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A Microscopic Derivation of Quantum Stochastic Differential Equations for A Non-Linear Damped Oscillator

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1 Introduction

For a dissipative non-linear quantum system, the effect of the non-linearity on its relaxation was considered [1]-[5] in deriving a quantum master equation for the system within the damping theory [6, 7].

Let us consider the system of a non-linear damped oscillator where the Hamiltonian of the relevant system is given by

$$H_S = \omega a^\dagger a + \frac{1}{2} g a^\dagger a^\dagger a a, \tag{1}$$

where a and $a^\dagger$ are boson operators satisfying the commutation relations

$$[a,\ a^\dagger] = 1, \quad [a,\ a] = 0. \tag{2}$$

In the *non-conventional* treatment [1]-[5] of the damping theory, the effect of the non-linearity within a relevant system on its relaxation is taken into account, which ensures that the density operator of the relevant system, $\rho_S(t)$, leads to the true final equilibrium state, i.e. $\rho_S(t) \to \mathrm{e}^{-\beta H_S}$ as $t \to \infty$. In the *conventional* treatment of the damping theory which ignores the effect of the non-linearity within a relevant system on its relaxation, $\rho_S(t)$ converges to the equilibrium state for a damped harmonic oscillator, i.e. $\rho_S(t) \to \mathrm{e}^{-\beta\omega a^\dagger a}$ as $t \to \infty$. This shows that the effect of the non-linearity within a relevant system on its relaxation plays the important role for its long time behavior. Haake et al. [5] derived the master equation for the non-linear damped oscillator in the non-conventional treatment within the damping theory.

Within the framework of Non-Equilibrium Thermo Field Dynamics (NETFD) [8]-[12], a unified canonical operator formalism of quantum stochastic differential equations was constructed including the quantum Langevin equation and the quantum stochastic Liouville equation [10]-[24]. Within this formalism, quantum stochastic differential equations for a non-linear damped oscillator are constructed [20].

Accardi et al. [25]-[29] gave a microscopic foundation to quantum stochastic processes. They considered a quantum system interacting with thermal reservoir which consists of boson fields. Then, they showed that, in the weak coupling limit (the van Hove limit) [30], suitably chosen boson fields of reservoir, called collective boson fields, converge to the quantum Wiener processes and the time-evolution equation of a wave function in the interaction representation to a quantum stochastic differential equation where the infinitesimal time-evolution generator contains the increments of the quantum Wiener processes.

In this paper, we will apply the procedure of Accardi et al. to a non-linear damped oscillator within the formalism of NETFD, and give a microscopic foundation of quantum stochastic differential equations for a non-linear damped oscillator where the effect of the non-linearity within a relevant system on its relaxation is taken into account.

2 Microscopic Model

We consider a non-linear oscillator interacting with a reservoir which is described by the following Hamiltonian

$$H = H_0 + H_1, \tag{3}$$

where

$$H_0 = H_S + H_R, \tag{4}$$

and

$$H_1 = i\lambda \sum_k (a^\dagger b_k - b_k^\dagger a). \tag{5}$$

Here, H_S is given by (1) and

$$H_R = \sum_k \epsilon_k b_k^\dagger b_k. \tag{6}$$

The operators a, $a^\dagger$ and b_k, $b_k^\dagger$ are boson operators satisfying the commutation relations (2) and

$$[b_k,\ b_l^\dagger] = \delta_{kl}, \quad [b_k,\ b_l] = 0. \tag{7}$$

We introduce operators with *tilde*, $\tilde{a}$, $\tilde{a}^\dagger$, $\tilde{b}_k$, $\tilde{b}_k^\dagger$. The *tilde conjugation* $\sim$ is defined by

$$(A_1 A_2)^\sim = \tilde{A}_1 \tilde{A}_2, \quad (c_1 A_1 + c_2 A_2)^\sim = c_1^* \tilde{A}_1 + c_2^* \tilde{A}_2, \tag{8}$$

$$(\tilde{A})^\sim = A, \quad (A^\dagger)^\sim = \tilde{A}^\dagger, \tag{9}$$

where A_1, A_2 and A are arbitrary operators and c_1 and c_2 are c-numbers. The representation space of $(a, a^\dagger, \tilde{a}, \tilde{a}^\dagger)$ will be denoted by $\mathcal{H}_S$, while that of $(b_k, b_k^\dagger, \tilde{b}_k, \tilde{b}_k^\dagger)$ by Γ_R.

