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Multiple-scale analysis of Duffing's equation and Fourier series expansion of Jacobi's elliptic functions

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0 Introduction

This note is concerned with an elementary ordinary differential equation with a small positive parameter ε:

$$\frac{d^2y}{dt^2}+y+\varepsilon y^3=0. \tag{1}$$

This equation is called Duffing's equation. Several textbooks of perturbation theory treat this equation to illustrate the method of multiple-scale analysis ([BO], [H], [KC], [N], for example). In these textbooks, the leading term of a perturbative solution of (1) is calculated by using the method of multiple-scale analysis. On the other hand, (1) can be solved by quadrature ([KC], [L]): the general solution can be written in terms of a Jacobi's elliptic function. The purpose of this article is to clarify the relation between these two solutions: a perturbative solution and the exact solution.

1 Regular perturbation expansions and an apparent paradox

In this section, we briefly illustrate why regular perturbation expansions are not appropriate for global analysis of (1). This section and the first half part of the next section are due to [BO].

A naïve perturbative solution of (1) with the initial conditions

$$y(0) = 1, \quad y'(0) = 0 \tag{2}$$

is obtained by expanding $y(t)$ as a power series in ε:

$$y(t) = \sum_{k=0}^{\infty} \varepsilon^k y_k(t). \tag{3}$$

Substituting this into (1) and equating coefficients of like powers of ε gives a sequence of linear differential equations:

$$y_0'' + y_0 = 0 \tag{4}$$

$$y_1'' + y_1 = -y_0^3 \tag{5}$$

$$y_2'' + y_2 = -3y_0^2 y_1 \tag{6}$$

and so on. The solution to (4) which satisfies $y_0(0) = 1$, $y_0'(0) = 0$ is

$$y_0(t) = \cos t.$$

Now the right-hand side of (5) is known:

$$-y_0^3 = -\cos^3 t.$$

Using the relation $\cos^3 t = \cos 3t/4 + 3\cos/4$, we have the solution of (5) with the initial conditions $y_1(0) = y_1'(0) = 0$:

$$y_1(t) = \frac{1}{32}\cos 3t - \frac{1}{32}\cos t - \frac{3}{8}t\sin t.$$

The last term $t \sin t$ of the right-hand side of y_1 is called a secular term since it is not bounded. The right-hand side of (6) is now given and we have the solution y_2 of (6) with the initial conditions $y_2(0) = y_2'(0) = 0$:

$$y_2(t) = \frac{1}{2^{10}}\cos 5t - \frac{3}{2^7}\cos 3t + \frac{23}{2^{10}}\cos t + \frac{t}{2^8}(-9\sin 3t + 24\sin t) - \frac{9}{2^7}t^2\cos t.$$

The last two terms are unbounded and we also call them secular terms. In a similar manner, we can obtain y_k's successively.

Thus we have a perturbative solution of (1) in arbitrary order. For example, the first-order perturbative solution is

$$y(t) = \cos t + \varepsilon\left(\frac{1}{32}\cos 3t - \frac{1}{32}\cos t - \frac{3}{8}t\sin t\right) + \mathrm{O}(\varepsilon^2).$$

The term $\mathrm{O}(\varepsilon)$ means that for fixed t the error between $y(t)$ and $y_0(t)+\varepsilon y_1(t)$ is at most of order ε^2 as $\varepsilon \to 0$. Hence it might be not small for values of t of order $1/\varepsilon$. For such large values of t, the secular terms in y_1 and in y_2 suggest that the amplitude of oscillation grows with t.

In spite of these observations, we can show that the solution to (1) with (2) is bounded for all t. Multiplying (1) by y' converts each term in the differential equation to an exact derivative:

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}y^2 + \frac{1}{4}\varepsilon y^4\right] = 0.$$

Thus,

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}y^2 + \frac{1}{4}\varepsilon y^4 = C, \tag{7}$$

where C is a constant. By (2), we have $C = \frac{1}{2} + \frac{1}{4}\varepsilon$. Therefore $|y(t)|$ is bounded for all t by $\sqrt{1+\varepsilon/2}$.

