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Semigroup semantics for orthomodular logic

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Abstract

Quantum logic is usually considered as a logic which is based on orthomodular lattices. Here we introduce a different type of semantics, in which we use particular semigroups, and show that these two ways of interpretation of formulas are equivalent.

0 Basic notions

First we will give here some basic notions.
The language of our logics consists of :

(i) a countable collection $\{ p_i \mid i < \omega \}$ of propositional variables,

(ii) the connectives $\neg$ and $\wedge$ of negation and conjunction,

(iii) parentheses (and).

The set Φ of formulas is defined in the usual way. That is, Φ is the minimum set which satisfies the following three conditions:

(i) for every $i < \omega$, $p_i \in \Phi$,

(ii) if $\alpha \in \Phi$, then $(\neg\alpha) \in \Phi$,

(iii) if $\alpha, \beta \in \Phi$, then $(\alpha \wedge \beta) \in \Phi$.

The letters α, β, etc. are used as metavariables ranging over Φ. Parentheses may be omitted by the convention that $\neg$ binds strongly than $\wedge$. The disjunction $\alpha \vee \beta$ of α and β can be introduced as the abbreviation of $\neg(\neg\alpha \wedge \neg\beta)$.

Definition 0.1 (Orthomodular lattice) An *orthomodular lattice* $\mathcal{A}$ is a structure $\langle$ A, $\leq$, $\sqcap$, $\sqcup$, $^\perp$, **1**, **0** $\rangle$, which satisfies the following conditions:

(i) $\langle$ A, $\leq$, $\sqcap$, $\sqcup$, **1**, **0** $\rangle$ is a lattice with **1**(maximum) and **0**(minimum). We denote, for any $x, y \in$ A, $\quad x \sqcap y := \inf \ \{x, y\}, \quad x \sqcup y := \sup \ \{x, y\}$.

(ii) The unary operation $^\perp$ (*orthocomplement*) satisfies the following conditions, (a), (b) and (c): for any $x, y \in$ A,

(a) $x \sqcap x^\perp = \mathbf{0}$

(b) $x^{\perp\perp} = x$

(c) $x \leq y \quad$ implies $\quad y^\perp \leq x^\perp$

(d) $x \leq y$ implies $y = x \sqcup (x^{\perp} \sqcap y)$

It is easy to see that $x \sqcup y = (x^{\perp} \sqcap y^{\perp})^{\perp}$ holds in any orthomodular lattice. ∎

Definition 0.2 (Valuation) A valuation is a function v, which associates with any formula $\alpha \in \Phi$ an element $v(\alpha)$ in an orthomodular lattice $\mathcal{A}$, and satisfies the following conditions:
for any formula α, β,

(i) $v(\neg\alpha) = (v(\alpha))^{\perp}$

(ii) $v(\alpha \wedge \beta) = v(\alpha) \sqcap v(\beta)$

We call this v an *orthomodular valuation.* ∎

It is easy to see that for any valuation v and for any formula α, the value $v(\alpha)$ is uniquely determined by the values $v(p_i)$ for propositional variables p_i appearing in α.

Definition 0.3 (Orthomodular logic) The *orthomodular logic* L is the set of pairs of formulas (α, β) satisfying the following conditions: for any orthomodular lattice $\mathcal{A}$ and for any orthomodular valuation v from Φ to A, $v(\alpha) \leq v(\beta)$. We denote $\alpha \vdash_L \beta$ in place of $(\alpha, \beta) \in L$. ∎

R.I Goldblatt proposed his "quantum model" for orthomodular logic in 1974[1].

Definition 0.4 (Quantum frame and quantum model) $\mathcal{F} = \langle \mathrm{X}, \perp, \xi \rangle$ is a *quantum frame* if it satisfies the following conditions (i),(ii) and (iii).

(i) X is a nonempty set.