Thermal vacuums $|0_R\rangle$ and $\langle 1_R|$ in Γ_R are characterized by $\langle 1_R|b_k^\dagger b_l|0_R\rangle = \bar{n}_k \delta_{kl}$ with the Planck distribution $\bar{n}_k = \frac{1}{e^{\epsilon_k/T}-1}$. The annihilation operators $(c_k, \tilde{c}_k)$ and creation operators $(c_k^{\natural}, \tilde{c}_k^{\natural})$ on Γ_R satisfying the relations

$$c_k|0_R\rangle = \tilde{c}_k|0_R\rangle = 0, \quad \langle 1_R|c_k^{\natural} = \langle 1_R|\tilde{c}_k^{\natural} = 0, \tag{10}$$

and the canonical commutation relations

$$[c_k,\ c_l^{\natural}] = [\tilde{c}_k,\ \tilde{c}_l^{\natural}] = \delta_{kl}, \tag{11}$$

are introduced by the *Bogoliubov transformation*

$$\begin{pmatrix} c_k \\ \tilde{c}_k^{\natural} \end{pmatrix} = \begin{pmatrix} \bar{n}_k + 1 & -\bar{n}_k \\ -1 & 1 \end{pmatrix} \begin{pmatrix} b_k \\ \tilde{b}_k^\dagger \end{pmatrix}. \tag{12}$$

The space Γ_R is spanned by the basic vectors introduced by cyclic operations of $(c_k^{\dagger}, \tilde{c}_k^{\dagger})$ on $|0_R\rangle$ and $(c_k, \tilde{c}_k)$ on $\langle 1_R|$.

We introduce the time-evolution generator $\hat{U}_\lambda(t)$ in the interaction picture defined by

$$\hat{U}_\lambda(t) = \mathrm{e}^{i\hat{H}_0 t}\mathrm{e}^{-i\hat{H}t}, \tag{13}$$

where

$$\hat{H} = H - \tilde{H}, \quad \hat{H}_0 = H_0 - \tilde{H}_0. \tag{14}$$

The generator $\hat{U}_\lambda(t)$ is the operator acting on the thermal space $\mathcal{H}_S \otimes \Gamma_R$. The time-evolution equation of $\hat{U}_\lambda(t)$ is given by

$$\frac{\partial}{\partial t}\hat{U}_\lambda(t) = -i\hat{H}_1^I(t)\hat{U}_\lambda(t), \tag{15}$$

with

$$\begin{aligned}\hat{H}_1^I(t) &= \mathrm{e}^{i\hat{H}_0 t}(H_1 - \tilde{H}_1)\mathrm{e}^{-i\hat{H}_0 t} \\ &= i\lambda\sum_k\left\{a^\dagger b_k \mathrm{e}^{-i[\epsilon_k-(\omega+ga^\dagger a)]t} - b_k^\dagger \mathrm{e}^{i[\epsilon_k-(\omega+ga^\dagger a)]t}a\right\} - \text{t.c.},\end{aligned} \tag{16}$$

where t.c. indicates tilde conjugates of the previous term.

We introduce vacuum states $|0,\tilde{0}\rangle$ and $\langle 0,\tilde{0}|$ by

$$a|0,\tilde{0}\rangle = \tilde{a}|0,\tilde{0}\rangle = 0, \quad \langle 0,\tilde{0}|a^\dagger = \langle 0,\tilde{0}|\tilde{a}^\dagger = 0, \tag{17}$$

and define ket- and bra-vectors

$$|m,\tilde{n}\rangle = \frac{(a^\dagger)^m}{\sqrt{m!}}\frac{(\tilde{a}^\dagger)^n}{\sqrt{n!}}|0,\tilde{0}\rangle, \quad \langle m,\tilde{n}| = \langle 0,\tilde{0}|\frac{a^m}{\sqrt{m!}}\frac{(\tilde{a})^n}{\sqrt{n!}}, \tag{18}$$

which satisfy the orthonormalization condition

$$\langle m,\tilde{n}|m',\tilde{n}'\rangle = \delta_{mm'}\delta_{nn'}, \tag{19}$$

and the completeness relation

$$\sum_{mn}|m,\tilde{n}\rangle\langle m,\tilde{n}| = I. \tag{20}$$

The representation space $\mathcal{H}_S$ can be spanned by the basic vectors $|m,\tilde{n}\rangle$ and $\langle m,\tilde{n}|$.

Using the vectors $|m,\tilde{n}\rangle$ and $\langle m,\tilde{n}|$, $\hat{H}_1^I(t)$ can be expressed as

$$\begin{aligned}\hat{H}_1^I(t) &= i\lambda\sum_{mn}\sum_k\Big\{\sqrt{m+1}|m+1,\tilde{n}\rangle\langle m,\tilde{n}|b_k\mathrm{e}^{-i(\epsilon_k-\phi_m)t} \\ &\quad -b_k^\dagger e^{i(\epsilon_k-\phi_m)t}\sqrt{m+1}|m,\tilde{n}\rangle\langle m+1,\tilde{n}|\Big\} - \text{t.c.},\end{aligned} \tag{21}$$

where we defined ϕ_n by $\phi_n = \omega + gn$. Note that $|m,\tilde{n}\rangle^{\sim} = |n,\tilde{m}\rangle$ and $\langle m,\tilde{n}|^{\sim} = \langle n,\tilde{m}|$.

