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Homogenization of Hamilton-Jacobi equations with Neumann and Dirichlet boundary conditions

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Introduction.

We describe some results which we have obtained jointly with Prof. H. Ishii in [HI].

We consider the limiting behavior, as $\varepsilon \to 0$, of solutions of the following boundary value problems for Hamilton-Jacobi equations,

$$(\mathrm{N})_\varepsilon \qquad \begin{cases} H\left(Du^\varepsilon(x), u^\varepsilon(x), x, \dfrac{x}{\varepsilon}\right) = 0 & \text{in } \Omega_\varepsilon, \\ B\left(Du^\varepsilon(x), u^\varepsilon(x), x, \dfrac{x}{\varepsilon}\right) = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}$$

and

$$(\mathrm{D})_\varepsilon \qquad \begin{cases} H\left(Du^\varepsilon(x), u^\varepsilon(x), x, \dfrac{x}{\varepsilon}\right) = 0 & \text{in } \Omega_\varepsilon, \\ u^\varepsilon(x) = b\left(x, \dfrac{x}{\varepsilon}\right) & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \varepsilon\Omega$ and Ω is a periodic domain of $\mathbf{R}^N$. In homogenization theory, one of important issues is the treatment of "domain with small holes" and many mathematicians studied this problem in linear cases ([A] and its references). We want to study the case of Hamilton-Jacobi equations via the viscosity solutions approach.

The study of homogenization based on viscosity solutions was initiated by [LPV] and then developed by [E1] and [E2]. In those papers, they considered equations of the following type

$$(1) \qquad u^\varepsilon(x) + |Du^\varepsilon(x)|^2 = V\left(\frac{x}{\varepsilon}\right) \quad \text{in } \mathbf{R}^N,$$

and examined the asymptotics of u^ε as $\varepsilon \to 0$ (or the corresponding evolution equations).

Thanks to [LPV], [E1] and [E2], we have a rather good comprehension of homogenization of (1) in the case $N = 1$. Assume that $V \in C(\mathbf{R})$, $V(y+1) = V(y)$ and $\min_{\mathbf{R}} V = 0$. Then, according to [LPV], the solution of (1) converges uniformly on $\mathbf{R}$ to the solution of the PDE

$$u(x) + \overline{H}(Du(x)) = 0 \quad \text{in } \mathbf{R}. \tag{2}$$

Here $\overline{H}$ is the function on $\mathbf{R}$ defined by

$$\overline{H}(p) = \begin{cases} 0 \ \text{ if } |p| \le \int_0^1 V^{\frac{1}{2}}(y)dy, \\ \lambda(p) \ge 0 \text{ is a solution of } |p| = \int_0^1 (V(y)+\lambda)^{\frac{1}{2}} dy \ \text{ if } |p| \ge \int_0^1 V^{\frac{1}{2}}(y)dy. \end{cases}$$

In this example $\overline{H}$ is given explicitly, but in general situations it is determined through the "cell problem" (see [LPV], [E1] or [E2]).

Through this paper, we will deal only with viscosity solutions and omit giving here the definitions (for example, see [CIL]).

1. Main results.

We give the list of assumptions of Ω, H, B and b.

(Ω1) Ω is a (connected) domain of $\mathbf{R}^N$.

(Ω2) $\partial\Omega \in C^1$.

(Ω3) $\Omega + e_i = \Omega$ for all $1 \le i \le N$, where $\{e_1, \cdots, e_N\}$ denote the standard basis of $\mathbf{R}^N$.

(H1) For each $R > 0$,

$$H \in BUC(B(0,R) \times [-R,R] \times \mathbf{R}^N \times \overline{\Omega}),$$

where $B(z,R)$ denotes the closed ball with radius R and center z.

(H2) $H(p,u,x,y+e_i)=H(p,u,x,y)$ for all $1\le i\le N$.

(H3) For some $\gamma>0$, the function $u\mapsto H(p,u,x,y)-\gamma u$ is nondecreasing.

(H4) For each $u\in\mathbf{R}$,

$$\lim_{r\to\infty}\inf\{H(p,u,x,y)\mid |p|>r,\ x\in\mathbf{R}^N,\ y\in\overline{\Omega}\}=\infty.$$

(B1) For each $R>0$,

$$B\in BUC(B(0,R)\times[-R,R]\times\mathbf{R}^N\times\partial\Omega).$$

(B2) $B(p,u,x,y+e_i)=B(p,u,x,y)$ for all $1\le i\le N$.

