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# Finite Hilbert Networks

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## 1 Introduction

Let $N = \{X, Y, K\}$ be a finite connected graph which has no self-loof. Namely $X$ is a finite set of nodes, $Y$ is a finite set of arcs and $K$ is the node-arc incidence matrix.

Let $\mathcal{H}$ be a real Hilbert space with an inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. Denote by $L(X;\mathcal{H})$ the set of all functions $u$ on $X$ such that $u(x) \in \mathcal{H}$. We call an element of $L(X,\mathcal{H})$ a $\mathcal{H}$-potential. The meaning of the notation $L(Y;\mathcal{H})$ is similar. Let $\mathcal{L}(\mathcal{H})$ be the set of all positive invertible linear operator from $\mathcal{H}$ to $\mathcal{H}$. Let $r \in L(Y;\mathcal{L}(\mathcal{H}))$. For each $y \in Y$, we have $r(y) \in \mathcal{L}(\mathcal{H})$ and there exists $\rho(y) > 0$ such that

$$(r(y)h, h) \geq \rho(y)\|h\|^2 \quad \text{for all} \quad h \in \mathcal{H}.$$

Here $r(y)h$ means the image of $h$ under $r(y)$, i.e., $r(y)(h)$. In this paper, we use this convention unless no confusion occurs from the context. Denote by $r(y)^{-1}$ the inverse operator of $r(y)$. Since $r(y) \in \mathcal{L}(\mathcal{H})$, there exists $\rho^*(y) > 0$ such that

$$(r(y)^{-1}h, h) \geq \rho^*(y)\|h\|^2 \quad \text{for all} \quad h \in \mathcal{H}.$$

By [1], we see that there exists a unique *square root* $r(y)^{1/2} \in \mathcal{L}(\mathcal{H})$ of $r(y)$ for each $y \in Y$, i.e.,

$$[r(y)^{1/2}]^2 = r(y).$$

**Definition 1. 1** *Let $e$ be a fixed element of $\mathcal{H}$ such that* $\|e\| = 1$.

**Lemma 1. 1** *For every $y \in Y$, the following relations hold:*
*(1)* $|(r(y)w(y), e)|^2 \leq (r(y)w(y), w(y))(r(y)e, e)$.
*(2)* $(r(y)^{-1}e, e)(r(y)e, e) \geq 1$;

**Proof.** By Schwarz's inequality, we have

$$\begin{aligned} |(r(y)w(y), e)|^2 &= |(r(y)^{1/2}w(y), r(y)^{1/2}e)|^2 \\ &\leq \|r(y)^{1/2}w(y)\|^2\|r(y)^{1/2}e\|^2 \\ &= (r(y)w(y), w(y))(r(y)e, e). \end{aligned}$$

2) follows from (1) by taking $w(y) := r(y)^{-1}e$. □

**Definition 1. 2** *For $u \in L(X;\mathcal{H})$, let $\delta u$ be the potential drop of $u$ and let $du$ be the discrete derivative of $u$:*

$$\delta u(y) := \sum_{x\in X} K(x,y)u(x)$$
$$du(y) := -r(y)^{-1}(\delta u(y)) = -r(y)^{-1}\delta u(y).$$

*The Dirichlet sum of $u$ is defined by*

$$D(u) := \sum_{y\in Y}(r(y)du(y), du(y)) = \sum_{y\in Y}(r(y)^{-1}\delta u(y), \delta u(y)).$$

**Definition 1. 3** *For $w \in L(Y;\mathcal{H})$, let $\partial w(x)$ be the divergence of $w$ and let $H(w)$ be the energy of $w$:*

$$\partial w(x) := \sum_{y\in Y} K(x,y)w(y)$$
$$H(w) := \sum_{y\in Y}(r(y)w(y), w(y)).$$

Notice that $D(u) = H(du)$.

