the solution globally exists in time.

We expect that the restriction of $L^1 - norm$ is not necessary. Because, T. Senba [8] deal with the limiting system as $\varepsilon \to 0$. I have given a result such that the solution is globally bounded in time, in the case of $\phi(v) = \log v$ and radial initial functions (u_0, v_0) on $\Omega = \{x \in \mathbf{R}^2; |x| < L\}$.

2. Proof of Theorem 2.

Lemma 2.1 Put

$$W(t) = \int_\Omega \left\{ u \log u - \chi uv + \frac{\chi}{2}\left(|\nabla v|^2 + v^2\right)\right\} dx.$$

Then we have

$$\frac{dW}{dt}(t) + \chi\varepsilon \int_\Omega (v_t)^2 dx + \int_\Omega u|\nabla \cdot (\log u - \chi v)|^2 dx = 0.$$

Proof. Multiplying $\log u - \chi v$ by the first equation and using Green's formula and the second equation, we have this lemma.

Lemma 2.2

(i) Let Ω be a bounded and connected domain in R^2 with smooth boundary. Then, $\exists C_\Omega > 0$ s.t.

$$\int_\Omega \exp|u| dx \le C_\Omega \exp\left\{\frac{1}{8\pi}\|\nabla u\|_2^2 + \frac{2}{|\Omega|}\|u\|_1\right\}$$

for $u \in H^1(\Omega)$. (S.Y.A. Chang and P.C. Yang [1])

(ii) Let $\Omega = \{x \in \mathrm{R}^2; |x| < L\}$. Then for $\forall\delta > 0$, $\exists C = C_\delta > 0$ s.t.

$$\int_\Omega \exp|u|dx \le C_\delta \exp\left\{\left(\frac{1}{16\pi} + \delta\right)\|\nabla u\|_2^2 + \frac{2}{|\Omega|}\|u\|_1\right\}.$$

for $u \in H^1(\Omega)$ with $u(x) = u(|x|)$.

Lemma 2.3 If $\|u_0\|_1 < \pi^*/\chi$, $\exists C$(independent of t) > 0 s.t. $\int_\Omega uvdx \le C$ and $|W(t)| \le C$,

$$\text{where } \pi^* = \begin{cases} 8\pi, & \text{in radially symmetric case,} \\ 4\pi, & \text{otherwise.} \end{cases}$$

Proof. Let $a > 0$. For fix $t \in (0,T)$, put $\psi(x,t) = \frac{M}{\mu}e^{av(x,t)}$, where

$$M = \int_\Omega u(x,t)dx \text{ and } \mu = \int_\Omega e^{av(x,t)}dx.$$

By Lemma 2.2 for $\forall\delta > 0$, $\exists C_\delta > 0$ s.t.

$$\log\mu \le \log C_\delta + \frac{2a}{|\Omega|}\|v\|_1 + \left\{\frac{1}{2\pi^*} + \delta\right\}a^2\|\nabla v\|_2^2.$$

By $\int_\Omega \frac{\psi}{u}\frac{u}{M}dx = 1$ and Jensen's inequality,

$$\begin{aligned} 0 &= -\log\int_\Omega \frac{\psi}{u}\frac{u}{M}dx \\ &\le \int_\Omega\left\{-\log\frac{\psi}{u}\right\}\frac{u}{M}dx = \frac{1}{M}\int_\Omega u\log\frac{u}{\psi}dx. \end{aligned}$$

Then

$$\begin{aligned} &\left\{\frac{\chi}{2} - M\left(\frac{1}{2\pi^*} + \delta\right)a^2\right\}\|\nabla v\|_2^2 + (a-\chi)\int_\Omega uvdx \\ &\le M\left\{\log C_\delta + \frac{2a}{|\Omega|}\|v\|_1 - \log M\right\} + W(t). \end{aligned}$$

Lemma 2.4 $\exists C$(independent of t) > 0 s.t. $\|u(\cdot,t)\|_2 \le C$.

Proof. For simplicity we put $\chi = \varepsilon = 1$. Multiply u by the first equation, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx + \int_\Omega|\nabla u|^2dx &= -\int_\Omega \nabla\cdot(u\nabla v)udx \\ = -\frac{1}{2}\int_\Omega u^2\Delta vdx &= -\frac{1}{2}\int_\Omega u^2(v_t + v - u)dx. \end{aligned}$$

We can show that

$$\begin{aligned}\|u\|_3 &\le \delta\|\nabla u\|_2^{2/3}\|u\log u\|_1^{1/3}\\ &\quad + C_\delta\left\{\|u\log u\|_1 + \|u\|_1^{1/3}\right\},\end{aligned}$$

By Hörder and Gagliardo-Nirenberg inequality, we have

$$\int_\Omega |u^2 v_t|dx \le \delta\|\nabla u\|_2^2 + C\left\{\|v_t\|_2^2 + \|v_t\|_2\right\}\|u\|_2^2,$$

which together above formulas leads to

$$\begin{aligned}\frac{d}{dt}\|u\|_2^2 + 2\|\nabla u\|_2^2 &\le \delta\|\nabla u\|_2^2 + C\left\{\|v_t\|_2^2 + \|v_t\|_2\right\}\|u\|_2^2\\ &\quad + \delta^3\|u\log u\|_1\|\nabla u\|_2^2 + C\left\{\|u\log u\|_1^3 + \|u\|_1\right\}.\end{aligned}$$

By the above inequality and Gronwall's inequality, we have this lemma.