We have arrived at an apparent paradox. We have proved that the exact solution $y(t)$ is bounded for all t while the first-order or the second-order perturbative solutions are unbounded. The resolution of this paradox lies in the summation of the perturbative series (3). The most secular term in $y_k(t)$ has the form

$$A_k t^k e^{it} + \overline{A}_k t^k e^{-it}$$

with

$$A_k = \frac{1}{2}\frac{1}{n!}\left(\frac{3i}{8}\right)^k.$$

The sum of the most secular terms becomes a cosine function:

$$\sum_{k=0}^{\infty}\frac{1}{2}\varepsilon^k t^k\left[\frac{1}{k!}\left(\frac{3i}{8}\right)^k e^{it} + \frac{1}{k!}\left(\frac{-3i}{8}\right)^k e^{-it}\right] = \cos\left[t\left(1+\frac{3}{8}\varepsilon\right)\right].$$

This expression is bounded for all t. Less secular terms are negligible. See [BO] for details.

Thus the paradox has disappeared. But this explanation needs lengthy calculation and requires an infinite sum. Such a lengthy calculation can be avoided by using the method of multiple-scale analysis.

2 Multiple-scale analysis of Duffing's equation

A shortcut for eliminating the secular terms to all orders begins by introducing a new variable $\tau = \varepsilon t$. Multiple-scale analysis seeks solutions of (1) in the form

$$y(t) = Y(t, \tau),$$

where the both variables t and τ are treated as independent variables. The t-derivative of $Y(t, \varepsilon t)$ is:

$$\frac{d}{dt} Y(t, \varepsilon t) = \left(\frac{\partial Y}{\partial t} + \varepsilon \frac{\partial Y}{\partial \tau} \right) \Bigg|_{\tau = \varepsilon t}.$$

Hence if Y satisfies the partial differential equation

$$\left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau} \right)^2 Y + Y + \varepsilon Y^3 = 0, \tag{8}$$

then $y(t) = Y(t, \varepsilon t)$ is a solution to (1). Equation (8) is rewritten in the form

$$\frac{\partial^2 Y}{\partial t^2} + Y = -\varepsilon \left(2 \frac{\partial^2 Y}{\partial t \partial \tau} + Y^3 \right) - \varepsilon^2 \frac{\partial^2 Y}{\partial \tau^2}. \tag{9}$$

We construct a perturbative solution of this partial differential equation in the following manner. Assume a perturbative expansion of the form

$$Y = \sum_{n=0}^{\infty} \varepsilon^n Y_n(t, \tau). \tag{10}$$

Substituting (10) into (9) and collecting powers of ε gives

$$\frac{\partial^2 Y_0}{\partial t^2} + Y_0 = 0, \tag{11}$$

$$(12)\qquad \frac{\partial^2 Y_1}{\partial t^2} + Y_1 = -2\frac{\partial^2 Y_0}{\partial t \partial \tau} - Y_0^3,$$

$$(13)\quad \frac{\partial^2 Y_k}{\partial t^2} + Y_k = -2\frac{\partial^2 Y_{k-1}}{\partial t \partial \tau} - \sum_{k_1+k_2+k_3=k} Y_{k_1} Y_{k_2} Y_{k_3} - \frac{\partial^2 Y_{k-2}}{\partial \tau^2} \quad (k \geq 2).$$

We forget the initial data and the reality of solutions for a while. The most general solution to (11) is

$$(14)\qquad Y_0 = a_1^{(0)} e^{it} + a_{-1}^{(0)} e^{-it},$$

where $a_{\pm 1}^{(0)} = a_{\pm 1}^{(0)}(\tau)$ are arbitrary functions of τ. $a_{\pm 1}^{(0)}$ will be determined by the condition that secular terms do not appear in the solution to (12). From (14), the right-hand side of (12) is

$$-a_1^{(0)^3} e^{3it} - \left[2i\frac{\partial a_1^{(0)}}{\partial \tau} + 3a_1^{(0)^2} a_{-1}^{(0)}\right] e^{it} + \left[2i\frac{\partial a_{-1}^{(0)}}{\partial \tau} - 3a_1^{(0)} a_{-1}^{(0)^2}\right] e^{-it} - a_{-1}^{(0)^3} e^{-3it}$$