**Lemma 1. 2** *Let $u \in L(X;\mathcal{H})$ and $w \in L(Y;\mathcal{H})$. Then*

$$\sum_{y\in Y}(w(y), \delta u(y)) \le H(w)^{1/2}D(u)^{1/2}.$$

**Proof.** We have by Schwarz's inequality

$$\begin{aligned}
\sum_{y\in Y}(w(y), \delta u(y)) &= \sum_{y\in Y}(r(y)^{1/2}w(y), r(y)^{-1/2}\delta u(y)) \\
&\le \sum_{y\in Y}\|r(y)^{1/2}w(y)\|\|r(y)^{-1/2}\delta u(y)\| \\
&\le [\sum_{y\in Y}\|r(y)^{1/2}w(y)\|^2]^{1/2}[\sum_{y\in Y}\|r(y)^{-1/2}\delta u(y)\|^2]^{1/2} \\
&= H(w)^{1/2}D(u)^{1/2}. \quad \square
\end{aligned}$$

To emphasize the analogy to [2], we put

$$D(N;\mathcal{H};a) := \{u \in L(X;\mathcal{H}); u(a) = 0\}.$$

Note that $D(u) < \infty$ for every $L(X;\mathcal{H})$, since $G$ is a finite graph. We see that $D(u)^{1/2}$ is a norm on $D(N;\mathcal{H};a)$ by the following lemma:

**Lemma 1. 3** *Let $a \in X$. For any $x \in X$, there exists a constant $M_x$ which satisfies:*

$$\|u(x)\| \le M_x D(u)^{1/2}$$

*for all $u \in L(X;\mathcal{H})$ with $u(a) = 0$.*

**Proof.** There exists a path $P$ from $a$ to $x$. Let $C_X(P)$ and $C_Y(P)$ be the sets of nodes and arcs on $P$ respectively (cf. [2]), i.e.,

$$C_X(P) := \{x_0, x_1, \cdots, x_n\} \ (x_0 = a, x_n = x)$$

$$C_Y(P) := \{y_1, y_2, \cdots, y_n\}, e(y_i) = \{x_{i-1}, x_i\} \ (i = 1, 2, \cdots, n).$$

Let $u \in L(X;\mathcal{H})$ and $u(a) = 0$. We have

$$\begin{aligned} D(u) &\geq \textstyle\sum_{y\in P}(r(y)^{-1}\delta u(y), \delta u(y)) \\ &= \sum_{i=1}^{n}(r(y_i)^{-1}\delta u(y_i), \delta u(y_i)) \\ &\geq \sum_{i=1}^{n}\rho^*(y_i)\|u(x_i) - u(x_{i-1})\|^2 \\ &\geq \sum_{i=1}^{n}\rho^*(y_i)[\|u(x_i)\| - \|u(x_{i-1})\|]^2, \end{aligned}$$

so that, for $i = 1, 2, \cdots$

$$\|u(x_i)\| - \|u(x_{i-1})\| \leq D(u)^{1/2}[\rho^*(y_i)]^{-1/2}.$$

Since $u(a) = 0$, we have

$$\|u(x)\| = \sum_{i=1}^{n}[\|u(x_i)\| - \|u(x_{i-1})\|] \leq M_x D(u)^{1/2}$$

with

$$M_x := \sum_{i=1}^{n}[\rho^*(y_i)]^{-1/2}.$$

This completes the proof. □

Since $G$ is a finite graph, the following fact is obvious:

**Proposition 1. 1** *$D(N;\mathcal{H}, a)$ is a Hilbert space with respect to the inner product:*

$$D(u_1, u_2) := \textstyle\sum_{y\in Y}(r(y)^{-1}\delta u_1(y), \delta u_2(y)).$$

*$L(Y;\mathcal{H})$ is a Hilbert space with respect to the inner product:*

$$H(w_1, w_2) := \textstyle\sum_{y\in Y}(r(y)w_1(y), w_2(y)).$$

## 2 $\mathcal{H}$-flows

**Definition 2. 1** *Let a and b be distinct nodes. We say that $w \in L(Y;\mathcal{H})$ is a $\mathcal{H}$-flow from a to b if*

$$\partial w(x) = 0 \quad \textit{for all} \quad x \in X, \quad x \neq a, b.$$

*Denote by $F(a,b;\mathcal{H})$ the set of all $\mathcal{H}$-flows from a to b.*

Notice that

$$\partial w(a) + \partial w(b) = 0,$$

since $G$ is a finite graph.