3 Evaluation of $\hat{U}_\lambda(t)$

We introduce exponential vectors in Γ_R defined by

$$|e(z,w)\rangle_R = \exp\left[\sum_k z_k c_k^\dagger + w_k^* \tilde{c}_k^\dagger\right] |0_R\rangle, \quad {}_R\langle e(z,w)| = \langle 1_R| \exp\left[\sum_k z_k^* c_k + w_k \tilde{c}_k\right], \tag{22}$$

where z_k, w_k are c-numbers. The exponential vectors have the properties that the actions of c_k, $c_k^\dagger$ and their tilde conjugates on them are as follows:

$$c_k|e(z,w)\rangle_R = z_k|e(z,w)\rangle_R, \quad \tilde{c}_k|e(z,w)\rangle_R = w_k^*|e(z,w)\rangle_R, \tag{23}$$

$${}_R\langle e(z,w)|c_k^\dagger = {}_R\langle e(z,w)|z_k^*, \quad {}_R\langle e(z,w)|\tilde{c}_k^\dagger = {}_R\langle e(z,w)|w_k, \tag{24}$$

which indicates that the exponential vectors are the coherent states. Let us introduce the exponential vectors (22) with z_k and w_k replaced with

$$z_k = \lambda \sum_n \int_{S_n/\lambda^2}^{T_n/\lambda^2} du\ z_{nk} e^{i(\epsilon_k - \phi_n)u}, \tag{25}$$

and

$$w_k = \lambda \sum_n \int_{S_n/\lambda^2}^{T_n/\lambda^2} du\ w_{nk} e^{i(\epsilon_k - \phi_n)u}, \tag{26}$$

respectively, and denote them by $|e_\lambda(z,w)\rangle_R$ and ${}_R\langle e_\lambda(z,w)|$. Here, z_{nk} and w_{nk} are c-numbers. The exponential vectors $|e_\lambda(z,w)\rangle_R$ and ${}_R\langle e_\lambda(z,w)|$ are called the *collective exponential vectors*.

Let $\hat{K}_\lambda(t)$ be defined by

$$\hat{K}_\lambda(t) = {}_R\langle e_\lambda(z_1,w_1)|\hat{U}_\lambda(t/\lambda^2)|e_\lambda(z_2,w_2)\rangle_R. \tag{27}$$

Using (15), we see that the equation of motion of $\hat{K}_\lambda(t)$ is given by

$$\frac{d}{dt}\hat{K}_\lambda(t) = \frac{1}{\lambda^2}\frac{d}{d(t/\lambda^2)}\hat{K}_\lambda(t) = {}_R\langle e_\lambda(z_1,w_1)|\frac{-i}{\lambda^2}\hat{H}_1^I(t/\lambda^2)\hat{U}_\lambda(t/\lambda^2)|e_\lambda(z_2,w_2)\rangle_R. \tag{28}$$

Substituting (21) with $(b_k, b_k^\dagger, \tilde{b}_k, \tilde{b}_k^\dagger)$ expressed by $(c_k, c_k^\dagger, \tilde{c}_k, \tilde{c}_k^\dagger)$ into (28), we have

$$\frac{d}{dt}\hat{K}_\lambda(t) = \hat{I}_\lambda + \hat{II}_\lambda, \tag{29}$$

where

$$\begin{aligned}\hat{I}_\lambda = \frac{1}{\lambda}{}_R\langle e_\lambda(z_1,w_1)| \sum_{mn}\sum_k \Big\{&\sqrt{m+1}|m+1,\tilde{n}\rangle\langle m,\tilde{n}|\bar{n}_k \tilde{c}_k^\dagger e^{-i(\epsilon_k-\phi_m)t/\lambda^2} \\ &-(\bar{n}_k+1)c_k^\dagger e^{i(\epsilon_k-\phi_m)t/\lambda^2}\sqrt{m+1}|m,\tilde{n}\rangle\langle m+1,\tilde{n}| + \text{t.c.}\Big\}\hat{U}_\lambda(t/\lambda^2)|e_\lambda(z_2,w_2)\rangle_R,\end{aligned} \tag{30}$$

and

$$\begin{aligned}\hat{II}_\lambda = \frac{1}{\lambda}{}_R\langle e_\lambda(z_1,w_1)| \sum_{mn}\sum_k \Big\{&\sqrt{m+1}|m+1,\tilde{n}\rangle\langle m,\tilde{n}|c_k e^{-i(\epsilon_k-\phi_m)t/\lambda^2} \\ &-\tilde{c}_k e^{i(\epsilon_k-\phi_m)t/\lambda^2}\sqrt{m+1}|m,\tilde{n}\rangle\langle m+1,\tilde{n}| + \text{t.c.}\Big\}\hat{U}_\lambda(t/\lambda^2)|e_\lambda(z_2,w_2)\rangle_R.\end{aligned} \tag{31}$$

