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Interpolation Theorem for L_{DBCC} and L_{DBCK}

Bayu Surarso (バユ スラルソ)
Graduate School of Information Engineering,
Hiroshima University, Higashi Hiroshima, 739, Japan
and
Department of Mathematics,
Faculty of Mathematics and Natural Sciences,
Diponegoro University, Semarang, Indonesia.

1 Introduction

It is known that the interpolation theorem holds for the logics **LR** and **LRW**, which are obtained from the relevant logic **R** and **RW** respectively by omitting the distributive axiom $A\wedge(B\vee C)\rightarrow(A\wedge B)\vee(A\wedge C)$ (see [7] and [2]). On the other hand, Urquhart proved in [13] that the interpolation theorem fails for **R**, **RW** and some other relevant logics. He also claims that the interpolation theorem fails for the positive versions of all the logics discussed, provided that either the language contains the constant t or the formula $((A\supset B)\wedge A)\supset B$ is provable. This fact shows that the distributive axiom seems to play a critical role in the interpolation problems for substructural logics. In the present study we will show that the interpolation theorem holds for the logics L_{DBCC} and L_{DBCK}, which are obtained from L_{BCC} and L_{BCK}, respectively, by adding the distributive law $A\wedge(B\vee C)\rightarrow(A\wedge B)\vee(A\wedge C)$ as an initial sequent. Ono and Komori proved in [11] that the interpolation theorem holds for L_{BCC} and L_{BCK}.

Slaney in [12] introduced sequent systems without cut rule which are equivalent to L_{DBCC} and L_{DBCK}. We will take Slaney's systems, but in a slightly modified form, and use essentially Maehara's method introduced in [6] to prove the interpolation theorem for these logics.

A note about the names of the logics discussed. To avoid any confusion with those

[11] and [12], here we also use the names of logics L_{BCC}, L_{BCK}, L_{DBCC} and L_{DBCK}. However it must be noticed that the letters C and K which appear in them have no connection with the combinators **C** and **K**. Better names for them can be found in [9] or [10]. In those papers Ono introduced the basic logical system FL (*full Lambek logic*) and then gave the names FL_w and FL_{ew} for L_{BCC} and L_{BCK} since they can be obtained from FL by adding the weakening rule and the exchange and weakening rules, respectively. By the reason mentioned above we will denote our systems, which are equivalent to L_{DBCC} and L_{DBCK}, by $L_\diamond L_{\mathrm{DBCC}}$ and $L_\diamond L_{\mathrm{DBCK}}$, respectively.

The full version of the present paper will appear as [1]. The authour would like to express his sincere gratitude to Professor Hiroakira Ono and Dr. Toshiyasu Arai for their suggestions and comments.

2 Gentzen sequent systems $L_\diamond L_{\mathrm{DBCC}}$ and $L_\diamond L_{\mathrm{DBCK}}$

Slaney in [12] introduced sequent systems without cut rule LL_{DBCC} and LL_{DBCK}, in the same way as relevant systems discussed in [3] and [5]. These systems are equivalent to L_{DBCC} and L_{DBCK}. They contain two types of structural connectives, the *extensional structural connective* "," which corresponds to the extensional conjunction and the *intensional structural connective* ";" which corresponds to the intensional conjunction or fusion. Having these two types of structural connective, two types of structural rules (extensional and intensional) will be formulated in these systems. By using these rules, the distributive law can be derived.

In the following, we will give a definition of these systems, but in a slightly modified form. As in [12] our language will contain the false constant $\perp$, implication $\supset$, disjunction $\vee$ and two kinds of conjunction, i.e. the extensional conjunction $\wedge$ and the intensional conjunction $*$.

First, for our sequent system $L_\diamond L_{\mathrm{DBCC}}$, *structures* (see [5]), which are called *bunches of premises* in [12], are defined recursively as follows;

1) any formula is a structure,

2) for $n \geq 2$, if each X_i is a structure for $i = 1, \dots, n$, then both sequences $(X_1, \dots, X_n)$ and $(X_1; \dots; X_n)$ are structures.