(B3) The function $u\mapsto B(p,u,x,y)$ is nondecreasing.

(B4) For some $\nu>0$, the function $t\mapsto B(p+tn(y),u,x,y)-\nu t$ is nondecreasing, where $n(y)$ denotes the unit normal vector at $y\in\partial\Omega$ outward to Ω.

(b1) $b\in BUC(\mathbf{R}^N\times\partial\Omega)$.

(b2) $b(x,y+e_i)=b(x,y)$ for all $1\le i\le N$.

We state an existence result for solutions of $(\mathrm{N})_\varepsilon$ and some of their properties.

Proposition 1. *Assume that* (Ω1)-(Ω3), (H1)-(H4) *and* (B1)-(B4) *hold. Then,* $(\mathrm{N})_\varepsilon$ *has a unique bounded Lipschitz continuous solution* u^ε. *It satisfies*

$$\sup_{0<\varepsilon<1}\|u^\varepsilon\|_{C(\overline{\Omega}_\varepsilon)}<\infty \tag{1.1}$$

and

$$\sup_{0<\varepsilon<1}\|Du^\varepsilon\|_{L^\infty(\Omega_\varepsilon)}<\infty. \tag{1.2}$$

To determine the effective Hamiltonian associated with the problem $(\mathrm{N})_\varepsilon$, we consider the following "cell problem".

Proposition 2. *(Cell problem) Assume that* $(\Omega1)$-$(\Omega3)$, (H1)-(H4) *and* (B1)-(B4) *hold. Then, for each* $(p,u,x) \in \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N$, *there exists a unique number* $\lambda_1 \in \mathbf{R}$ *such that the problem*

$$\text{(CPN)} \qquad \begin{cases} H(p + D_y v_1(y), u, x, y) = \lambda_1 & \text{in } \Omega, \\ B(p + D_y v_1(y), u, x, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

has a bounded solution $v_1 \in C^{0,1}(\overline{\Omega})$.

We put $\lambda_1 = \tilde{H}(p,u,x)$ and call $\tilde{H}$ the effective Hamiltonian associated with the problem $(\mathrm{N})_\varepsilon$. This Hamiltonian $\tilde{H}$ satisfies the following properties.

Proposition 3. *Assume that* $(\Omega1)$-$(\Omega3)$, (H1)-(H4) *and* (B1)-(B4) *hold. Then:*
(1) *For each* $R > 0$,

$$\tilde{H} \in BUC(B(0,R) \times [-R,R] \times \mathbf{R}^N).$$

(2) *For some* $\gamma > 0$, *the function* $u \mapsto \tilde{H}(p,u,x,y) - \gamma u$ *is nondecreasing.*

Theorem 1. *The Hamilton-Jacobi equation*

$$\tilde{H}(Du(x), u(x), x) = 0 \quad \text{in } \mathbf{R}^N, \tag{1.3}$$

has a unique solution $u \in BUC(\mathbf{R}^N)$ *and*

$$\lim_{\varepsilon \searrow 0} \sup_{x \in \Omega_\varepsilon} |u^\varepsilon(x) - u(x)| = 0. \tag{1.4}$$

Proposition 4. *Assume that* (Ω1)-(Ω3), (H1)-(H4) *and* (B1)-(B4) *hold. Then,* $(\mathrm{D})_\varepsilon$ *has a unique bounded Lipschitz continuous solution* u^ε. *It satisfies*

$$\sup_{0<\varepsilon<1} \|u^\varepsilon\|_{C(\overline{\Omega}_\varepsilon)} < \infty \tag{1.5}$$

and

$$\sup_{0<\varepsilon<1} \|Du^\varepsilon\|_{L^\infty(\Omega_\varepsilon)} < \infty. \tag{1.6}$$

Proposition 5. *(Cell problem) Assume that* (Ω1)-(Ω3), (H1)-(H4) *and* (B1)-(B4) *hold. Then, for each* $(p,u,x) \in \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N$, *there exists a unique number* $\lambda_2 \in \mathbf{R}$ *such that the problem*

$$\text{(CPD)} \qquad \begin{cases} H(p + D_y v_2(y), u, x, y) \leq \lambda_2 & \text{in } \Omega, \\ H(p + D_y v_2(y), u, x, y) \geq \lambda_2 & \text{in } \overline{\Omega}, \end{cases}$$

has a bounded solution $v \in C^{0,1}(\overline{\Omega})$.