Making use of (23) and (24) together with the relations

$$c_k\hat{U}_\lambda(t/\lambda^2) = \hat{U}_\lambda(t/\lambda^2)c_k + [c_k,\ \hat{U}_\lambda(t/\lambda^2)], \tag{32}$$

$$\tilde{c}_k\hat{U}_\lambda(t/\lambda^2) = \hat{U}_\lambda(t/\lambda^2)\tilde{c}_k + [\tilde{c}_k,\ \hat{U}_\lambda(t/\lambda^2)], \tag{33}$$

we evaluate the limits of $\hat{I}_\lambda$ and $\hat{II}_\lambda$ as $\lambda \to 0$, which gives

$$\begin{aligned}
\frac{d}{dt}\hat{K}(t) &= \lim_{\lambda\to 0}\frac{d}{dt}\hat{K}_\lambda(t)\left(\hat{I}_\lambda + \hat{II}_\lambda\right) \\
&= -i\sum_{mn}\Big\{i\sqrt{m+1}|m+1,\tilde{n}\rangle\langle m,\tilde{n}|2\kappa(\phi_m)\bar{n}(\phi_m)w_{1m}(\phi_m)\chi_{[S'_{1m},T'_{1m}]}(t) \\
&\quad -i2\kappa(\phi_m)[\bar{n}(\phi_m)+1]z^*_{1m}(\phi_m)\chi_{[S_{1m},T_{1m}]}(t)\sqrt{m+1}|m,\tilde{n}\rangle\langle m+1,\tilde{n}| \\
&\quad +i\sqrt{n+1}|m,\widetilde{n+1}\rangle\langle m,\tilde{n}|2\kappa(\phi_n)\bar{n}(\phi_n)z^*_{1n}(\phi_n)\chi_{[S_{1n},T_{1n}]}(t) \\
&\quad -i2\kappa(\phi_n)[\bar{n}(\phi_m)+1]w_{1n}(\phi_n)\chi_{[S'_{1n},T'_{1n}]}(t)\sqrt{n+1}|m,\tilde{n}\rangle\langle m,\widetilde{n+1}|\Big\}\hat{K}(t) \\
&\quad -i\sum_{mn}\Big\{i\sqrt{m+1}|m+1,\tilde{n}\rangle\langle m,\tilde{n}|2\kappa(\phi_m)z_{2m}(\phi_m)\chi_{[S_{2m},T_{2m}]}(t) \\
&\quad -i\sqrt{m+1}|m,\tilde{n}\rangle\langle m+1,\tilde{n}|2\kappa(\phi_m)w^*_{2m}(\phi_m)\chi_{[S'_{2m},T'_{2m}]}(t) \\
&\quad +i\sqrt{n+1}|m,\widetilde{n+1}\rangle\langle m,\tilde{n}|2\kappa(\phi_n)w^*_{2n}(\phi_n)\chi_{[S'_{2n},T'_{2n}]}(t) \\
&\quad -i\sqrt{n+1}|m,\tilde{n}\rangle\langle m,\widetilde{n+1}|2\kappa(\phi_n)z_{2n}(\phi_n)\chi_{[S_{2n},T_{2n}]}(t)\Big\}\hat{K}(t) \\
&\quad -i\left(\hat{\Delta}+i\hat{\Pi}\right)\hat{K}(t),
\end{aligned} \tag{34}$$

where $\hat{K}(t) = \lim_{\lambda\to 0}\hat{K}_\lambda(t)$.* Here, $\chi_{[S,T]}(t) = \theta(t-S)\theta(T-t)$, with the step function $\theta(t)$ defined by

$$\theta(t) = \begin{cases} 1, & \text{for } t \geq 0, \\ 0, & \text{for } t \leq 0, \end{cases} \tag{35}$$

and we introduced the operators $\hat{\Delta}$ and $\hat{\Pi}$ as

$$\begin{aligned}
\hat{\Delta} = \mathcal{P}\int d\epsilon\ \sum_{mn}\Big\{&[\bar{n}(\epsilon)+1]\frac{\rho(\epsilon)}{\phi_m-\epsilon}(m+1)|m+1,\tilde{n}\rangle\langle m+1,\tilde{n}| \\
&-(m+1)|m,\tilde{n}\rangle\langle m,\tilde{n}|\bar{n}(\epsilon)\frac{\rho(\epsilon)}{\phi_m-\epsilon} - \text{t.c.}\Big\},
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
\hat{\Pi} = &-\sum_{mn}\{\kappa(\phi_m)[\bar{n}(\phi_m)+1](m+1)|m+1,\tilde{n}\rangle\langle m+1,\tilde{n}| \\
&+(m+1)|m,\tilde{n}\rangle\langle m,\tilde{n}|\kappa(\phi_m)\bar{n}(\phi_m) + \text{t.c.}\} \\
&+2\sum_{m}\Big\{(m+1)|m+1,\widetilde{m+1}\rangle\langle m,\tilde{m}|\kappa(\phi_m)\bar{n}(\phi_m) \\
&+\kappa(\phi_m)[\bar{n}(\phi_m)+1](m+1)|m,\tilde{m}\rangle\langle m+1,\widetilde{m+1}|\Big\}.
\end{aligned} \tag{37}$$

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*In this paper, we assume the convergence of $\hat{K}_\lambda(t)$ as $\lambda \to 0$.

In deriving (34), we changed the summation with respect to k to the integral with respect to ϵ with a density of states $\rho(\epsilon)$ defined by

$$\sum_k \delta(\epsilon - \epsilon_k) = \rho(\epsilon), \tag{38}$$

and used the relation

$$\int_{-\infty}^{\infty} dv\ e^{\pm i(\epsilon - \phi_n)v} = 2\pi\delta(\epsilon - \phi_n). \tag{39}$$

4 Quantum Wiener Processes

In this section, we construct the quantum Wiener processes affected by the non-linearity within a relevant system.