Therefore, if one of the coefficients of $e^{\pm it}$ on the right-hand side of (12) is nonzero, then the solution $Y_1(t,\tau)$ will be secular (unbounded) in t. To preclude the appearance of secularity, we require that the coefficients of $e^{\pm it}$ on the right-hand side of (12) to be equal to zero:

$$(15)\qquad 2i\frac{\partial a_1^{(0)}}{\partial \tau} + 3a_1^{(0)^2} a_{-1}^{(0)} = 0,$$

$$(16)\qquad 2i\frac{\partial a_{-1}^{(0)}}{\partial \tau} - 3a_1^{(0)} a_{-1}^{(0)^2} = 0.$$

This system of differential equations for $a_{\pm 1}^{(0)}$ is easily solved and we have

$$(17)\qquad a_1^{(0)}(\tau) = \alpha e^{3i\alpha\beta\tau/2},$$

$$(18)\qquad a_{-1}^{(0)}(\tau) = \beta e^{-3i\alpha\beta\tau/2},$$

where α, β are arbitrary constants. The right-hand side of (12) becomes

$$-\alpha^3 e^{3i(3\alpha\beta\tau/2+t)} - \beta^3 e^{-3i(3\alpha\beta\tau/2+t)}.$$

Therefore,

$$(19)\qquad Y_1 = \frac{1}{8}\alpha^3 e^{3i(3\alpha\beta\tau/2+t)} + a_1^{(1)} e^{it} + a_{-1}^{(1)} e^{-it} + \frac{1}{8}\beta^3 e^{-3i(3\alpha\beta\tau/2+t)}.$$

Here $a_{\pm 1}^{(1)}$ are arbitrary functions in τ. $a_{\pm 1}^{(1)}$ will be determined by the non-secularity condition in the next step: the coefficients of $e^{\pm it}$ in the right-hand side of (13) for $k = 2$ are equal to zero. This condition turns out to be a system of linear inhomogeneous differential equations for $a_{\pm 1}^{(1)}$ and we can find easily a special solution of this system as follows:

$$a_1^{(1)}(\tau) = \frac{5}{16}\alpha^2\beta e^{3i\alpha\beta\tau/2}, \tag{20}$$

$$a_{-1}^{(1)}(\tau) = \frac{5}{16}\alpha\beta^2 e^{-3i\alpha\beta\tau/2}. \tag{21}$$

Thus we have obtained Y_1 that satisfies (12) and the non-secularity condition for (13) for $k = 2$.

Suppose that $Y_0, Y_1, \ldots, Y_{k-1}$ are obtained in the form

$$Y_j = b_{2j+1}^{(j)} e^{(2j+1)\Phi} + b_{2j-1}^{(j)} e^{(2j-1)\Phi} + \cdots + b_{-2j-1}^{(j)} e^{-(2j+1)\Phi}, \tag{22}$$

where $b_l^{(j)}$ $(j = 0, \ldots, k-1; l = 2j+1, 2j-1, \ldots, -2j-1)$ are constants and $\Phi = i(3\alpha\beta\tau/2 + t)$ so that the non-secularity condition of (13) is satisfied. Then we can solve (13) and get Y_k in the form

$$\begin{aligned} Y_k \;=\; & b_{2k+1}^{(k)} e^{(2k+1)\Phi} + \cdots + b_3^{(k)} e^{3\Phi} + a_1^{(k)} e^{it} + a_{-1}^{(k)} e^{-it} \\ & + b_{-3}^{(k)} e^{3\Phi} + \cdots + b_{-2k-1}^{(k)} e^{-(2k+1)\Phi}, \end{aligned} \tag{23}$$

where $b_j^{(k)}$ $(j = \pm 3, \ldots, \pm(2k+1))$ are determined and $a_{\pm 1}^{(k)}$ are arbitrary functions of τ. $a_{\pm 1}^{(k)}$ will be determined by the non-secularity condition of (13) for $k + 1$. This condition can be written down in the form of a system of linear inhomogeneous differential equations of first order whose inhomogeneous terms are given and the homogeneous equations are the same as the equations for $a_{\pm 1}^{(1)}$. We can choose a special solution of this system of the form:

$$a_1^{(k)}(\tau) = \alpha c^{(k)} e^{3i\alpha\beta\tau/2}, \tag{24}$$

$$a_{-1}^{(k)}(\tau) = \beta c^{(k)} e^{-3i\alpha\beta\tau/2}, \tag{25}$$

where $c^{(k)}$ is a constant. Thus we have obtained Y_k so that the non-secularity condition of (13) for $k + 1$ is satisfied.