(ii) $\perp$ is an irreflexive and symmetric binary relation. (*orthogonality relation*)

- For P $\subseteq$ X, $x \perp$P means that $x \perp y$ for all $y \in$ P.
- P ($\subseteq$ X) is $\perp$-*closed* iff the following condition holds:
$$\forall x \in \mathrm{X}(x \notin \mathrm{P}), \exists y \in \mathrm{X}\ [y \perp \mathrm{P} \text{ and not}(y \perp x)]$$
- P ($\subseteq$ X) is $\perp$-*closed* in Q (Q $\subseteq$ X) iff the following condition holds:
$$\forall x \in \mathrm{Q}(x \notin \mathrm{P}), \exists y \in \mathrm{Q}\ [y \perp \mathrm{P} \text{ and not}(y \perp x)]$$

(iii) ξ is a nonempty collection of $\perp$-closed subsets of X, such that

(a) ξ is closed under set-inclusion and the following operation †.
$$\mathrm{P}^{\dagger} = \{x \in \mathrm{X} | x \perp \mathrm{P}\}$$

(b) For any P, Q in ξ, if P $\subseteq$ Q then P is $\perp$-closed in Q.

$\mathcal{Q} = \langle \mathrm{X}, \perp, \xi, \mathrm{V} \rangle$ is a *quantum model* if it satisfies the following:

(i) $\mathcal{F} = \langle \mathrm{X}, \perp, \xi \rangle$ is a quantum frame.

(ii) V is a function assigning to each propositional variables p_i a member V(p_i) of ξ.

The notion of truth in quantum models is defined inductively as follows: the symbol '$\mathcal{Q}\models_x \alpha$' is read as "formula α is true at x in $\mathcal{Q}$".

(i) $\mathcal{Q}\models_x p_i$ iff $p_i \in \mathrm{V}(p_i)$,

(ii) $\mathcal{Q}\models_x \alpha \wedge \beta$ iff $\mathcal{Q}\models_x \alpha$ and $\mathcal{Q}\models_x \beta$,

(iii) $\mathcal{Q}\models_x \neg\alpha$ iff for any $y \in \mathrm{X}$, ($\mathcal{Q}\models_y \alpha \Rightarrow x \perp y$).

- α *implises* β in a model $\mathcal{Q}$ iff for all x in the model $\mathcal{Q}$, either $\mathcal{Q}\models_x \alpha$ does not hold, or $\mathcal{Q}\models_x \beta$ holds.

∎

Using his quantum models, Goldblatt showed the following completeness theorem.

Theorem 0.5 (Completeness Theorem) For given formulas α and β, the statements (P) and (Q) are mutually equivalent, that is

(P): for any orthomodular lattice $\mathcal{A}$ and any valuation $v : \Phi \to \mathrm{A}$, $v(\alpha) \leq v(\beta)$ holds.

(Q): for any quantum model $\mathcal{Q}$, $\mathcal{Q} : \alpha \models \beta$ holds.

□

In study of orthomodular lattice, D.J.Foulis [2] found in 1960 the following representation theorem for orthomodular lattices with a particular kind of semigroups.

Theorem 0.6 (Foulis's representation theorem) Let $\mathcal{A}$ be an orthomodular lattice. Then $\mathcal{G}(\mathcal{A}) = \langle \mathrm{G(A)}, \cdot, * \rangle$ is a Rickart * semigroup and A is isomorphic to $\mathrm{P}_c(\mathrm{G(A)})$. □

We will give another type of models for orthomodular logic using this representation theorem.

1 Rickart * semigroups

Now we introduce a special type of semigroups called Rickart * semigroups and lead some properties of them.

Definition 1.1 (Rickart * semigroups) A Rickart * semigroup is a structure $\mathcal{G} = \langle \mathrm{G}, \cdot, * \rangle$ which satisfies the following conditions (i), (ii), (iii) and (iv).

(i) $\langle \mathrm{G}, \cdot \rangle$ is a semigroup, that is,
   (a) $\cdot$ is a binary operation on G.
   (b) For any $x, y, z \in \mathrm{G}$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

(ii) There exists the unique element 0 (zero element) in G such that $0 \cdot x = x \cdot 0 = 0$ holds for any $x \in \mathrm{G}$.

(iii) $*$ is a unary operation on G, which satisfies the following:
   For any $x, y \in \mathrm{G}$, (a): $(x^*)^* = x$. (b): $(x \cdot y)^* = y^* \cdot x^*$.

Before introducing the conditon (iv), it is necessary to introduce some other notions.

- An element $e \in \mathrm{G}$ is called a *projection* iff it satisfies $e^* = e \cdot e = e$.
  We denote the set of all projections in G by P(G).