We introduce boson operators $c_{t,k}(\phi_n)$, $c^{\dagger}_{t,k}(\phi_n)$ and their tilde conjugates satisfying the commutation relations

$$[c_{t,k}(\phi_n),\ c^{\dagger}_{t',k'}(\phi_{n'})] = 2\pi\delta(\epsilon_k - \phi_n)\delta(t-t')\delta_{kk'}\delta_{nn'}, \tag{40}$$

$$[\tilde{c}_{t,k}(\phi_n),\ \tilde{c}^{\dagger}_{t',k'}(\phi_{n'})] = 2\pi\delta(\epsilon_k - \phi_n)\delta(t-t')\delta_{k,k'}\delta_{nn'}, \tag{41}$$

and define the vacuums $|\rangle$ and $\langle|$ by

$$c_{t,k}(\phi_n)|\rangle = \tilde{c}_{t,k}(\phi_n)|\rangle = 0, \quad \langle|c^{\dagger}_{t,k}(\phi_n) = \langle|\tilde{c}^{\dagger}_{t,k}(\phi_n) = 0. \tag{42}$$

Let the Fock space built on the basic ket- and bra-vectors made by cyclic operations of $(c^{\dagger}_{t,k}(\phi_n), \tilde{c}^{\dagger}_{t,k}(\phi_n))$ on $|\rangle$ and of $(c_{t,k}(\phi_n), \tilde{c}_{t,k}(\phi_n))$ on $\langle|$ be denoted by Γ^{β}.

We introduce the exponential vectors defined by

$$|e(z,w)\rangle = \exp\left[\sum_n\sum_k\left\{\int_{S_n}^{T_n} du\ z_{nk}c^{\dagger}_{u,k}(\phi_n) + \int_{S'_n}^{T'_n} du\ w^*_{nk}\tilde{c}^{\dagger}_{u,k}(\phi_n)\right\}\right]|\rangle, \tag{43}$$

and

$$\langle e(z,w)| = \langle|\exp\left[\sum_n\sum_k\left\{\int_{S_n}^{T_n} du\ z^*_{nk}c_{u,k}(\phi_n) + \int_{S'_n}^{T'_n} du\ w_{nk}\tilde{c}_{u,k}(\phi_n)\right\}\right]. \tag{44}$$

Introducing the operators[†]

$$c_t(\phi_n) = \sum_k c_{t,k}(\phi_n), \quad c^{\dagger}_t(\phi_n) = \sum_k c^{\dagger}_{t,k}(\phi_n), \tag{45}$$

and their tilde conjugates, we have

$$c_t(\phi_n)|e(z,w)\rangle = \chi_{[S_n,T_n]}(t)2\kappa(\phi_n)z_n(\phi_n)|e(z,w)\rangle, \tag{46}$$

$$\tilde{c}_t(\phi_n)|e(z,w)\rangle = \chi_{[S'_n,T'_n]}(t)2\kappa(\phi_n)w^*_n(\phi_n)|e(z,w)\rangle, \tag{47}$$

$$\langle e(z,w)|c^{\dagger}_t(\phi_n) = \langle e(z,w)|\chi_{[S_n,T_n]}(t)2\kappa(\phi_n)z^*_n(\phi_n), \tag{48}$$

and

$$\langle e(z,w)|\tilde{c}^{\dagger}_t(\phi_n) = \langle e(z,w)|\chi_{[S'_n,T'_n]}(t)2\kappa(\phi_n)w_n(\phi_n). \tag{49}$$

[†]The operators $c_t(\phi_n)$, $c^{\dagger}_t(\phi_n)$ and their tilde conjugates correspond to the annihilation and creation operators $c(t)$, $c^{\dagger}(t)$ in the reference [31], which are regarded as quantum white noises.

From the commutation relations (40) and (41), we see that

$$[c_t(\phi_n),\ c^{♀}_{t'}(\phi_{n'})] = 2\kappa(\phi_n)\delta(t-t')\delta_{nn'}, \tag{50}$$

$$[\tilde{c}_t(\phi_n),\ \tilde{c}^{♀}_{t'}(\phi_{n'})] = 2\kappa(\phi_n)\delta(t-t')\delta_{nn'}. \tag{51}$$

Here, we changed the summation with respect to k to the integral with respect to ϵ with the density of states (38).