By this procedure, we can calculate Y_k term by term as many as we like.

Thus we have constructed a family of perturbative solutions of (1)

$$y = y(t,\alpha,\beta) = \sum_{n=0}^{\infty} \varepsilon^n Y_n(t,\varepsilon t) \tag{26}$$

including two arbitrary constants α, β.

In each step of constructing Y_k's, we have chosen special solutions $a_{\pm 1}^{(k)}$ to the inhomogeneous system of linear differential equations. Hence there is some ambiguity for $a_{\pm 1}^{(k)}$ in each step. But we can see that such ambiguity is equivalent to replacing α, β by arbitrary power series in ε:

$$\alpha = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \cdots,$$

$$\beta = \beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + \cdots.$$

Bender and Orszag wrote in their book [BO] that not only τ but infinitely many scales are needed to construct a perturbative solution of (1) with (2) in all order (see PROBLEM 11.7 in [BO]). But as we have shown, forgetting initial conditions permits us to construct the general perturbative solutions by using double scales (t,τ) only. If we choose α, β so that (2) holds:$y(0,\alpha,\beta) = 1$, $y'(0,\alpha,\beta) = 0$, then we have $\alpha = \beta = 1/2 - 7\varepsilon/128 + \cdots$ and hence

$$\Phi = i\left(1 + \frac{3}{8}\varepsilon - \frac{21}{256}\varepsilon^2 + \cdots\right)t.$$

The first two terms in the right-hand side of Φ recovers the frequency shift that we calculated by using the naïve perturvative solution. Thus taking an infinite sum of the most secular terms in the naïve perturvative solution has been already built in the leading term Y_0.

Remark 1 Since Duffing's equation (1) is autonomous, we can choose one free constant as the freedom of translation. In other words,

$$y(t,\alpha,\beta) = y(t+c,\gamma,\gamma)$$

holds for

$$c = \frac{\log\alpha - \log\beta}{i(3\alpha\beta\varepsilon + 1)}, \quad \gamma = (\alpha\beta)^{\frac{1}{2}}.$$

Remark 2 First six Y_k's are as follows:

$$
\begin{aligned}
Y_0 &= \alpha\, e^{\Phi} + \beta e^{-\Phi} \\
\\
Y_1 &= \frac{\alpha^3}{8} e^{3\,\Phi} + \frac{5\,\alpha^2\,\beta}{16} e^{\Phi} + \frac{5\,\alpha\,\beta^2}{16} e^{-\Phi} + \frac{\beta^3}{8} e^{-3\Phi} \\
Y_2 &= \frac{\alpha^5}{64} e^{5\,\Phi} - \frac{27\,\alpha^4\,\beta}{128} e^{3\,\Phi} + \frac{11\,\alpha^3\,\beta^2}{512} e^{\Phi} + \frac{11\,\alpha^2\,\beta^3}{512} e^{-\Phi} \\
&\quad - \frac{27\,\alpha\,\beta^4}{128} e^{-3\,\Phi} + \frac{\beta^5}{64} e^{-5\,\Phi} \\
Y_3 &= \frac{\alpha^7}{512} e^{7\,\Phi} - \frac{61\,\alpha^6\,\beta}{1024} e^{5\,\Phi} + \frac{1419\,\alpha^5\,\beta^2}{4096} e^{3\,\Phi} + -\frac{799\,\alpha^4\,\beta^3}{8192} e^{\Phi} \\
&\quad - \frac{799\,\alpha^3\,\beta^4}{8192} e^{-\Phi} + \frac{1419\,\alpha^2\,\beta^5}{4096} e^{-3\,\Phi} - \frac{61\,\alpha\,\beta^6}{1024} e^{-5\,\Phi} \frac{\beta^7}{512} e^{-7\,\Phi} \\
Y_4 &= \frac{\alpha^9}{4096} e^{9\,\Phi} - \frac{95\,\alpha^8\,\beta}{8192} e^{7\,\Phi} + \frac{5255\,\alpha^7\,\beta^2}{32768} e^{5\,\Phi} \\
&\quad - \frac{33119\,\alpha^6\,\beta^3}{65536} e^{3\,\Phi} + \frac{61043\,\alpha^5\,\beta^4}{524288} e^{\Phi} \\
&\quad + \frac{61043\,\alpha^4\,\beta^5}{524288} e^{-\Phi} - \frac{33119\,\alpha^3\,\beta^6}{65536} e^{-3\,\Phi} \\
&\quad + \frac{5255\,\alpha^2\,\beta^7}{32768} e^{-5\,\Phi} - \frac{95\,\alpha\,\beta^8}{8192} e^{-7\,\Phi} + \frac{\beta^9}{4096} e^{-9\,\Phi} \\
Y_5 &= \frac{\alpha^{11}}{32768} e^{11\,\Phi} - \frac{129\,\alpha^{10}\,\beta}{65536} e^{9\,\Phi} + \frac{11403\,\alpha^9\,\beta^2}{262144} e^{7\,\Phi} \\
&\quad - \frac{188565\,\alpha^8\,\beta^3}{524288} e^{5\,\Phi} + \frac{2644083\,\alpha^7\,\beta^4}{4194304} e^{3\,\Phi} - \frac{458133\,\alpha^6\,\beta^5}{8388608} e^{\Phi} \\
&\quad - \frac{458133\,\alpha^5\,\beta^6}{8388608} e^{-\Phi} + \frac{2644083\,\alpha^4\,\beta^7}{4194304} e^{-3\,\Phi} - \frac{188565\,\alpha^3\,\beta^8}{524288} e^{-5\,\Phi} \\
&\quad + \frac{11403\,\alpha^2\,\beta^9}{262144} e^{-7\,\Phi} - \frac{129\,\alpha\,\beta^{10}}{65536} e^{-9\,\Phi} + \frac{\beta^{11}}{32768} e^{-11\,\Phi}
\end{aligned}
$$

We constructed the general perturbative solution

$$y(t,\alpha,\beta)=\sum_{n=0}^{\infty}\varepsilon^n Y_n(t,\varepsilon t), \tag{27}$$

where

$$Y_n(t,\tau)=\sum_{k=0}^{2n+1} b_{2n+1-2k}^{(n)} e^{(2n+1-2k)\Phi},\quad \Phi=i\left(\frac{3}{2}\alpha\beta\tau+t\right).$$

At least formally, we may change the order of summation in (27) and write

$$y(t,\alpha,\beta)=\sum_{n=-\infty}^{\infty} c_{2n+1} e^{i\left(\frac{3}{2}\alpha\beta\varepsilon+1\right)(2n+1)t}, \tag{28}$$

where

$$c_{2n+1}=\sum_{k=0}^{\infty} b_{2n+1}^{(n+k)}\varepsilon^{n+k}. \tag{29}$$

Proposition 1 *If $\varepsilon|\alpha\beta|<1$, then (29) converges for every n and*

$$\overline{\lim}|c_{2n+1}|^{\frac{1}{2n+1}}<1$$

holds. Hence (28) makes sense as a Fourier series of a periodic analytic function on the real line with a period $4\pi/(3\alpha\beta\varepsilon+2)$.

3 Exact solution of Duffing's equation and its Fourier expansion

In this section, we will see what really is the analytic function obtained in Proposition 1. To see this, we first solve (1) with initial conditions

$$y(0)=a,\quad y'(0)=0 \tag{30}$$

by quadrature (cf. [KC], [L]). By (7) and (30), we have

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2+\frac{1}{2}y^2+\frac{1}{4}\varepsilon y^4=C, \tag{31}$$

where

$$C = \frac{a^2}{4}(2 + \varepsilon a^2).$$

Hence we have

$$\left(\frac{dy}{dt}\right)^2 = (a^2 - y^2)\left(1 + \frac{1}{2}\varepsilon a^2 + \frac{1}{2}\varepsilon y^2\right) \tag{32}$$

and integrating this to obtain t, we find

$$\begin{aligned} t &= \sqrt{\frac{2}{\varepsilon}} \int_y^a \frac{dy}{\sqrt{(a^2 - y^2)(2/\varepsilon + a^2 + y^2)}} \\ &= \frac{1}{\sqrt{1 + \varepsilon a^2}} \mathrm{cn}^{-1}\left(\frac{y}{a}, \sqrt{\frac{\varepsilon a^2}{2 + 2\varepsilon a^2}}\right). \end{aligned} \tag{33}$$