- For an element $x \in \mathrm{G}$, the set $\{\ x\ \}^{(r)} := \{\ y \in \mathrm{G} \mid x \cdot y = 0\ \}$ is called the *right annihilator* for x.

By using these two notions, we formulate the conditon (iv) as follows:

(iv) For any $x \in \mathrm{G}$, there exists a projection e such that the right annihilator for x can be expressed as : $\{\ x\ \}^{(r)} = e \cdot \mathrm{G} = \{\ e \cdot y \mid y \in \mathrm{G}\}$. We call this e a *right annihilating projection* for x.

∎

Lemma 1.2 (Properties of P(G)) Let $\mathcal{G} = \langle \mathrm{G}, \cdot, * \rangle$ be a Rickart * semigroup.

(i) For any $x \in \mathrm{G}$, the right annihilating projection for x is uniquely determined. Hereafter, this will be written as x^{r}.

(ii) There is the unit element in G, that is, an element 1 satisfying that for any $x \in \mathrm{G}$, $x \cdot 1 = 1 \cdot x = x$.

(iii) Both 0 and 1 are projections.

(iv) For any $e, f \in \mathrm{P(G)}$, the following three conditions are equivalent.
(a) $e \cdot f = e$.
(b) $f \cdot e = e$.
(c) $e \cdot \mathrm{G} \subseteq f \cdot \mathrm{G}$.

Proof :

(i) Using the properties of the operation $*$.

(ii) We can show that 0^{r} is the unit element 1.

(iii) By operating $*$ to both sides of the equation $0 = 0 \cdot 0^*$, we get that $0^* = 0$. Similarly we can show that $1^* = 1$.

(iv) Not so hard.

□

The above Lemma 1.2 (iv) assures us the possibility of introducing a partially order on P(G).

Definition 1.3 (Order on P(G)) Let $\mathcal{G} = \langle \mathrm{G}, \cdot, * \rangle$ be a Rickart * semigroup. Define a partial order $\leq$ on P(G) as follows: for $e, f \in \mathrm{P(G)}$, $e \leq f$ iff $e \cdot f = e$. ∎

It is obvious that 1 is the maximum and that 0 is the minimum with respect to this order. Hence P(G) can be regarded as a bounded partial ordered set.

In the proof of Lemma 1.2, we have defined the unary operation $^{\mathrm{r}}$ from G to P(G). Here we will see some of the basic properties of the operation $^{\mathrm{r}}$in detail, which will be used in the later discussion.

Lemma 1.4 (Properties of the operation $^{\mathrm{r}}$) Let $\mathcal{G} = \langle \mathrm{G}, \cdot, * \rangle$ be a Rickart * semigroup. For any $x, y \in \mathrm{G}$ and for any $e, f \in \mathrm{P(G)}$, the following statements can be verified.

(i)	$0^{\mathrm{r}} = 1$, and $1^{\mathrm{r}} = 0$.	(v)	If $e \leq f$, then $f^{\mathrm{r}} \leq e^{\mathrm{r}}$.
(ii)	$x \cdot x^{\mathrm{r}} = 0$, and $x^{\mathrm{r}} \cdot x^* = 0$.	(vi)	$x = x \cdot x^{\mathrm{rr}}$, and $e \leq e^{\mathrm{rr}}$.
(iii)	If $x \cdot e = 0$, then $e \leq x^{\mathrm{r}}$.	(vii)	$x^{\mathrm{r}} = x^{\mathrm{rrr}}$.
(iv)	$x^{\mathrm{r}} \leq (y \cdot x)^{\mathrm{r}}$.	(viii)	If $e \cdot x = x \cdot e$, then $e^{\mathrm{r}} \cdot x = x \cdot e^{\mathrm{r}}$.

Proof : Here we prove only (vi) and (viii). Rest is not so hard.