We introduce the quantum Wiener processes defined by

$$C_t(\phi_n) = \int_0^t ds\ c_s(\phi_n), \quad C^{♀}_t(\phi_n) = \int_0^t ds\ c^{♀}_s(\phi_n), \tag{52}$$

and their tilde conjugates. We now investigate the product rule of the increments $dC_t(\phi_n)$, $dC^{♀}_t(\phi_n)$, $d\tilde{C}_t(\phi_n)$, $d\tilde{C}^{♀}_t(\phi_n)$ defined by

$$dC_t(\phi_n) = C_{t+dt}(\phi_n) - C_t(\phi_n) = \int_t^{t+dt} ds\ c_s(\phi_n), \tag{53}$$

$$dC^{♀}_t(\phi_n) = C^{♀}_{t+dt}(\phi_n) - C^{♀}_t(\phi_n) = \int_t^{t+dt} ds\ c^{♀}_s(\phi_n), \tag{54}$$

and their tilde conjugates. It can be done by evaluating the matrix elements of the products such as $dC^{♀}_t(\phi_n)dC_t(\phi_{n'})$ with respect to the exponential vectors. By making use of (46)–(49), we then have

$$dC_t(\phi_n)dC^{♀}_t(\phi_{n'}) = 2\kappa(\phi_n)\delta_{nn'}dt, \quad d\tilde{C}_t(\phi_n)d\tilde{C}^{♀}_t(\phi_{n'}) = 2\kappa(\phi_n)\delta_{nn'}dt, \tag{55}$$

and other products vanish[‡].

We introduce the quantum Wiener processes $B_t(\phi_n)$, $B^{\dagger}_t(\phi_n)$, $\tilde{B}_t(\phi_n)$, $\tilde{B}^{\dagger}_t(\phi_n)$ defined by

$$B_t(\phi_n) = C_t(\phi_n) + \bar{n}(\phi_n)\tilde{C}^{♀}_t(\phi_n), \quad B^{\dagger}_t(\phi_n) = \tilde{C}_t(\phi_n) + [\bar{n}(\phi_n)+1]C^{♀}_t(\phi_n), \tag{56}$$

and their tilde conjugates. The definitions (56) of $B_t(\phi_n)$ and $B^{\dagger}_t(\phi_n)$ together with the product rules (55) give us the following product rules of the increments $dB_t(\phi_n)$, $dB^{\dagger}_t(\phi_n)$ and their tilde conjugates:

$$dB_t(\phi_n)dB^{\dagger}_t(\phi_{n'}) = 2\kappa(\phi_n)[\bar{n}(\phi_n)+1]\delta_{nn'}dt, \quad dB_t(\phi_n)d\tilde{B}_t(\phi_{n'}) = 2\kappa(\phi_n)\bar{n}(\phi_n)\delta_{nn'}dt, \tag{57}$$

$$dB^{\dagger}_t(\phi_n)dB_t(\phi_{n'}) = 2\kappa(\phi_n)\bar{n}(\phi_n)\delta_{nn'}dt, \quad dB^{\dagger}_t(\phi_n)d\tilde{B}^{\dagger}_t(\phi_{n'}) = 2\kappa(\phi_n)[\bar{n}(\phi_n)+1]\delta_{nn'}dt, \tag{58}$$

$$d\tilde{B}_t(\phi_n)dB_t(\phi_{n'}) = 2\kappa(\phi_n)\bar{n}(\phi_n)\delta_{nn'}dt, \quad d\tilde{B}_t(\phi_n)d\tilde{B}^{\dagger}_t(\phi_{n'}) = 2\kappa(\phi_n)[\bar{n}(\phi_n)+1]\delta_{nn'}dt, \tag{59}$$

$$d\tilde{B}^{\dagger}_t(\phi_n)dB^{\dagger}_t(\phi_{n'}) = 2\kappa(\phi_n)[\bar{n}(\phi_n)+1]\delta_{nn'}dt, \quad d\tilde{B}^{\dagger}_t(\phi_n)d\tilde{B}_t(\phi_{n'}) = 2\kappa(\phi_n)\bar{n}(\phi_n)\delta_{nn'}dt, \tag{60}$$

and other products vanish.

‡The operators $C_t(\phi_n)$ and $C^{♀}_t(\phi_n)$ correspond to the annihilation and creation processes in the reference [32].

5 Stochastic Time-Evolution Generator

We define the operator $\hat{U}(t)$ such that

$$\hat{K}(t) = \langle e(z_1, w_1)|\hat{U}(t)|e(z_2, w_2)\rangle. \tag{61}$$

Using the properties (46)–(49), we see from (34) that $\hat{U}(t)$ satisfies the quantum stochastic differential equation

$$\begin{aligned} d\hat{U}(t) &= -i\Big\{\left(\hat{\Delta} + i\hat{\Pi}\right)\hat{U}(t)dt \\ &\quad +i\sum_{mn}\Big[\sqrt{m+1}|m+1,\tilde{n}\rangle\langle m,\tilde{n}|\hat{U}(t)\circ dC_t(\phi_m) - \sqrt{m+1}|m,\tilde{n}\rangle\langle m+1,\tilde{n}|\hat{U}(t)\circ d\tilde{C}_t(\phi_m) \\ &\quad +\sqrt{n+1}|m,\widetilde{n+1}\rangle\langle m,\tilde{n}|\hat{U}(t)\circ d\tilde{C}_t(\phi_n) - \sqrt{n+1}|m,\tilde{n}\rangle\langle m,\widetilde{n+1}|\hat{U}(t)\circ dC_t(\phi_n)\Big] \\ &\quad +i\sum_{mn}\Big[d\tilde{C}_t^{\natural}(\phi_m)\sqrt{m+1}|m+1,\tilde{n}\rangle\langle m,\tilde{n}|\bar{n}(\phi_m) - dC_t^{\natural}(\phi_m)\sqrt{m+1}|m,\tilde{n}\rangle\langle m+1,\tilde{n}| \\ &\quad \times[\bar{n}(\phi_m)+1] \\ &\quad +dC_t^{\natural}(\phi_n)\sqrt{n+1}|m,\widetilde{n+1}\rangle\langle m,\tilde{n}|\bar{n}(\phi_n) - d\tilde{C}_t^{\natural}(\phi_n)\sqrt{n+1}|m,\tilde{n}\rangle\langle m,\widetilde{n+1}|[\bar{n}(\phi_n)+1]\Big] \\ &\quad \circ\hat{U}(t)\Big\}, \end{aligned} \tag{62}$$