Structures of the form $(X_1, \dots, X_n)$ and of the form $(X_1; \dots; X_n)$ are said to be *extensional* and *intensional*, respectively. Each structure X_i is called an *immediate constituent* of $(X_1, \dots, X_n)$ and $(X_1; \dots; X_n)$. Here, if X_k is of the form $(Y_1, \dots, Y_{m_1})$

for some structures Y_j, $j = 1, \ldots, m_1$, then the above $(X_1, \ldots, X_n)$ should be understood as $(X_1, \ldots, X_{k-1}, Y_1, \ldots, Y_m, X_{k+1}, \ldots, X_n)$. Similarly, if X_l is of the form $(Y_1; \ldots; Y_{m_2})$ for some structures Y_j, $j = 1, \ldots, m_2$, then the above $(X_1; \ldots; X_n)$ should be understood as $(X_1; \ldots; X_{l-1}; Y_1; \ldots; Y_{m_2}; X_{l+1}; \ldots; X_n)$. Thus, we will assume that no extensional (intensional) structures have an extensional (intensional) structure as their immediate constituent.

In the sequel, the letters X, Y, Z, U and W with or without subscripts will denote structures. We will omit parentheses when no confusions will occur.

Substructures of a given structure X in $L_\diamond L_{\mathrm{DBCC}}$ are defined as follows;

1) if a structure $X_1, \ldots, X_n$ occurs in X then
1.1) each X_i is a substructure of X for $i = 1, \ldots, n$,
1.2) any subsequence of the sequence $X_1, \ldots, X_n$ is a substructure of X,
2) if a structure $X_1; \ldots; X_n$ occurs in X then
2.1) each X_i is a substructure of X for $i = 1, \ldots, n$,
2.2) any subsequence of the sequence $X_1; \ldots; X_n$ is a substructure of X.

Here, *subsequences* are defined as usual. Thus, suppose $X = Y, (Z; U), W$. Then $(Z; U), W$ and X itself are examples of subsequences of X. On the other hand, Z and Y, Z, W are examples of sequences which are not subsequences of X.

Following [12], an expression like $\Gamma(X)$ is used for denoting the structure with an indicated substructure-occurrence X in it. Then $\Gamma(Y)$ denotes the structure obtained from $\Gamma(X)$ by replacing the indicated substructure-occurence X in it by a structure Y. A *sequent* is an expression of the form $X \to A$, where X is a structure (possibly empty) and A is a formula. Then $L_\diamond L_{\mathrm{DBCC}}$ will be given as follows:

It consists of the initial sequents $\quad A \to A \quad$ and $\quad \bot \to A$,

the following structural rules

$$\frac{\Gamma(Y, X) \to C}{\Gamma(X, Y) \to C}\ (E - exchange) \qquad \frac{\Gamma(X) \to C}{\Gamma(X, Y) \to C}\ (E - weakening)$$

$$\frac{\Gamma(X, X) \to C}{\Gamma(X) \to C}\ (E - contraction) \qquad \frac{\Gamma(X) \to C}{\Gamma(X; Y) \to C}\ (I - weakening)$$

and the following rules for logical connectives

$$\frac{X; A \to B}{X \to A \supset B}\ (\to\supset) \qquad \frac{X \to A \quad \Gamma(B) \to C}{\Gamma(A \supset B; X) \to C}\ (\supset\to)$$

$$\frac{X \to A}{X \to A \vee B}\ (\to \vee 1) \qquad \frac{X \to B}{X \to A \vee B}\ (\to \vee 2) \qquad \frac{\Gamma(A) \to C \quad \Gamma(B) \to C}{\Gamma(A \vee B) \to C}\ (\vee \to)$$

$$\frac{X \to A \quad Y \to B}{X, Y \to A \wedge B}\ (\to \wedge) \qquad \frac{\Gamma(A, B) \to C}{\Gamma(A \wedge B) \to C}\ (\wedge \to)$$

$$\frac{X \to A \quad Y \to B}{X; Y \to A * B}\ (\to *) \qquad \frac{\Gamma(A; B) \to C}{\Gamma(A * B) \to C}\ (* \to).$$

For instance, applying (E – *contraction*) to a sequent of the form $(Y, X, X, Z) \to C$, we can get the sequent $(Y, X, Z) \to C$. Thus, X, X in $\Gamma(X, X)$ will be understood not as a substructure but as a *subexpression*. We will use these sloppy definitions, simply to avoid unnecessary complications. (See the footnotes 28 and 29 in Dunn [4].)