(vi) By (ii), $x^* \in \{x^{\mathrm{r}}\}^{(\mathrm{r})} = x^{\mathrm{rr}} \cdot \mathrm{G}$. Then there exists some $s \in \mathrm{G}$, such that $x^* = x^{\mathrm{rr}} \cdot s$. By operating * to this equation, we have that $x = x^{**} = s^* \cdot x^{\mathrm{rr}*} = s^* \cdot x^{\mathrm{rr}}$. Further operating x^{rr} from the right to the equation $x = s^* \cdot x^{\mathrm{rr}}$, we can derive that $x \cdot x^{\mathrm{rr}} = (s^* \cdot x^{\mathrm{rr}}) \cdot x^{\mathrm{rr}} = s^* \cdot x^{\mathrm{rr}} = x$. In particular, when x is equal to a projection e, we have that $e \cdot e^{\mathrm{rr}} = e$, that is, $e \leq e^{\mathrm{rr}}$.

(viii) Suppose that $e \cdot x = x \cdot e$. Then we have $e \cdot x \cdot e^{\mathrm{r}} = x \cdot e \cdot e^{\mathrm{r}} = 0$, since $e \cdot e^{\mathrm{r}} = 0$. So $x \cdot e^{\mathrm{r}} \in \{e\}^{(\mathrm{r})} = e^{\mathrm{r}} \cdot \mathrm{G}$, and there exists some $s \in \mathrm{G}$ satisfying that $x \cdot e^{\mathrm{r}} = e^{\mathrm{r}} \cdot s$. By multiplying e^{r} from the left to both sides of this equation, we have that

$$e^{\mathrm{r}} \cdot x \cdot e^{\mathrm{r}} = e^{\mathrm{r}} \cdot e^{\mathrm{r}} \cdot s = e^{\mathrm{r}} \cdot s = x \cdot e^{\mathrm{r}} \qquad \cdots (1)$$

On the other hand, by operating * to the supposition $e \cdot x = x \cdot e$, so we have that $x^* \cdot e = e \cdot x^*$. Then $e \cdot x^* \cdot e^{\mathrm{r}} = x^* \cdot e \cdot e^{\mathrm{r}} = 0$, which means that $x^* \cdot e^{\mathrm{r}} \in \{e\}^{(\mathrm{r})} = e^{\mathrm{r}} \cdot \mathrm{G}$. So there exists some $t \in \mathrm{G}$ such that $x^* \cdot e^{\mathrm{r}} = e^{\mathrm{r}} \cdot t$. By multiplying e^{r} from the left to both sides of this equation, we have that $e^{\mathrm{r}} \cdot x^* \cdot e^{\mathrm{r}} = e^{\mathrm{r}} \cdot e^{\mathrm{r}} \cdot t = e^{\mathrm{r}} \cdot t = x^* \cdot e^{\mathrm{r}}$. Further operating * again, we get that

$$e^{\mathrm{r}} \cdot x \cdot e^{\mathrm{r}} = e^{\mathrm{r}} \cdot x \qquad \cdots (2).$$

From (1) and (2), we can conclude that $x \cdot e^{\mathrm{r}} = e^{\mathrm{r}} \cdot x$.

□

Now we will consider a particular class of projections, called *closed projections.*

Definition 1.5 (Closed projection) A projection $f \in \mathrm{P(G)}$ is called *closed* iff there exists an element $x \in \mathrm{G}$ such that f is the right annihilating projection for x. This means that a closed projection f can be written as $f = x^{\mathrm{r}}$ for some element $x \in \mathrm{G}$. We denote the set of all closed projections in G by $\mathrm{P}_c(\mathrm{G})$. ▮

In other words, the set $\mathrm{P}_c(\mathrm{G})$ is the range of the function $^{\mathrm{r}}$ from G to P(G). We give here a necessary and sufficient condition on a projection to be closed.

Proposition 1.6 For any $e \in \mathrm{P(G)}$, $e \in \mathrm{P}_c(\mathrm{G})$ if and only if $e^{\mathrm{rr}} = e$. □

We will show that in $\mathrm{P}_c(\mathrm{G})$ we can always find the supremum and the infimum of any two elements of it and hence this partially ordered set forms a lattice. Moreover we can show that $\mathrm{P}_c(\mathrm{G})$ is an orthomodular lattice.

Lemma 1.7 (Existence of meet in $\mathrm{P}_c(\mathrm{G})$)

(i) For any closed projections e and f such that $e \cdot f = f \cdot e$, $e \cdot f \in \mathrm{P}_c(\mathrm{G})$ holds, and there exists the infimum $(e \sqcap f)$ of e, f, which satisfies the equation $e \sqcap f = e \cdot f$.