where the symbol $\circ$ indicates the Stratonovich product. Here, we interpreted (62) as the stochastic differential equation of the Stratonovich type, because it was derived from the ordinary operator-valued differential equation (15) where the ordinary calculus rule can be applied.

Using the relations between the Stratonovich and the Ito products

$$X_t \circ dC_t(\phi_n) = X_t dC_t(\phi_n) + \frac{1}{2}dX_t dC_t(\phi_n), \quad \text{etc.}, \tag{63}$$

we find that (62) becomes

$$d\hat{U}(t) = -i\left(\hat{\Delta}dt + d\hat{M}_t^I\right)\circ\hat{U}(t), \tag{64}$$

with $d\hat{M}_t^I$ defined by

$$d\hat{M}_t^I = i\sum_{mn}\Big[\sqrt{m+1}|m+1,\tilde{n}\rangle\langle m,\tilde{n}|dB_t(\phi_m) - dB_t^{\dagger}(\phi_m)\sqrt{m+1}|m,\tilde{n}\rangle\langle m+1,\tilde{n}|\Big] - \text{t.c.}. \tag{65}$$

6 Quantum Stochastic Differential Equations

6.1 Quantum Stochastic Liouville Equation

Let us introduce the stochastic time-evolution generator $\hat{V}_f(t)$ defined by

$$\hat{V}_f(t) = e^{-i\hat{H}_S t}\hat{U}(t), \tag{66}$$

with $\hat{H}_S = H_S - \tilde{H}_S$. The time-evolution equation of $\hat{V}_f(t)$ is given by

$$d\hat{V}_f(t) = -i\hat{H}_{f,t}dt\circ\hat{V}_f(t), \tag{67}$$

where

$$\hat{H}_{f,t}dt = (\hat{H}_S + \hat{\Delta})dt + d\hat{M}_t, \tag{68}$$

with

$$\begin{aligned} d\hat{M}_t &= \mathrm{e}^{-i\hat{H}_S t} d\hat{M}_t^I \mathrm{e}^{i\hat{H}_S t} \\ &= i\sum_{mn} \left[\sqrt{m+1}|m+1,\tilde{n}\rangle\langle m,\tilde{n}|\mathrm{e}^{-i\phi_m} dB_t(\phi_m) - dB_t^\dagger(\phi_m)\mathrm{e}^{i\phi_m}\sqrt{m+1}|m,\tilde{n}\rangle\langle m+1,\tilde{n}|\right] \\ &\quad -\text{t.c..} \end{aligned} \tag{69}$$

Using the relation (63), we can transform the equation (67) of the Stratonovich type into that of the Ito type

$$d\hat{V}_f(t) = -i\hat{\mathcal{H}}_{f,t}dt\hat{V}_f(t), \tag{70}$$

where

$$\hat{\mathcal{H}}_{f,t}dt = \hat{H}_{f,t}dt - i\frac{1}{2}\hat{H}_{f,t}dt\hat{H}_{f,t}dt. \tag{71}$$

Evaluating $\hat{H}_{f,t}dt\hat{H}_{f,t}dt$ in terms of the product rules (57)–(60), we have

$$\hat{H}_{f,t}dt\hat{H}_{f,t}dt = d\hat{M}_t d\hat{M}_t = -2\hat{\Pi}dt. \tag{72}$$

Therefore, we find that $\hat{\mathcal{H}}_{f,t}dt$ is given by

$$\hat{\mathcal{H}}_{f,t}dt = (\hat{H}_S + \hat{\Delta} + i\hat{\Pi})dt + d\hat{M}_t. \tag{73}$$

We define the thermal vacuum

$$|0_f(t)\rangle = \hat{V}_f(t)|0_f(0)\rangle. \tag{74}$$

In terms of the time-evolution equation (70), we obtain the quantum stochastic Liouville equation of the Ito type

$$d|0_f(t)\rangle = -i\hat{\mathcal{H}}_{f,t}dt|0_f(t)\rangle, \tag{75}$$

where $\hat{\mathcal{H}}_{f,t}dt$ is given by (73).