**Definition 2. 2** *For $w \in F(a,b;\mathcal{H})$, we define two real valued functions:*

$$\begin{aligned} I_e(w) &:= (\partial w(b), e) = -(\partial w(a), e), \\ I(w) &:= \|\partial w(a)\| = \|\partial w(b)\|. \end{aligned}$$

## 3 Extremum problems

Let us consider several extremum problems related to $\mathcal{H}$-potentials and $\mathcal{H}$-flows:

$$\begin{aligned} d(a,b;\mathcal{H},e) &:= \inf\{D(u); u \in L(X;\mathcal{H}), u(a) = 0,\ u(b) = e\} \\ d_e(a,b;\mathcal{H}) &:= \inf\{D(u); u \in L(X;\mathcal{H}), (u(a),e) = 0,\ (u(b),e) = 1\} \\ d(a,b;\mathcal{H}) &:= \inf\{D(u); u \in L(X;\mathcal{H}), u(a) = 0,\ \|u(b)\| = 1\} \\ d^*(a,b;\mathcal{H},e) &:= \inf\{H(w); w \in F(a,b;\mathcal{H}),\ Kw(b) = e\} \\ d_e^*(a,b;\mathcal{H}) &:= \inf\{H(w); w \in F(a,b;\mathcal{H}),\ I_e(w) = 1\} \\ d^*(a,b;\mathcal{H}) &:= \inf\{H(w); w \in F(a,b;\mathcal{H}),\ I(w) = 1\} \end{aligned}$$

Clearly

$$d_e(a,b;\mathcal{H}) \leq d(a,b;\mathcal{H},e), \quad d(a,b;\mathcal{H}) \leq d(a,b;\mathcal{H},e),$$
$$d_e^*(a,b;\mathcal{H}) \leq d^*(a,b;\mathcal{H},e), \quad d^*(a,b;\mathcal{H}) \leq d^*(a,b;\mathcal{H},e).$$

**Lemma 3. 1** *Let u be a feasible solution for $d(a,b;\mathcal{H},e)$ and w be a feasible solution for $d_e^*(a,b;\mathcal{H})$. Then $1 \leq H(w)^{1/2}D(u)^{1/2}$.*

**Proof.** By definition and Lemma 1. 2

$$\begin{aligned} 1 = I_e(w) &= (Kw(b), e) = \textstyle\sum_{x\in X}(Kw(x), u(x)) \\ &= \textstyle\sum_{y\in Y}(w(y), \delta u(y)) \\ &\leq H(w)^{1/2}D(u)^{1/2}. \quad \square \end{aligned}$$

Similarly we can prove

**Lemma 3. 2** *Let $u$ be a feasible solution for $d_e(a,b;\mathcal{H})$ and $w$ be a feasible solution for $d^*(a,b;\mathcal{H},e)$. Then $1 \le H(w)^{1/2}D(u)^{1/2}$.*

By the above observation, we obtain

**Theorem 3. 1** *The following relations hold:*
*(1)* $1 \le d(a,b;\mathcal{H},e)d_e^*(a,b;\mathcal{H})$,
*(2)* $1 \le d_e(a,b;\mathcal{H})d^*(a,b;\mathcal{H},e)$.

**Lemma 3. 3** *There exists a unique optimal solution for $d(a,b;\mathcal{H},e)$.*

**Proof.** Let $\{u_n\}$ be a minimizing sequence for $d(a,b;\mathcal{H},e)$, i.e., $\{u_n\} \subset L(X;\mathcal{H})$, $u_n(a) = 0$, $u_n(b) = e$ and $D(u_n) \to d(a,b;\mathcal{H},e)$ as $n \to \infty$. Since $(u_n + u_m)/2$ is a feasible solution for $d(a,b;\mathcal{H},e)$, we have

$$\begin{aligned} d(a,b;\mathcal{H},e) &\le D((u_n+u_m)/2) \\ &\le D((u_n+u_m)/2) + D((u_n-u_m)/2) \\ &= [D(u_n)+D(u_m)]/2 \to d(a,b;\mathcal{H},e) \end{aligned}$$

as $m,n \to \infty$. Therefore $D(u_n - u_m) \to 0$ as $n,m \to \infty$. It follows from Lemma 1. 3 that $\{u_n(x)\}$ is a Cauchy sequence in $\mathcal{H}$ for each $x \in X$. Therefore $\{u_n(x)\}$ converges strongly to $\tilde{u}(x) \in \mathcal{H}$ for each $x \in X$. we see easily that $\tilde{u}(a) = 0$, $\tilde{u}(b) = e$ and $d(a,b;\mathcal{H},e) = D(\tilde{u})$. Namely $\tilde{u}$ is an optimal solution. We omit the proof of the uniqueness of the optimal solution. □

Now we study some properties of the optimal solution $\tilde{u}$ of $d(a,b;\mathcal{H},e)$.