(ii) In general, for any closed projections e and f, there exists the infimum $(e \sqcap f)$ of e, f and the equation $e \sqcap f = e \cdot (f^{\mathrm{r}} \cdot e)^{\mathrm{r}} = (f^{\mathrm{r}} \cdot e)^{\mathrm{r}} \cdot e = e \sqcap (f^{\mathrm{r}} \cdot e)^{\mathrm{r}}$ holds.

Proof :

(i) Suppose that $e \cdot f = f \cdot e$. We show that $e \cdot f \in \mathrm{P}_c(\mathrm{G})$. Since $e, f \in \mathrm{P}(\mathrm{G})$ and $e \cdot f = f \cdot e$, we can derive:

$$(e \cdot f)^* = f^* \cdot e^* = f \cdot e = e \cdot f, \quad \text{and} \quad (e \cdot f) \cdot (e \cdot f) = e \cdot e \cdot f \cdot f = e \cdot f.$$

Thus, $e, f \in \mathrm{P}(\mathrm{G})$. To prove that $e \cdot f \in \mathrm{P}_c(\mathrm{G})$, by Proposition 1.5, it is enough to show that $(e \cdot f)^{\mathrm{rr}} = e \cdot f$. Then we have only to show that $(e \cdot f)^{\mathrm{rr}} \leq e \cdot f$ as the converse inequality holds always by Lemma 1.4 (vi). Considering the Lemma 1.4 (iv), we have that $e^{\mathrm{r}} \leq (e \cdot f)^{\mathrm{r}}$. Then by the Lemma 1.4 (v), we can derive that $(e \cdot f)^{\mathrm{rr}} \leq e^{\mathrm{rr}} = e$, which means $e \cdot (e \cdot f)^{\mathrm{rr}} = (e \cdot f)^{\mathrm{rr}}$. Similarly we can derive that $f \cdot (e \cdot f)^{\mathrm{rr}} = (e \cdot f)^{\mathrm{rr}}$. Therefore $e \cdot f \cdot (e \cdot f)^{\mathrm{rr}} = e \cdot (e \cdot f)^{\mathrm{rr}} = (e \cdot f)^{\mathrm{rr}}$. Thus $(e \cdot f)^{\mathrm{rr}} \leq e \cdot f$.

It is easy to see that $e \cdot f$ is the infimum of e and f.

(ii) We put $u := f^{\mathrm{r}} \cdot e$. By Lemma 1.4 (iv), we have that $e^{\mathrm{r}} \leq (f^{\mathrm{r}} \cdot e)^{\mathrm{r}} = u^{\mathrm{r}}$. This means that $e^{\mathrm{r}} \cdot u^{\mathrm{r}} = e^{\mathrm{r}} = u^{\mathrm{r}} \cdot e^{\mathrm{r}}$. By applying Lemma 1.4 (viii), we have that $e \cdot u^{\mathrm{r}} = u^{\mathrm{r}} \cdot e$. Then by (i) of the present lemma, we can conclude that $e \cdot u^{\mathrm{r}} \in \mathrm{P}_c(\mathrm{G})$, and that $e \sqcap u^{\mathrm{r}} = e \cdot u^{\mathrm{r}}$. So it remains to show that $e \sqcap f = e \cdot u^{\mathrm{r}}$.

(a) Clearly, $e \cdot (e \cdot u^{\mathrm{r}}) = e \cdot u^{\mathrm{r}}$. So we have $e \cdot u^{\mathrm{r}} \leq e$. On the other hand, $f^{\mathrm{r}} \cdot e \cdot u^{\mathrm{r}} = f^{\mathrm{r}} \cdot e \cdot (f^{\mathrm{r}} \cdot e)^{\mathrm{r}} = 0$. So from Lemma 1.4 (iii), we derive that $e \cdot u^{\mathrm{r}} \leq f^{\mathrm{rr}} = f$. Thus $e \cdot u^{\mathrm{r}}$ is a lower bound of $\{e, f\}$.