For our sequent system $L_\diamond L_{\mathrm{DBCK}}$, if we define intensional structures as sequences, some difficulties will occur in the proof of the interpolation theorem given in the next section. So, instead of taking sequences, we will take multisets, since the exchange law holds in it. Thus for $L_\diamond L_{\mathrm{DBCK}}$ we will modify the definition of structures as follows;
1) any formula is a structure,
2) for $n \geq 2$, if each X_i is a structure for $i = 1, \ldots, n$, then both the sequence $(X_1, \ldots, X_n)$ and the multiset $\{X_1; \ldots; X_n\}$ are structures.

Substructures are defined similarly to that in the case for $L_\diamond L_{\mathrm{DBCC}}$. Since we define intensional structures as multisets, we can dispense with the intensional exchange rule. Thus, $L_\diamond L_{\mathrm{DBCK}}$ will have the same initial sequents, structural rules and rules for logical connectives as the above $L_\diamond L_{\mathrm{DBCC}}$.

For the equivalence of $L_\diamond L_{\mathrm{DBCC}}$ ($L_\diamond L_{\mathrm{DBCK}}$) and Hilbert system H_{DBCC} (H_{DBCK}), see the proof of the equivalence of LL_{DBCC} (LL_{DBCK}) and H_{DBCC} (H_{DBCK}) in [12].

3 Interpolation theorem for L_{DBCC} and L_{DBCK}

We will show that the interpolation theorem holds for L_{DBCC} and L_{DBCK} by using the systems $L_\diamond L_{\mathrm{DBCC}}$ and $L_\diamond L_{\mathrm{DBCK}}$, respectively. In the following the expressions $V(X)$ denotes the set of propositional variables which occur in X.

Ono and Komori proved in [11] that the interpolation theorem holds for L_{BCC} and L_{BCK} by showing that interpolation theorem of the following form holds for them:

If a sequent $X;Y;Z \to D$ is provable, then there is a formula C such that
1) $Y \to C$ is provable,
2) $X;C;Z \to D$ is provable,
3) $V(C) \subset V(Y) \cap [V(X;Z) \cup V(D)]$.
Here, all of X, Y and Z are sequences of formulas of the form $A_1; \ldots ; A_n$.

Thus for $L_\diamond L_{\text{DBCC}}$ and $L_\diamond L_{\text{DBCK}}$, a desirable form of interpolation theorem might be of the following:

Let $X \to D$ be a provable sequent. Suppose that Z is a substructure-occurrence in X. Then there is a formula C such that
1) $Z \to C$ is provable,
2) $X_{\{C/Z\}} \to D$ is provable,
3) $V(C) \subset V(Z) \cap [V(X_{\{-/Z\}}) \cup V(D)]$.
Here $X_{\{C/Z\}}$ denotes the structure obtained from X by replacing Z by C and $X_{\{-/Z\}}$ denotes the structure obtained from X by deleting Z.

In fact, even the following stronger form of interpolation theorem holds for them.

Theorem 1 *Let Z_i be a substructure-occurrence in a structure Z for $i = 1, \ldots, n$. Suppose that 1) Z_j and Z_k do not intersect each another when $j \neq k$ and 2) there is no structure-occurrences of the form $Z'; Z''$ in Z such that Z' contains Z_j and Z'' contains Z_k for some j and k. Then, if the sequent $Z \to D$ is provable, there exist formulas C_i for $i = 1, \ldots, n$ such that*
1) each $Z_j \to C_j$ is provable for $j = 1, \ldots, n$,
2) $Z_{\{C_i/Z_i\}_i} \to D$ is provable,
3) for $j = 1, \ldots, n$, $V(C_j) \subset V(Z_j) \cap [V(Z_{\{-/Z_i\}_i}) \cup V(D)]$,
Here $Z_{\{C_i/Z_i\}_i}$ denotes the structure obtained from Z by replacing Z_i by C_i for every $i = 1, \ldots, n$, and $Z_{\{-/Z_i\}_i}$ denotes the structure obtained from Z by deleting Z_i for every $i = 1, \ldots, n$.