**Lemma 3. 4** *Let $\tilde{w}(y) := d\tilde{u}(y)$. Then $\tilde{w} \in F(a,b;\mathcal{H})$ and $I_e(\tilde{w}) = D(\tilde{u})$.*

**Proof.** Let $f \in L(X;\mathcal{H})$ satisfy $f(a) = 0$ and $f(b) = 0$. Then for any $t \in \mathbf{R}$, $\tilde{u} + tf$ is a feasible solution for $d(a,b;\mathcal{H},e)$, we have

$$\begin{aligned} D(\tilde{u}) &\le D(\tilde{u}+tf) \\ &= D(\tilde{u}) + 2tD(\tilde{u},f) + t^2D(f). \end{aligned}$$

By the standard variational argument, we have

$$D(\tilde{u},f) = 0.$$

On the other hand, we have

$$\begin{aligned} D(\tilde{u},f) &= \sum_{y\in Y}(\tilde{w}(y), \sum_{z\in X} K(z,y)f(z)) \\ &= \sum_{z\in X}\sum_{y\in Y}(K(z,y)\tilde{w}(y), f(z)) \\ &= \sum_{z\in X}(\partial\tilde{w}(z), f(z)). \end{aligned}$$

Denote by $\varepsilon_x$ the characteristic function of $\{x\}$, i.e., $\varepsilon_x(x) = 1$ and $\varepsilon_x(z) = 0$ for $z \neq x$. Let $x \neq a, b$. For any $h \in \mathcal{H}$, we may take $\varepsilon_x h$ for $f$, and hence

$$(\partial \tilde{w}(x), h) = 0.$$

Therefore $\partial \tilde{w}(x) = 0$ for $x \neq a, b$. Let $\hat{e} \in L(X; \mathcal{H})$ such that $\hat{e}(x) = e$ for all $x \in X$. By taking $\hat{e} - \tilde{u} - \varepsilon_a e$ for $f$, we obtain

$$D(\tilde{u}, \hat{e} - \tilde{u} - \varepsilon_a e) = 0,$$

so that

$$D(\tilde{u}) = -D(\tilde{u}, \varepsilon_a e) = -(\partial \tilde{w}(a), e).$$

Therefore $I_e(\tilde{w}) = D(\tilde{u})$. □

**Theorem 3. 2** $d(a, b; \mathcal{H}, e) d_e^*(a, b; \mathcal{H}) = 1.$

**Proof.** It suffices to show that $d(a, b; \mathcal{H}, e) d_e^*(a, b; \mathcal{H}) \leq 1$. Let $\tilde{u}$ be the optimal solution for $d(a, b; \mathcal{H}, e)$ and put $\tilde{w}(y) := d\tilde{u}(y)$. Then we see by the above observation that $\tilde{w}(y)/D(\tilde{u})$ is a feasible solution for $d_e^*(a, b; \mathcal{H})$, so that

$$\begin{aligned} d_e^*(a, b; \mathcal{H}) &\leq H(\tilde{w}(y)/D(\tilde{u})) \\ &= D(\tilde{u})/D(\tilde{u})^2 \\ &= 1/D(\tilde{u}) = 1/d(a, b; \mathcal{H}, e). \quad \square \end{aligned}$$

**Lemma 3. 5** *There exists a unique optimal solution $\tilde{w}$ for $d^*(a, b; \mathcal{H}, e)$.*

**Proof.** There exists a minimizing sequence $\{w_n\}$ for $d^*(a, b; \mathcal{H}, e)$. Since $(w_n + w_m)/2$ is a feasible solution for $d^*(a, b; \mathcal{H}, e)$, we have

$$\begin{aligned} d^*(a, b; \mathcal{H}, e) &\leq H((w_n + w_m)/2) \\ &\leq H((w_n + w_m)/2) + H((w_n - w_m)/2) \\ &= [H(w_n) + H(w_m)]/2 \to d^*(a, b; \mathcal{H}, e) \end{aligned}$$

as $m, n \to \infty$. Therefore $\{w_n\}$ is a Cauchy sequence in the Hilbert space $L(Y; \mathcal{H})$ and converges to $\tilde{w} \in L(Y; \mathcal{H})$. Then we see easily that $\tilde{w}$ is an optimal solution for $d^*(a, b; \mathcal{H}, e)$. We omit the proof of the uniqueness of the optimal solution. □

**Definition 3. 1** *We say that $\omega \in L(Y; \mathcal{H})$ is a cycle if $\partial\omega(x) = 0$ for all $x \in X$. Denote by $C(Y; \mathcal{H})$ the set of cycles on $N$.*