The problem (CPD) is of the *state-constraint* type. Problems of this type naturally arises in optimal control, where the dynamic programing equations have convex Hamiltonians H in the first variable p. Here, the interesting point is that the function H is not assumed to be convex in p.

We put $\lambda_2 = \overline{H}(p,u,x)$ and call $\overline{H}$ the effective Hamiltonian associated with the problem $(\mathrm{D})_\varepsilon$. The effective Hamiltonian $\overline{H}$ has the following properties.

Proposition 6. *Assume that* (Ω1)-(Ω3), (H1)-(H4) *and* (B1)-(B4) *hold. Then:*
(1) *For each* $R > 0$,

$$\overline{H} \in BUC(B(0,R) \times [-R,R] \times \mathbf{R}^N).$$

(2) *For some* $\gamma > 0$, *the function* $u \mapsto \overline{H}(p,u,x,y) - \gamma u$ *is nondecreasing.*

Theorem 2. *The Hamilton-Jacobi equation*

$$\max\{u(x) - \overline{b}(x), \overline{H}(Du(x), u(x), x)\} = 0 \quad in \ \mathbf{R}^N, \tag{1.7}$$

where $\overline{b}(x) = \min_{y \in \partial\Omega} b(x,y)$, *has a unique solution* $u \in BUC(\mathbf{R}^N)$ *and*

$$\lim_{\varepsilon \searrow 0} \sup_{x \in \Omega_\varepsilon} |u^\varepsilon(x) - u(x)| = 0. \tag{1.8}$$

2. Proof of main results.

We only sketch the proof in the case of the Dirichlet problem $(\mathrm{D})_\varepsilon$.

Proof of Proposition 4. Note that, by (H4), a bounded subsolution of $(\mathrm{D})_\varepsilon$ is Lipschitz continuous. Moreover, if (1.5) holds, then solutions satisfy (1.6). Noting that

$$u_1(x) = -A_1 \quad \text{and} \quad u_2(x) = A_1,$$

where $A_1 > 0$ is large enough are, respectively, a subsolution and a supersolution of $(\mathrm{D})_\varepsilon$. Then, using Perron's method and standard comparison arguments, we see that $(\mathrm{D})_\varepsilon$ has a unique bounded Lipschitz solution u^ε. Moreover, noting that the constant A_1 can be chosen independently of $\varepsilon > 0$, we conclude (1.5) and (1.6). ■

Outline of proof of Proposition 5. For $0 < \alpha < 1$, we consider the following approximate problem

$$(\mathrm{CP})_\alpha \qquad \begin{cases} \alpha w^\alpha(y) + H(p + D_y w^\alpha(y), u, x, y) \le 0 & \text{in } \Omega, \\ \alpha w^\alpha(y) + H(p + D_y w^\alpha(y), u, x, y) \ge 0 & \text{in } \overline{\Omega}. \end{cases}$$

Since

$$w_1(y) = -\frac{A_2}{\alpha} \quad \text{and} \quad w_2(y) = \frac{A_2}{\alpha}$$

are, respectively, a subsolution and a supersolution of $(\mathrm{CP})_\alpha$ if the constant A_2 is large enough, we get a unique Lipschitz solution w^α of $(\mathrm{CP})_\alpha$ by Perron's method for each $0 < \alpha < 1$. Morerover, by (H2), we have $w^\alpha(y + e_i) = w^\alpha(y)$ for all $1 \le i \le N$.

It follows from the construction of the solution that

$$\sup_{\alpha} \|\alpha w^{\alpha}\|_{C(\overline{\Omega})} < \infty.$$

By using this inequality, we obtain

$$\sup_{\alpha} \|Dw^{\alpha}\|_{L^{\infty}(\Omega)} < \infty.$$

We put $v^{\alpha}(y) = w^{\alpha}(y) - \min w^{\alpha}$. Then we have

$$\sup_{\alpha} \|v^{\alpha}\|_{C^{0,1}(\overline{\Omega})} < \infty.$$

Therefore,

$$v^{\alpha} \longrightarrow v_2 \quad \text{and} \quad \alpha w^{\alpha} \longrightarrow -\lambda_2 \quad \text{uniformly}$$

along a sequence as $\alpha \to 0$, for some $v \in C^{0,1}(\overline{\Omega})$ and $\lambda_2 \in \mathbf{R}$. This way we get a solution (v_2, λ_2). We omit giving the proof of the uniqueness of λ (see [E2]). ∎

We omit giving the proof of Proposition 6 (see [I4] or [HI]). Next, we will prove Theorem 2, where we use both the perturbed test function method (see [E1] and [E2]) and the test function used in the proof of comparison results (see [I2]).