To understand the conditions of Z_i in the above theorem, let us consider the case where $X = (X_1; X_2; X_3), X_4, (X_5; X_6)$. Here the above conditions are not satisfied if we take $n = 2$, $Z_1 = X_1$ and $Z_2 = X_3$, for X_1 and X_3 are substructures of $Z' = X_1$ and $Z'' = X_2; X_3$, respectively. On the other hand, if we take $n = 2$, $Z_1 = X_1$ and $Z_2 = X_5$, then the above conditions are satisfied. In this case the above theorem says that if $X \to D$ is provable, then there exist formulas C_1 and C_2 such that

1) both $X_1 \to C_1$ and $X_5 \to C_2$ are provable,
2) $(C_1; X_2; X_3), X_4, (C_2; X_6) \to D$ is provable,
3) $V(C_1) \subset V(X_1) \cap [V((X_2; X_3), X_4, X_6) \cup V(D)]$ and
$V(C_2) \subset V(X_5) \cap [V((X_2; X_3), X_4, X_6) \cup V(D)]$.

Next, let us consider the proof of Theorem 1 for $L_\diamond L_{\mathrm{DBCC}}$. As usual, the theorem is proved by induction on the number l of inferences in the proof figure of the sequent $Z \to D$. Here we will show the proof for the following case.

Case 1. $l > 0$ and the last inference is $(\vee \to)$. Here $Z \to D$ will be of the form $\Gamma(A \vee B) \to D$ and the last inference will be of the following form;

$$\frac{\Gamma(A) \to D \quad \Gamma(B) \to D}{\Gamma(A \vee B) \to D} (\vee \to).$$

Suppose that Z_i is substructure-occurence in $\Gamma(A \vee B)$ for $i = 1, \ldots, n$, such that the conditions in the theorem are satisfied. Here we will consider the following subcase;

Subcase 1.1 The 'displayed' $A \vee B$ in $\Gamma(A \vee B)$ occcurs in Z_k for some k.

Let $U_k = Z_{k\{A/A \vee B\}}$ and $U_i = Z_i$ when $i \neq k$. Then by the hypothesis of induction there exist formulas C_i for $i = 1, \ldots, n$ such that
1a) each $U_j \to C_j$ is provable for $j = 1, \ldots, n$,
2a) $\Gamma(A)_{\{C_i/U_i\}_i} \to D$ is provable,
3a) for $j = 1, \ldots n$, $V(C_j) \subset V(U_j) \cap [V(\Gamma(A)_{\{-/U_i\}_i}) \cup V(D)]$.

Let $W_k = Z_{k\{B/A \vee B\}}$ and $W_i = Z_i$ when $i \neq k$. Then by the hypothesis of induction there exist formulas C'_i for $i = 1, \ldots, n$ such that
1b) each $W_j \to C'_j$ is provable for $j = 1, \ldots, n$,
2b) $\Gamma(B)_{\{C'_i/W_i\}_i} \to D$ is provable,
3b) for $j = 1, \ldots, n$, $V(C'_j) \subset V(W_j) \cap [V(\Gamma(B)_{\{-/W_i\}_i}) \cup V(D)]$.

Now, for $i \neq k$ by applying $(\to \wedge)$ to $U_i \to C_i$ and $W_i \to C'_i$, we can get $U_i, W_i \to C_i \wedge C'_i$. Note that when $i \neq k$, $U_i = W_i = Z_i$. Then, by applying $(E-contraction)$ to this sequent we can get $Z_i \to C_i \wedge C'_i$;

$$\cfrac{\cfrac{U_i \to C_i \quad W_i \to C'_i}{U_i, W_i \to C_i \wedge C'_i} (\wedge \to)}{Z_i \to C_i \wedge C'_i} (E-contraction).$$

By applying $(\to \vee 1)$ to $U_k \to C_k$ and $(\to \vee 2)$ to $W_k \to C'_k$ we can get $U_k \to C_k \vee C'_k$ and

$W_k \to C_k \vee C_k'$, respectively. Then by applying $(\vee \to)$ to them we can get $Z_k \to C_k \vee C_k'$;

$$\dfrac{\dfrac{Z_{k\{A/A\vee B\}} \to C_k}{Z_{k\{A/A\vee B\}} \to C_k \vee C_k'}(\to \vee 1) \quad \dfrac{Z_{k\{B/A\vee B\}} \to C_k'}{Z_{k\{B/A\vee B\}} \to C_k \vee C_k'}(\to \vee 2)}{Z_{k\{A\vee B/A\vee B\}} \to C_k \vee C_k'}(\vee \to).$$

So by 1a) and 1b), $Z_k \to C_k \vee C_k'$ and $Z_i \to C_i \wedge C_i'$ are provable when $i \neq k$.