By the standard variational argument, we have

**Lemma 3. 6** *Let $\tilde{w}$ be the optimal solution of $d^*(a,b;\mathcal{H},e)$. For any cycle $\omega \in C(Y;\mathcal{H})$,*

$$H(\tilde{w},\omega) := \sum_{y\in Y}(r(y)\tilde{w}(y),\omega(y)) = 0.$$

**Definition 3. 2** *Let $\mathbf{P}_{a,x}$ the set of all paths from $a$ to $x (x \neq a)$.*

**Theorem 3. 3** $d_e(a,b;\mathcal{H})d^*(a,b;\mathcal{H},e) = 1$.

**Proof.** Let $\tilde{w}$ be the optimal solution of $d^*(a,b;\mathcal{H},e)$. Let $h \in \mathcal{H}$ and let $P_1, P_2 \in \mathbf{P}_{a,x}$. Then

$$\omega(y) = (p_1(y) - p_2(y))h \in C(Y;\mathcal{H}),$$

where $p_1$ and $p_2$ are path indices of $P_1$ and $P_2$ respectively. By Lemma 3. 6 , we have $H(\tilde{w},p_1h) = H(\tilde{w},p_2h)$. We set $\tilde{u}(a) = 0$. For $x \neq a$ and a path index $p_x$ of a path $P \subset \mathbf{P}_{a,x}$, the function $\tilde{u} \in L(X)$ defined by $\tilde{u}(a) = 0$ and

$$\tilde{u}(x) := \sum_{y\in Y} p_x(y)\tilde{w}(y)$$

is well-defined by the above observation. Then we have $\delta\tilde{u}(y) = -\tilde{w}(y)$. In case $P \in \mathbf{P}_{a,b}$, $\tilde{w} - pe$ is a feasible solution for $d^*(a,b;\mathcal{H},e)$, so that $H(\tilde{w},\tilde{w}-pe) = 0$ or

$$H(\tilde{w}) = H(\tilde{w},pe) = (\tilde{u}(b),e).$$

Now $\tilde{u}/\beta$ is a feasible solution for $d_e(a,b;\mathcal{H})$ and

$$d_e(a,b;\mathcal{H}) \leq D(\tilde{u}) = H(\tilde{w})/H(\tilde{w})^2 = 1/H(\tilde{w}) = 1/d^*(a,b;\mathcal{H},e). \quad \Box$$

# 4 Extremal length

Let $a$ and $b$ be distinct two nodes. The extremal length $EL(a,b;\mathcal{L}(\mathcal{H}))$ is defined by the inverse of the value of the extremum problem $(EL)$:

$$\begin{gathered} Minimize \quad H(w) \quad \text{subject to} \\ w \in L(Y;\mathcal{H}), \\ \sum_{y\in P}\|r(y)w(y)\| \geq 1 \quad \text{for all} P \in \mathbf{P}_{a,b}. \end{gathered}$$

The extremal length $EL_e(a,b;\mathcal{L}(\mathcal{H}))$ is defined by the inverse of the value of the extremum problem $(EL_e)$:

$$\begin{gathered} Minimize \quad H(w) \quad \text{subject to} \\ w \in L(Y;\mathcal{H}), \\ \sum_{y\in P}|(r(y)w(y),e)| \geq 1 \quad \text{for all} P \in \mathbf{P}_{a,b}. \end{gathered}$$

Since $|(r(y)w(y),e)| \leq \|r(y)w(y)\|\|e\| = \|r(y)w(y)\|$, we have

(4.1) $$EL(a,b;\mathcal{L}(\mathcal{H})) \geq EL_e(a,b;\mathcal{L}(\mathcal{H})).$$

**Lemma 4. 1** $d_e(a,b;\mathcal{H}) \geq EL_e(a,b;\mathcal{H})^{-1}$.

**Proof.** Let $u$ be any feasible solution for $d_e(a,b;\mathcal{H})$. Then

$$w(y) := r(y)^{-1}\delta u(y) \in \mathcal{H}$$

for each $y \in Y$. As in the proof of Lemma 1.3, for $P \in \mathbf{P}_{a,b}$ let

$$C_X(P) := \{x_0, x_1, \cdots, x_n\} \; (x_0 = a, x_n = b)$$

$$C_Y(P) := \{y_1, y_2, \cdots, y_n\}, e(y_i) = \{x_{i-1}, x_i\} \; (i = 1, 2, \cdots, n).$$

