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Sufficient Condition for Multivalently Starlikeness

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Abstract

It is the purpose of the present paper to obtain a sufficient condition for multivalently starlikeness.

1 Introduction.

Let $A(p)$ be the class of functions of the form

$$f(z)=z^{p}+\sum_{n=p+1}^{\infty}a_{n}z^{n}, \qquad (p\in N=1,2,3,\ldots)$$

which are analytic in $U=\{z:|z|<1\}$.

A function $f(z)\in A(p)$ is said to be p-valently starlike if and only if

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>0 \qquad in \quad U.$$

In [4], R.Singh and S. Singh proved the following result.

Theorem A. If $f(z)\in A(1)$ satisfies

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)}<\frac{3}{2} \qquad in \quad U,$$

then $f(z)$ is starlike in U or

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>0 \quad in \quad U.$$

Nunokawa [1] and Owa [3] generalized Theorem A independently.
Owa [3] proved the following result.

Theorem B. If $f(z) \in A(p)$ satisfies

$$1 + \operatorname{Re}\frac{zf''(z)}{f'(z)} < p + \frac{1}{2} \quad in \quad U,$$

then $f(z)$ is p-valently starlike in U and

$$0 < \operatorname{Re}\frac{zf'(z)}{f(z)} < \frac{2p(p+1)}{2p+1} \quad in \quad U.$$

2 Main result.

Theorem. Let $f(z) \in A(p)$ and suppose that

$$(1) \qquad 1 + \operatorname{Re}\frac{zf''(z)}{f'(z)} < \alpha \quad in \quad U$$

where $p < \alpha < p + \frac{1}{2}$.

Then we have

$$\operatorname{Re}\frac{f(z)}{zf'(z)} > \frac{2\alpha}{p(2\alpha+1)} \quad in \quad U$$

and

$$0 < \operatorname{Re}\frac{zf'(z)}{f(z)} < \frac{p(2\alpha+1)}{2\alpha} \quad in \quad U.$$

Proof. Let us put

$$(2) \qquad p(z) = p\frac{1+\beta}{q(z)+\beta},$$

where $p(z)$ is analytic in U, $p(0) = p$, $\beta = 2\alpha$, and $2p < \beta < 2p+1$, then we have $q(0) = 1$.

Then it follows that

$$p(z) + \frac{zp'(z)}{p(z)} = p\frac{1+\beta}{q(z)+\beta} - \frac{zq'(z)}{q(z)+\beta}.$$

If there exists a point $z_0 \in U$ such that

(3) $$\mathrm{Re}q(z) > 0 \quad for \quad |z| < |z_0|, \quad \mathrm{Re}q(z_0) = 0 \quad and \quad q(z_0) = ia,$$

then from [2, p.152], we have

$$-z_0 q'(z_0) \geq \frac{1}{2}(1+a^2).$$

Therefore we have

$$\begin{aligned}\mathrm{Re}(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}) &= \mathrm{Re}(p\frac{1+\beta}{ia+\beta} - \frac{z_0 q'(z_0)}{ia+\beta}) \\ &\geq \frac{p\beta(1+\beta)}{\beta^2+a^2} + \frac{1}{2}(1+a^2)\frac{\beta}{\beta^2+a^2} \\ &= \frac{\beta}{2(\beta^2+a^2)}\left\{2p(1+\beta)+(1+a^2)\right\}.\end{aligned}$$

Putting

$$g(x) = \frac{\beta}{2(\beta^2+x^2)}\left\{2p(1+\beta)+(1+x^2)\right\} \quad for \quad -\infty < x < \infty,$$

then it follows that

$$g'(x) = \frac{\beta x}{(\beta^2+x^2)^2}(\beta^2 - 2\beta p - 2p - 1).$$

This shows that $g(x)$ takes its minimum at $x = \infty$ and $x = -\infty$ and so

(4) $$g(x) \geq \frac{\beta}{2} = \alpha \quad for \quad -\infty < x < \infty.$$

On the other hand, putting

$$p(z) = \frac{zf'(z)}{f(z)},$$

then it follows that

(5) $$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}.$$

From (2), (3), (4) and (5), this contradicts (1).
Therefore we must have

$$\mathrm{Re}q(z) > 0 \quad in \quad U.$$

Then we easily have

$$\mathrm{Re}\frac{f(z)}{zf'(z)} > \frac{2\alpha}{p(2\alpha+1)} \quad in \quad U$$

and

$$0 < \mathrm{Re}\frac{zf'(z)}{f(z)} < \frac{p(2\alpha+1)}{2\alpha} \quad in \quad U$$

where $p < \alpha < p + \frac{1}{2}$.

Remark. Putting $\alpha = p + \frac{1}{2}$ in the Theorem, we have Theorem B.

References

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