Next, from $\Gamma(A)_{\{C_i/U_i\}_i} \to D$, by applying $(E - weakening)$ and $(\wedge \to)$, $n-1$ times, we can get $\Gamma(A)_{\{E_i/U_i\}_i} \to D$, where $E_k = C_k$ and $E_i = C_i \wedge C_i'$ when $i \neq k$. Also, from $\Gamma(B)_{\{C_i'/W_i\}_i} \to D$, by applying $(E - weakening)$, $(E - exchange)$ and $(\wedge \to)$, $n-1$ times, we can get $\Gamma(B)_{\{E_i'/W_i\}_i} \to D$, where $E_k' = C_k'$ and $E_i' = C_i \wedge C_i'$ when $i \neq k$. Note again that when $i \neq k$, $U_i = W_i = Z_i$. Then by applying $(\vee \to)$ to $\Gamma(A)_{\{E_i/U_i\}_i} \to D$ and $\Gamma(B)_{\{E_i'/W_i\}_i} \to D$ we can get $\Gamma(A \vee B)_{\{E_i''/Z_i\}_i} \to D$, where $E_k'' = C_k \vee C_k'$ and $E_i'' = E_i' = E_i = C_i \wedge C_i'$ when $i \neq k$. So by 2a) and 2b), we can get the proof of $\Gamma(A \vee B)_{\{E_i''/Z_i\}_i} \to D$ as follows;

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma(A)_{\{C_i/U_i\}_i} \to D}}{\text{applications of } (E - weakening)}}{\dfrac{\text{applications of } (\wedge \to)}{\Gamma(A)_{\{E_i/U_i\}_i} \to D}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma(B)_{\{C_i'/W_i\}_i} \to D}}{\text{applications of } (E - weakening)}}{\text{applications of } (E - exchange)}}{\dfrac{\text{applications of } (\wedge \to)}{\Gamma(B)_{\{E_i'/W_i\}_i} \to D}}}{\Gamma(A \vee B)_{\{E_i''/Z_i\}_i} \to D}(\vee \to).$$

Lastly, by 3a) and 3b) we can easily show that
a) for $h = 1, \ldots, k-1, k+1, \ldots, n$, $V(C_h \wedge C_h') \subset V(Z_h) \cap [V(\Gamma(A \vee B)_{\{-/Z_i\}_i}) \cup V(D)]$,
b) $V(C_k \vee C_k') \subset V(Z_k) \cap [V(\Gamma(A \vee B)_{\{-/Z_i\}_i}) \cup D]$.

Thus $C_h \wedge C_h'$ for $h = 1, \ldots, k-1, k+1, \ldots, n$, and $C_k \vee C_k'$ become the *interpolants.*

The proof of Theorem 1 for $L_\diamond L_{\mathrm{DBCK}}$ goes similarly to the above proof of Theorem 1 for $L_\diamond L_{\mathrm{DBCC}}$. In fact as we define intensional structures by multisets, we can omit some subcases in the proof.

Corollary 2 *The interpolation theorem holds for L_{DBCC} and L_{DBCK}. More precisely, if the formula $A \supset B$ is provable (in L_{DBCC} or L_{DBCK}), then there is a formula C such that both $A \supset C$ and $C \supset B$ are provable and $V(C) \subset [V(A) \cap V(B)]$.*

As an important application of Theorem 1, we can get the following theorem, which says that the Maksimova's principle of variable separation holds for $L_{\diamond}L_{\mathrm{DBCC}}$ and $L_{\diamond}L_{\mathrm{DBCK}}$. The detail of the proof will be announced in [8]. In fact, our interpolation theorem in a stronger form is necessary for proving this.

Theorem 3 *Suppose that $A_1 \supset A_2$ and $B_1 \supset B_2$ have no propositional variables in common. Then the following holds for $L_{\diamond}L_{\mathrm{DBCC}}$ and $L_{\diamond}L_{\mathrm{DBCK}}$.*
1) if the sequent $A_1 \wedge B_1 \to A_2 \vee B_2$ is provable, then either $A_1 \to A_2$ or $B_1 \to B_2$ is provable,
2) if the sequent $A_1 \wedge B_1 \to A_2$ is provable, then either $A_1 \to A_2$ or $B_1 \to$ is provable,
3) if the sequent $A_1 \to A_2 \vee B_2$ is provable, then either $A_1 \to A_2$ or $\to B_2$ is provable.

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