Then we have

$$\begin{aligned} \textstyle\sum_{y\in P} |(r(y)w(y), e)| &= \sum_{i=1}^{n} |(r(y_i)w(y_i), e)| \\ &= \sum_{i=1}^{n} |(\delta u(y_i), e)| \\ &\geq \sum_{i=1}^{n} |(u(x_i) - u(x_{i-1}), e)| \\ &\geq (u(b), e) - (u(a), e) = 1. \end{aligned}$$

Therefore

$$EL_e(a,b;\mathcal{H})^{-1} \leq H(w) = D(u),$$

so that $EL_e(a,b;\mathcal{H})^{-1} \leq d_e(a,b;\mathcal{H})$. □

**Lemma 4. 2** *Let $w$ be a feasible solution for the problem $(EL_e)$. Then*

$$d_e(a,b;\mathcal{H}) \leq \textstyle\sum_{y\in Y}(r(y)w(y), w(y))(r(y)e, e)(r(y)^{-1}e, e).$$

**Proof.** Put $V(y) := |(r(y)w(y), e)|$. Then

$$\textstyle\sum_{y\in P} V(y) \geq 1 \quad \text{for all} \quad P \in \mathbf{P}_{a,b}.$$

By the duality between the max-potential problem and the min-work problem, there exists $\beta \in L(X;\mathbf{R})$ such that $\beta(a) = 0$, $\beta(b) = 1$ and $|\delta\beta(y)| \leq V(y)$ on $Y$. Let $u(x) := \beta(x)e$. Then $u \in L(X;\mathcal{H})$, $u(a) = 0$ and $u(b) = e$, so that by Lemma 1. 1 (1)

$$\begin{aligned} d_e(a,b;\mathcal{H}) &\leq D(u) = \textstyle\sum_{y\in Y}(r(y)^{-1}\delta u(y), \delta u(y)) \\ &= \textstyle\sum_{y\in Y}(\delta\beta(y))^2(r(y)^{-1}e, e) \\ &\leq \textstyle\sum_{y\in Y} V(y)^2(r(y)^{-1}e, e) \\ &\leq \textstyle\sum_{y\in Y}(r(y)w(y), w(y))(r(y)e, e)(r(y)^{-1}e, e) \quad \square \end{aligned}$$

**Theorem 4. 1** *Let* $M(r) := \sup\{(r(y)e, e)(r(y)^{-1}e, e); y \in Y\}$. *Then*

$$EL_e(a, b; \mathcal{H})^{-1} \leq d_e(a, b; \mathcal{H}) \leq M(r) EL_e(a, b; \mathcal{H})^{-1}.$$

**Corollary 4. 1** *Assume that* $(r(y)e, e)(r(y)^{-1}e, e) = 1$ *for all* $y \in Y$. *Then* $d_e(a, b; \mathcal{H}) = EL_e(a, b; \mathcal{H})^{-1}$.

**Remark 1.** Let $I$ be the identity map of $\mathcal{H}$ and let $\gamma \in L(Y; \mathbf{R})$ be positive. Then $r(y) = \gamma(y) I$ is positive and invertible. Clearly, we have $(r(y)e, e) = \gamma(y)$ and $(r(y)^{-1}e, e) = 1/\gamma(y)$, so that the condition in the above theorem holds in this case.

We show by an example that the equality $d_e(a, b; \mathcal{H}) = EL_e(a, b; \mathcal{H})^{-1}$ does not hold in general.

**Example** Let $X = \{x_0, x_1, x_2\}, Y = \{y_1, y_2\}$,

$$K(x_i, y_i) = 1, \ K(x_{i-1}, y_i) = -1 \ (i = 1, 2)$$

and $K(x, y) = 0$ for any other pair. Then $\{X, Y, K\}$ is a finite graph. Take $\mathcal{H}$ as $\mathbf{R}^2$ and define $r(y)$ by

$$r(y_1) := \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, \quad r(y_2) := \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}.$$

Then

$$r(y_1)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/s \end{pmatrix}, \quad r(y_2)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/t \end{pmatrix}.$$