(b) Take any $g \in \mathrm{P}_c(\mathrm{G})$ such that $g \cdot e = g$ and $g \cdot f = g$. Then because $f \cdot f^{\mathrm{r}} = 0$, we have that $g \cdot f \cdot f^{\mathrm{r}} \cdot e = 0$. By our assumption on g, $g \cdot f^{\mathrm{r}} \cdot e = 0$, which means that $g \cdot u = 0$. By Lemma 1.4 (iii), we can derive that $u \leq g^{\mathrm{r}}$. So by Lemma 1.4 (v), $g = g^{\mathrm{rr}} \leq u^{\mathrm{r}}$. This is equivalent to $g \cdot u^{\mathrm{r}} = g$. Again using the assumption on g, $g \cdot e \cdot u^{\mathrm{r}} = g$. So we have derived that $g \leq e \cdot u^{\mathrm{r}}$.

Thus we have shown that $e \sqcap f = e \cdot u^{\mathrm{r}}$.

□

Therefore we have the following Proposition.

Proposition 1.8 For any $e, f \in \mathrm{P}_c(\mathrm{G})$, the following equation holds:

$$e \cdot \mathrm{G} \cap f \cdot \mathrm{G} = (e \sqcap f) \cdot \mathrm{G}.$$

□

Next we will see that $\mathrm{P}_c(\mathrm{G})$ is an orthomodular lattice.

Theorem 1.9 $\mathrm{P}_c(\mathrm{G})$ forms an orthomodular lattice, where the orthocomplement is the operation $^{\mathrm{r}}$.

Proof : We can easily check the conditions in Definition 0.1 . □

Next, in Section 2, we will introduce a semantics for orthomodular logic by using Rickart * semigroups, and prove the soundness.

2 Semigroup semantics and soundness theorem

Definition 2.1 **(Orthomodular model)** $\mathcal{M} = \langle \mathcal{G}, \mathrm{u} \rangle$ is a *orthomodular model* (OM model for short) iff $\mathcal{G} = \langle \mathrm{G}, \cdot, * \rangle$ is a Rickart * semigroup and u is a function assigning to each propositional variable p_i an element $\mathrm{u}(p_i)$ of $\mathrm{P}_c(\mathrm{G})$.

The notion of truth in OM models is defined inductively as follows: the symbol '$(\mathcal{M}, x) \models \alpha$' is read as " a formula α is true at x in $\mathcal{M}$".

(i) $(\mathcal{M}, x) \models p_i$ iff $x \in \mathrm{u}(p_i) \cdot \mathrm{G}$.

(ii) $(\mathcal{M}, x) \models \alpha \wedge \beta$ iff $(\mathcal{M}, x) \models \alpha$ and $(\mathcal{M}, x) \models \beta$.

(iii) $(\mathcal{M}, x) \models \neg\alpha$ iff $\forall y \in \mathrm{G}$, [$(\mathcal{M}, y) \models \alpha$ only if $y^* \cdot x = 0$].

∎

For each formula α, define $\|\alpha\|^{\mathcal{M}} := \{x \in \mathrm{G} \mid (\mathcal{M}, x) \models \alpha\}$. Then we can restate the above conditions in the following way:

(i) $\|p_i\|^{\mathcal{M}} = \mathrm{u}(p_i) \cdot \mathrm{G}$.

(ii) $\|\alpha \wedge \beta\|^{\mathcal{M}} = \|\alpha\|^{\mathcal{M}} \cap \|\beta\|^{\mathcal{M}}$.

(iii) $\|\neg\alpha\|^{\mathcal{M}} = \{x \in \mathrm{G} \mid \forall y \in \|\alpha\|^{\mathcal{M}} (y^* \cdot x = 0) \}$.

Definition 2.2 Let α and β be formulas.

(i) α *implies* β at x in an OM model $\mathcal{M}$ ($(\mathcal{M}, x) : \alpha \models \beta$) iff either $(\mathcal{M}, x) \models \alpha$ does not hold or $(\mathcal{M}, x) \models \beta$ holds.

(ii) α *implies* β in an OM model $\mathcal{M}$ ($\mathcal{M} : \alpha \models \beta$) iff for all x in the model $\mathcal{M}$, $(\mathcal{M}, x) : \alpha \models \beta$ holds.