Here cn is a Jacobi's elliptic function. Therefore we have the solution of (1) with (30):

$$y = a\,\mathrm{cn}\left(\sqrt{1 + \varepsilon a^2}t, k\right), \quad k = \sqrt{\frac{\varepsilon a^2}{2 + 2\varepsilon a^2}}. \tag{34}$$

Taking the freedom of translation into account, we have the exact general solution to (1):

$$y = a\,\mathrm{cn}\left(\sqrt{1 + \varepsilon a^2}(t + b), \sqrt{\frac{\varepsilon a^2}{2 + 2\varepsilon a^2}}\right), \tag{35}$$

where a and b are arbitrary constants. If we take the modulus of the elliptic function to be one of the constants, we may write the general solution in the form

$$y = k\sqrt{\frac{2}{\varepsilon(1 - 2k^2)}}\,\mathrm{cn}\left(\frac{t + b}{\sqrt{1 - 2k^2}}, k\right), \tag{36}$$

where b and k are arbitrary constants. We consider real solutions and b and k are taken to be real and k should satisfy $|k| < 1/\sqrt{2}$. Real period of the general solution (36) is given by $T = 4\sqrt{1 - 2k^2}K$, where K is the complete elliptic integral of the first kind:

$$K = K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}.$$

The general solution (36) is a periodic function on the real line. Hence it has a Fourier expansion which is given as follows in the case $b = 0$ ([L]):

$$(37)\quad y = \frac{\pi}{K}\sqrt{\frac{2}{\varepsilon(1-2k^2)}}\sum_{n=0}^{\infty}\frac{q^{n+\frac{1}{2}}}{1+q^{2n+1}}2\cos\left((2n+1)\frac{\pi t}{2K\sqrt{1-2k^2}}\right),$$

where $q = e^{-\pi K'/K}$, $K' = K(\sqrt{1-k^2})$.

By the construction of the general solution by quadrature, fixing a real period specifies an even solution to the Duffing equation uniquely up to signature. Thus we have arrived at the following

Theorem 2 *Let $y(t,\alpha,\beta)$ denote the general perturbative solution constructed in Section 2. Let ε, α and k satisfy $\varepsilon|\alpha|^2 < 1$, $\alpha k > 0$ and*

$$\frac{3}{2}\alpha^2\varepsilon + 1 = \frac{\pi}{2K\sqrt{1-2k^2}}.$$

Then

$$y(t,\alpha,\alpha) = k\sqrt{\frac{2}{\varepsilon(1-2k^2)}}\,\mathrm{cn}\left(\frac{t}{\sqrt{1-2k^2}},k\right)$$

holds. Hence the coefficient c_{2n+1} of $e^{i(3\alpha^2\varepsilon/2+1)(2n+1)t}$ in $y(t,\alpha,\alpha)$ is written in terms of k:

$$c_{2n+1} = \frac{\pi}{K}\sqrt{\frac{2}{\varepsilon(1-2k^2)}}\frac{q^{n+\frac{1}{2}}}{1+q^{2n+1}}\quad (n\in\mathbf{Z}).$$

4 Concluding remarks

For Duffing's equation, the perturbative solutions constructed by multiple-scale method coincide with the Fourier expansion of the solutions obtained by quadrature. The method of multiple-scales can be applied to construction of formal solutions of Painlevé equations ([A1], [A2], [AKT], [JK]). But in this case, the meaning of convergence is not clear. Costin [C] proved the convergence in the case where one of the two free parameters is equal to zero. (As a matter of fact, he treats more general equations.) We hope that our discussion gives some insight into the convergence problem of formal solutions to Painlevé equations.

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