Let $a = x_0$, $b = x_2$ in the above setting and let $e = (e_1, e_2)^T \in \mathbf{R}^2$. Let $u \in L(X, \mathbf{R}^2)$ be a fesible solution for $d_e(a, b; \mathbf{R}^2)$ and set $u(x_1) = (\alpha, \beta)^T$. Then

$$D(u) = \alpha^2 + \beta^2/s + (\alpha - e_1)^2 + (\beta - e_2)^2/t.$$

It is easily seen that

$$d_e(a, b; \mathbf{R}^2) = \frac{e_1^2}{2} + \frac{e_2^2}{s + t}$$

and $\tilde{u}(x_1) := (e_1/2, e_2 s/(s + t))^T$ is the optimal solution. For $w \in L(Y, \mathbf{R}^2)$, set $w(y_1) = (p_1, q_1)^T$, $w(y_2) = (p_2, q_2)^T$. Then

$$H(w) = p_1^2 + s q_1^2 + p_2^2 + t q_2^2.$$

Clearly, $\mathbf{P}_{a,b}$ is a singleton. The feasibility of $w \in L(Y, \mathbf{R}^2)$ for the problem $(EL_e)$ implies

$$(p_1 + p_2)e_1 + (s q_1 + b q_2)e_2 \geq 1.$$

Therefore we have

$$EL_e(a,b;\mathbf{R}^2)^{-1} = \frac{1}{2e_1^2 + (s+t)e_2^2}$$

and the optimal solution is given by

$$p_1 = p_2 = \frac{e_1\lambda}{2}, \quad q_1 = q_2 = \frac{e_2\lambda}{2} \quad \text{with} \quad \lambda := \frac{2}{2e_1^2 + (s+t)e_2^2}.$$

We have

$$d_e(a,b;\mathbf{R}^2) - EL_e(a,b;\mathbf{R}^2)^{-1} = \frac{e_1^2 e_2^2 (c-2)^2}{2c(2e_1^2 + ce_2^2)} \geq 0 \quad \text{with} \quad c = s+t.$$

Thus the equality holds only if $c = 2$ or $e_1 = 0$ or $e_2 = 0$.

# 5 Extremal width

Let $a$ and $b$ be distinct two nodes. Denote by $\mathbf{Q}_{a,b}$ the set of all cuts between $a$ and $b$ (cf. [2]). For $Q \in \mathbf{Q}_{a,b}$, there exist two disjoint subsets $Q(a)$ and $Q(b)$ of $X$ such that $a \in Q(a)$, $b \in Q(b)$, $X = Q(a) \cup Q(b)$ and $Q = Q(a) \ominus Q(b)$. The index function $u_Q \in L(X;\mathcal{H})$ of $Q$ is defined by

$$u := \epsilon_{Q(A)} e = \sum\nolimits_{z \in Q(A)} \varepsilon_z e.$$

The characteristic function $s_Q$ of $Q$ is defined by

$$s_Q := \delta u_Q e = \sum\nolimits_{z \in Q(A)} \delta\varepsilon_z e.$$

Notice that $|\delta\varepsilon_{Q(A)}(y)| = 1$ if $y \in Q$ and $\delta\varepsilon_{Q(A)}(y) = 0$ otherwise. Observe that $\|s_Q(y)\| = 1$ if $y \in Q$ and $\|s_Q(y)\| = 0$ otherwise.

The extremal width $EW(a,b;\mathcal{H})$ is defined by the inverse of the value of the extremum problem $(EW)$:

$$\text{Minimize} \quad H(w) \quad \text{subject to}$$
$$w \in L(Y;\mathcal{H}),$$
$$\sum\nolimits_{y \in Q} \|w(y)\| \geq 1 \quad \text{for all} \quad Q \in \mathbf{Q}_{a,b}.$$

The extremal width $EW_e(a,b;\mathcal{H})$ is defined by the inverse of the value of the extremum problem $(EW_e)$:

$$\text{Minimize} \quad H(w) \quad \text{subject to}$$
$$w \in L(Y;\mathcal{H}),$$

$$\sum_{y\in Q}|(w(y),e)|\geq 1 \quad \text{for all} \quad Q\in \mathbf{Q}_{a,b}.$$

Since $|(w(y),e)|\leq \|w(y)\|\|e\| = \|w(y)\|$, we have

$$EW(a,b;\mathcal{H})\geq EW_e(a,b;\mathcal{H}) \tag{5.1.}$$

**Lemma 5. 1** $d_e^*(a,b;\mathcal{H})\geq EW_e(a,b;\mathcal{H})^{-1}$.

**Proof.** Let $w$ be any feasible solution for $d_e^*(a,b;\mathcal{H})$ and let $Q\in \mathbf{Q}_{a,b}$ Then

$$\begin{aligned}1=I_e(w) &= -(\partial w(a),e) = -\sum_{x\in X}(\partial w(x),\varepsilon_Q(x)e)\\ &= -\sum_{y\in Y}(w(y),\delta\varepsilon_Q(y)e)\\ &\leq \sum_{y\in Q}|(w(y),e)|\end{aligned}$$