Applying a thermal bra-vacuum $\langle|$ in Γ^β to the stochastic Liouville equation (75) of the Ito type, we see that

$$d\langle|0_f(t)\rangle = -i\langle|\hat{\mathcal{H}}_{f,t}dt|0_f(t)\rangle = -i\hat{H}dt\langle|0_f(t)\rangle, \tag{76}$$

where we defined $\hat{H}$ by

$$\hat{H} = \hat{H}_S + \hat{\Delta} + i\hat{\Pi}. \tag{77}$$

Here, under the assumption that $|0_f(0)\rangle = |0_S\rangle|\rangle$ with the thermal vacuum $|0_S\rangle$ of relevant system at $t = 0$, we evaluated as $\langle|d\hat{M}_t|0_f(t)\rangle = \langle|d\hat{M}_t\hat{V}_f(t)|0_f(0)\rangle = 0$ with the help of the properties of the Ito type

$$\langle|dB_t(\phi_n)\hat{V}_f(t)|\rangle = 0, \quad \langle|dB_t^\dagger(\phi_n)\hat{V}_f(t)|\rangle = 0, \tag{78}$$

$$\langle|d\tilde{B}_t(\phi_n)\hat{V}_f(t)|\rangle = 0, \quad \langle|d\tilde{B}_t^\dagger(\phi_n)\hat{V}_f(t)|\rangle = 0. \tag{79}$$

Therefore, putting $|0(t)\rangle = \langle|0_f(t)\rangle$, we obtain the quantum master equation

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle, \tag{80}$$

with $\hat{H}$ given by (77).

6.2 Quantum Langevin Equation

For any relevant system operator A, we define the Heisenberg operator by

$$A(t) = \hat{V}_f^{-1}(t) A \hat{V}_f(t). \tag{81}$$

With the help of the calculus rule of the Ito type together with (70) and the equation of $\hat{V}_f^{-1}(t)$ of the Ito type

$$d\hat{V}_f^{-1}(t) = i\hat{V}_f^{-1}(t)\hat{\mathcal{H}}_{f,t}^{(-1)}dt, \tag{82}$$

with

$$\hat{\mathcal{H}}_{f,t}^{(-1)}dt = \hat{\mathcal{H}}_{f,t}dt + id\hat{M}_t d\hat{M}_t, \tag{83}$$

we have the quantum Langevin equation of the Ito type

$$\begin{aligned} dA(t) &= d\hat{V}_f^{-1}(t)A\hat{V}_f(t) + \hat{V}_f^{-1}(t)Ad\hat{V}_f(t) + d\hat{V}_f^{-1}(t)Ad\hat{V}_f(t) \\ &= i\left[\hat{\mathcal{H}}_f(t)dt,\ A(t)\right] - d\hat{M}(t)\left[d\hat{M}(t),\ A(t)\right], \end{aligned} \tag{84}$$

where $\hat{\mathcal{H}}_f(t)dt = \hat{V}_f^{-1}(t)\hat{\mathcal{H}}_{f,t}dt\hat{V}_f(t)$ and $d\hat{M}(t) = \hat{V}_f^{-1}(t)d\hat{M}_t\hat{V}_f(t)$.

Applying $\langle\!\langle 1| = \langle|\langle 1_S|$ to the equation (84), we have

$$d\langle\!\langle 1|A(t) = -i\langle\!\langle 1|A(t)\left[\hat{H}_S(t)dt + i\hat{\Pi}(t)dt + d\hat{M}(t)\right], \tag{85}$$

where we used the property $\langle\!\langle 1|\hat{\mathcal{H}}_f(t) = 0$ and $\langle\!\langle 1|d\hat{M}(t) = 0$. Applying $|0\rangle\!\rangle = |0_S\rangle|\rangle$ to the equation (85), we obtain the equation of motion of expectation value

$$\frac{d}{dt}\langle\!\langle 1|A(t)|0\rangle\!\rangle = i\langle\!\langle 1|\left[H_S(t),\ A(t)\right]|0\rangle\!\rangle + \langle\!\langle 1|A(t)\hat{\Pi}(t)|0\rangle\!\rangle, \tag{86}$$

where we used the thermal state condition $\langle 1|\tilde{A}^\dagger(t) = \langle 1|A(t)$ for any operator A of relevant system, and the properties (78) and (79). The equation (86) can be also derived from the quantum master equation (80).

7 Summary and Discussion

In this paper, applying the procedure of Accardi et al. to a non-linear oscillator interacting with thermal reservoir, we obtained the quantum stochastic differential equations for the non-linear damped oscillator.

We showed that, in the weak coupling limit, the equation of motion of the matrix element of the time-evolution generator with respect to collective exponential vectors in reservoir space converges to the equation of motion of the matrix element of the stochastic time-evolution generator with respect to exponential vectors in the space of quantum Wiener processes. In the sense of the matrix elements, we found that the stochastic time-evolution generator satisfies a quantum stochastic differential equation. This indicates that the convergence of the time evolution equation is the weak convergence in the sense of the matrix elements, in other words, the change of the equation for the time-evolution generator to the one for the stochastic time-evolution generator can be interpreted as the change of a representation space.

Taking account of the effect of the non-linearity within a relevant system, we constructed quantum Wiener processes together with their representation space. The effect of the quantum Wiener processes on the time-evolution equation is appeared in the expression of quantum master equation and the equation of motion of expectation value of an observable. A further consideration of the effect will be given in the future.

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