Proof of Theorem 2. We put

$$\overline{u}(x) = \lim_{\varepsilon \to 0} \sup\{u^{\delta}(y) \mid |x-y| \le \varepsilon,\ 0 < \delta < \varepsilon\}$$

and

$$\underline{u}(x) = \lim_{\varepsilon \to 0} \inf\{u^{\delta}(y) \mid |x-y| \le \varepsilon,\ 0 < \delta < \varepsilon\}$$

for $x \in \mathbf{R}^N$. We will show that $\overline{u}$ and $\underline{u}$ are, respectively, a subsolution and a supersolution of (1.7).

Let $\varphi \in C^{1,1}(\mathbf{R}^N)$ and $\hat{x}$ be a maximum point of $\overline{u} - \varphi$. We may assume that $\lim_{|x|\to\infty} \varphi(x) = \infty$ and that $\overline{u} - \varphi$ attains a strict maximum at $\hat{x} \in \mathbf{R}^N$. Let $v \in C^{0,1}(\overline{\Omega})$ be a solution of Proposition 5 with $\lambda_2 = \overline{H}(D\varphi(\hat{x}), \overline{u}(\hat{x}), \hat{x})$. Then, we can find maximum points $x^\varepsilon \in \overline{\Omega}_\varepsilon$ of $u^\varepsilon(x) - \varphi(x) - \varepsilon v\left(\frac{x}{\varepsilon}\right)$ satisfying $x^\varepsilon \to \hat{x}$ as $\varepsilon \to 0$. We are concerned with the case $x^\varepsilon \in \partial\Omega_\varepsilon$; the other case can be argued similarily and more easily.

By (Ω2), there exist $\eta = \eta(x^\varepsilon) \in \mathbf{R}^N$ and $b > 0$ such that $B(x^\varepsilon + t\eta, tb) \subset \overline{\Omega}_\varepsilon$ for all $0 \le t < b$. For $\alpha > 0$, we put

$$\Phi(x,y) = u^\varepsilon(x) - \varphi(x) - \varepsilon v\left(\frac{y}{\varepsilon}\right) - \left|\frac{x-y}{\alpha} - \eta\right|^2 - |y - x^\varepsilon|^2$$

on $\overline{\Omega}_\varepsilon \times \overline{\Omega}_\varepsilon$. Let $(x^\varepsilon_\alpha, y^\varepsilon_\alpha) \in \overline{\Omega}_\varepsilon \times \overline{\Omega}_\varepsilon$ be a maximum point of Φ. Then $x^\varepsilon_\alpha, y^\varepsilon_\alpha \to x^\varepsilon$ as $\alpha \to 0$. Since $\Phi(y^\varepsilon_\alpha + \alpha\eta, y^\varepsilon_\alpha) \le \Phi(x^\varepsilon_\alpha, y^\varepsilon_\alpha)$, we have

$$\left|\frac{x^\varepsilon_\alpha - y^\varepsilon_\alpha}{\alpha} - \eta\right| \le C\alpha$$

for some $C > 0$ independent of $\varepsilon > 0$. Moreover, we may assume that $x^\varepsilon_\alpha \in \Omega_\varepsilon$.

Since u^ε is a solution of $(\mathrm{D})_\varepsilon$, we obtain

$$H\left(D\varphi(x^\varepsilon_\alpha) + \frac{2}{\alpha}\left(\frac{x^\varepsilon_\alpha - y^\varepsilon_\alpha}{\alpha} - \eta\right), u^\varepsilon(x^\varepsilon_\alpha), x^\varepsilon_\alpha, \frac{x^\varepsilon_\alpha}{\varepsilon}\right) \le 0$$

and

$$H\left(D\varphi(\hat{x}) + \frac{2}{\alpha}\left(\frac{x^\varepsilon_\alpha - y^\varepsilon_\alpha}{\alpha} - \eta\right) - 2(y^\varepsilon_\alpha - x^\varepsilon), \overline{u}(\hat{x}), \hat{x}, \frac{y^\varepsilon_\alpha}{\varepsilon}\right) \ge \overline{H}(D\varphi(\hat{x}), \overline{u}(\hat{x}), \hat{x}).$$