Therefore $EW_e(a,b;\mathcal{H})^{-1}\leq H(w)$, so that $EW_e(a,b;\mathcal{H})^{-1}\leq d_e^*(a,b;\mathcal{H})$. □

**Lemma 5. 2** *Let $w$ be a feasible solution for the problem $(EW_e)$. Then*

$$d_e^*(a,b;\mathcal{H})\leq \sum_{y\in Y}(r(y)w(y),w(y))(r(y)e,e)(r(y)^{-1}e,e).$$

**Proof.** Put $V(y):=|(w(y),e)|$. Then

$$\sum_{y\in Q}V(y)\geq 1 \quad \text{for all} \quad Q\in \mathbf{Q}_{a,b}.$$

By the duality between the max-flow problem and the min-cut problem, there exists $\varphi\in L(Y;\mathbf{R})$ such that $|\varphi(y)|\leq V(y)$ on $Y$,

$$\partial\varphi(x)=0 \quad \text{for} \quad x\in X\setminus\{a,b\} \quad \text{and} \quad -\partial\varphi(a)=\partial\varphi(b)=1.$$

Let $w(y):=\varphi(y)e$. Then $w\in F(a,b;\mathcal{H})$ and $I_e(w)=1$. Therefore

$$\begin{aligned}d_e^*(a,b;\mathcal{H}) &\leq H(w)=\sum_{y\in Y}(r(y)\varphi(y)e,\varphi(y)e)\\ &= \sum_{y\in Y}[\varphi(y)]^2(r(y)e,e)\\ &\leq \sum_{y\in Y}|(w(y),e)|^2(r(y)e,e)\\ &\leq \sum_{y\in Y}(r(y)w(y),w(y))(r(y)^{-1}e,e)(r(y)e,e). \quad \square\end{aligned}$$

**Theorem 5. 1** *Let $M(r):=\sup\{(r(y)e,e)(r(y)^{-1}e,e);y\in Y\}$. Then*

$$EW_e(a,b;\mathcal{H})^{-1}\leq d_e^*(a,b;\mathcal{H})\leq M(r)EW_e(a,b;\mathcal{H})^{-1}.$$

**Corollary 5. 1** *Assume that* $(r(y)e, e)(r(y)^{-1}e, e) = 1$ *for all* $y \in Y$. *Then* $d_e^*(a, b; \mathcal{H}) = EW_e(a, b; \mathcal{H})^{-1}$.

We recall the example in Section 4 and calculate $EW_e(a, b; \mathcal{H})^{-1}$ and $d_e^*(a, b; \mathcal{H})$ in this case. If $w \in F(a, b; \mathbf{R}^2)$, then $w(y_1) = w(y_2) = (p, q)^T$ and

$$H(w) = 2p^2 + (s+t)q^2, \quad I_e(w) = pe_1 + qe_2.$$

By a simple calculus, we see easily that

$$d_e^*(a, b; \mathbf{R}^2) = \frac{1}{e_1^2/2 + e_2^2/(s+t)}.$$

On the other hand, if $w$ is feasible for $EW_e(a, b; \mathbf{R}^2)^{-1}$, then we have

$$(*) \qquad p_1 e_1 + q_1 e_1 \le 1, \quad p_2 e_1 + q_2 e_1 \le 1$$

with $w(y_1) = (p_1, q_1)^T$, $w(y_2) = (p_2, q_2)$. Minimizing $H(w)$ subject to the condition $(*)$, we have

$$EW_e(a, b; \mathbf{R}^2)^{-1} = \frac{s}{se_1^2 + e_2^2} + \frac{t}{te_1^2 + e_2^2}.$$

Therefore

$$d_e^*(a, b; \mathbf{R}^2) - EW_e(a, b; \mathbf{R}^2)^{-1} = \frac{(s-t)^2 e_1^2 e_2^2}{[(s+t)e_1^2 + 2e_2^2](te_1^2 + e_2^2)(se_1^2 + e_2^2)} \ge 0$$

and the equality holds if $s = t$ or $e_1 = 0$ or $e_2 = 0$.

# References


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