By applying the estimes of $\|u\|_2$ and standard argments to the second equation, we have the boundedness of $\|\nabla v\|_\infty$ and $\|v\|_\infty$, which yields the bondedness of $\|u\|_\infty$ by applying Moser's technique to the first equation.

3. Proof of Theorem 3. Since the proof of (ii) is similar to one of (i), we shall prove only (ii).

Lemma 3.1 Let a be a positive constant. Then we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (u\log u - au\log v)\,dx + \frac{a}{\varepsilon}\int_\Omega \frac{u^2}{v}dx\\ &+ \int_\Omega u\left\{|\nabla \log u|^2 - \left(\chi + \frac{2a}{\varepsilon}\right)\nabla\log u\cdot\nabla\log v\right.\\ &\qquad\left. + \frac{a}{2}(\chi+1)|\nabla\log v|^2\right\}dx = \frac{a}{\varepsilon}|u_0|_1\end{aligned}$$

Proof. Multiplying $\log u - a\log v$ by the first equation and using Green's formula and the second equation, we have this lemma.

Lemma 3.2 For $\forall p \ge 1$, $\exists C_p$(independent of t) > 0 s.t.

$$\|v(\cdot,t)\|_p \le C_p(\|u_0\|_1 + \|v_0\|_p)$$

Proof. Using the following estimet of Green's function G.

$$|G(x,y,t)| \le \frac{\varepsilon C}{t}\exp\left(-c\varepsilon\frac{|x-y|^2}{t} - \frac{t}{\varepsilon}\right).$$

Lemma 3.3 If $\chi < 1$, $\exists C$(independent of t) > 0 s.t.

$$\int_\Omega u \log u dx \text{ and } \int_0^t \int_\Omega u|\nabla \log u| dx ds \leq Ct.$$

Proof. By using Lemma 3.1 with $a = \varepsilon/2$,

$$\frac{d}{dt}\int_\Omega \left(u \log u - \frac{\varepsilon}{2} u \log v\right) dx + \frac{1-\chi}{2}\int_\Omega u|\nabla \log u|^2 dx \leq \frac{1}{2}||u_0||_1.$$

Put $\psi(x) = \frac{M}{\mu} v^p$, where
$M = ||u||_1$ and $\mu = ||v||_p^p$. By $\displaystyle\int_\Omega \frac{\psi}{u}\frac{u}{M} dx = 1$ and Jensen's inequality,

$$0 = -\log \int_\Omega \frac{\psi}{u}\frac{u}{M} dx \leq \int_\Omega \left\{-\log \frac{\psi}{u}\right\} \frac{u}{M} dx$$

Then

$$p \int_\Omega u \log v dx \leq \int_\Omega u \log u dx + M \log \frac{\mu}{M}.$$

Combining the first eq. and the last eq. implies this lemma.

Proof of Theorem 3 By Gagliardo-Nirenberg inequality, we have

$$||u||_2^2 = ||\sqrt{u}||_4^4 \leq \left(||\nabla\sqrt{u}||_2^2 + ||\sqrt{u}||_2^2\right) ||\sqrt{u}||_2^2.$$

The above inequality and Lemma 3.3 implies that

$$\int_0^t \int_\Omega u^2 dx ds \leq Ct. \tag{1}$$

Mulitplying $-\Delta v$ by the second equation, we have

$$\frac{d}{dt}||\nabla v||_2^2 + ||\Delta v||_2^2 + ||\nabla v||2^2 \leq ||u||_2||\Delta v||_2.$$

Combining the above inequality and (1) implies that

$$\int_0^t ||\Delta v||_2^2 \text{ and } ||\nabla v||_2^2 \leq Ct. \tag{2}$$

Mulitiply u by the first equation and using Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
\frac{d}{dt}||u||_2^2 + ||\nabla u||_2^2 &= -\chi \int_\Omega u\nabla u \cdot \nabla \log v dx \\
= \frac{\chi}{2}\int_\Omega u^2 \Delta \log v dx &= \frac{\chi}{2}\int_\Omega \frac{u^2}{v}(\Delta v - |\nabla \log v|^2) dx \\
&\leq \frac{\chi}{2V_m}||\Delta v||_2 ||\nabla u||_2 ||u||_2,
\end{aligned}$$

,where $V_m(t) = \min_\Omega v(\cdot, t) \geq V_m(0)e^{-t}$, which together (2) implies

$$\|u\|_2^2 \leq C \exp\left(cte^{2t}\right).$$

By applying the estimes of $\|u\|_2$ and standard argments to the second equation, we have the boundedness of $\|\nabla v\|_\infty$ and $\|v\|_\infty$, which yields the bondedness of $\|u\|_\infty$ by using Moser's technique for the first equation.

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