Sending $\alpha \to 0$ first and $\varepsilon \to 0$, we get

$$\overline{H}(D\varphi(\hat{x}), \overline{u}(\hat{x}), \hat{x}) \le 0.$$

Now, we show that $\overline{u}(x) \le \overline{b}(x)$. If there exists $\tilde{x} \in \mathbf{R}^N$ such that $\overline{u}(\tilde{x}) > \overline{b}(\tilde{x})$, then we can show that there exist $\varepsilon > 0$ and $\tilde{x}_\varepsilon \in \partial\Omega_\varepsilon$ such that $u^\varepsilon(\tilde{x}_\varepsilon) > b\left(\tilde{x}_\varepsilon, \frac{\tilde{x}_\varepsilon}{\varepsilon}\right)$. Let $r > 0$, $A > 0$ and x_A be a maximum point of $u^\varepsilon(x) - A|x - \tilde{x}_\varepsilon - rn(\tilde{x}_\varepsilon)|$. Since $x_A \to \tilde{x}_\varepsilon$ as $r = \frac{1}{A}$ and $A \to \infty$ and

$$H\left(A\frac{x_A - \tilde{x}_\varepsilon - rn(\tilde{x}_\varepsilon)}{|x_A - \tilde{x}_\varepsilon - rn(\tilde{x}_\varepsilon)|}, u^\varepsilon(x_A), x_A, \frac{x_A}{\varepsilon}\right) > 0$$

for $A > 0$ large enough by (H4), we have $x_A \in \partial\Omega_\varepsilon$ and $u^\varepsilon(x_A) \le b\left(x_A, \frac{x_A}{\varepsilon}\right)$. Therefore, sending $A \to \infty$, we get a contradiction. Thus we have proved that $\overline{u}$ is a subsolution of (1.7).

Similarly, we can prove that $\underline{u}$ is a supersolution of (1.7). By comparison arguments, we have $\overline{u} = \underline{u}$ and conclude the proof. ■

References

[A] H. Attouch, *Variational Convergence for Functions and Operators* (New York: Pitman, 1984).

[BL] G. Barles and P.-L. Lions, Fully nonlinear Neumann type boundary conditions for first-order Hamilton-Jacobi equations, Nonlinear Anal. Theory Methods Appl. **16** (1991), 143–153.

[BP] G. Barles and B. Perthame, Discontinuous solutions of deterministic optimal stopping- time problems, Modèl. Math. Anal. Num. **21** (1987), 557–579.

[BLP] A. Bensoussan, J. L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures* (Amsterdam: Noth Holland, 1978).

[C1] M. C. Concordel, Periodic homogenization of Hamilton-Jacobi equations: 1. Additive eigenvalues and variational formula, preprint.

[C2] M. C. Concordel, Periodic homogenization of Hamilton-Jacobi equations: 2. Eikonal equations, preprint.

[CL] M. G. Crandall, and P.-L. Lions, Viscosity solutions of Hamilton-Jacobi equations, Trans. Amer. Math. Soc., **277** (1983), 1–42.

[CIL] M. G. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. **27** (1992), 1–67.

[DI] P. Dupuis and H. Ishii, On oblique derivative problems for fully nonlinear second order elliptic equations on nonsmooth domains, Nonlinear Anal. Theory Methods Appl. **12** (1991), 1123–1138.

[E1] L. C. Evans, The perturbed test function technique for vicosity solutions of partial differential equations, Proc. Roy. Soc. Edinburgh Sect. A, **111** (1989), 359–375.

[E2] L. C. Evans, Periodic homogenisation of fully nonlinear partial differential equations, Proc. Roy. Soc. Edinburgh Sect. A, **120** (1992), 245–265.

[HI] K. Horie and H. Ishii, Homogenization of Hamilton-Jacobi equations on domains with small scale periodic structure, preprint.

[I1] H. Ishii, Perron's method for Hamilton-Jacobi equations, Duke Math. J., **55** (1987), 369–384.

[I2] H. Ishii, A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations, Ann. Scuola Norm. Sup. Pisa **16** (1989), 105–135.

[I3] H. Ishii, Fully nonlinear oblique derivative problems for nonlinear second-order elliptic PDE's, Duke Math. J., **62** (1991), 633–661.

[I4] 石井仁司，粘性解とその応用，数学，46 (1995), 97–110.

[L] P.-L. Lions, Neumann type boundary conditions for Hamilton-Jacobi equations, Duke Math. J., **52** (1985), 793–820.

[LVP] P.-L. Lions, G. Papanicolaou and S. R. S. Varadhan, Homogenization of Hamilton-Jacobi equations, Preprint.

[S] H. M. Soner, Optimal control with state-space constraint **I**, SIAM J. Control Optim. **24** (1986), 552–562.