∎

It is easy to see that $\mathcal{M} : \alpha \models \beta$ is equivalent to $\|\alpha\|^{\mathcal{M}} \subseteq \|\beta\|^{\mathcal{M}}$.

Lemma 2.3 Let $\mathcal{M} = \langle \mathcal{G}, \mathrm{u} \rangle$ be an orthomodular model and e such an orthomodular valuation from Φ to $\mathrm{P}_c(\mathrm{G})$ that $e(p_i) = \mathrm{u}(p_i)$ holds for all propositional variables. Then for any formula α, $\|\alpha\|^{\mathcal{M}} = e(\alpha) \cdot \mathrm{G}$ holds.

Proof : Induction on the construction of the formula α. □

Now we can prove the soundness theorem.

Theorem 2.4 (Soundness theorem) For given formulas α and β, let (S) and (T) be the statements as follows:
(S): for any orthomodular lattice $\mathcal{A}$ and any orthomodular valuation $v : \Phi \to \mathrm{A}$, $v(\alpha) \leq v(\beta)$.
(T): for any orthomodular model $\mathcal{M}$, $\mathcal{M} : \alpha \models \beta$.
Then (S) implies (T). □

3 Monotone, residuated maps on an ordered set

Next, we will prove the Completeness Theorem. To show the direction ((S) $\Leftarrow$ (T)), we need to know how to build up an orthomodular model from a given orthomodular lattice. To do this, we need some preparations.

Definition 3.1 (Residuated, monotone maps on an ordered set) Let $\langle \mathrm{A}, \leq \rangle$ be an ordered set.

(i) A map φ from A to A is called *monotone* iff it satisfies the following condition: for any $x, y \in \mathrm{A}$, if $x \leq y$, then $\varphi(x) \leq \varphi(y)$.

We denote the set of all monotone maps from A to A by $\overline{\mathrm{G}}(\mathrm{A})$.

(ii) A map $\varphi \in \overline{\mathrm{G}}(\mathrm{A})$ is called *residuated* iff there exists a map $\varphi^{\sharp} \in \overline{\mathrm{G}}(\mathrm{A})$ such that for any $x \in \mathrm{A}$, $\varphi^{\sharp}(\varphi(x)) \geq x$ and $\varphi(\varphi^{\sharp}(x)) \leq x$.

We call this map $\varphi^{\sharp}$ a residual map for φ, and denote the set of all residuated, monotone maps on A by G(A). ∎

Lemma 3.2 (Properties of residual maps) Let $\langle \mathrm{A}, \leq \rangle$ be an ordered set. Then the following holds.

(i) For any $\varphi \in \mathrm{G}(\mathrm{A})$, the residual map for φ is uniquely determined.

(ii) For any $\varphi, \psi \in \mathrm{G}(\mathrm{A})$, $(\varphi \cdot \psi)^{\sharp} = \psi^{\sharp} \cdot \varphi^{\sharp}$ holds, where $\cdot$ means the composition operator for maps. Therefore G(A) is closed under this operation $\cdot$.

Proof : Using the monotonicity and the inequations which hold for $\varphi \in \mathrm{G}(\mathrm{A})$ and its residual map $\varphi^{\sharp}$. □

It is guaranteed by (i) of Lemma 3.2 that we can write the resudual map for φ as $\varphi^{\sharp}$. And (ii) of Lemma 3.2 means that G(A) is a semigroup with respect to the operation $\cdot$.

Lemma 3.3 Let $\langle A, \leq, \mathbf{0}, \mathbf{1}\rangle$ be an ordered set with the minimun element $\mathbf{0}$ and the maximum element $\mathbf{1}$ and let θ be a map defined by the condition: for all $x \in A$, $\theta(x) = \mathbf{0}$. Then θ is the zero element in the semigroup G(A). □

Lemma 3.4 Let $\mathcal{A} = \langle A, \leq, \sqcap, \sqcup, {}^{\perp}, \mathbf{1}, \mathbf{0}\rangle$ be an ortholattice. Let $*$ be defined by the following: for any $\varphi \in G(A)$, $\varphi^*(x) := (\varphi^{\sharp}(x^{\perp}))^{\perp}$ for any $x \in A$. Then $\varphi^* \in G(A)$. Moreover the following condtions hold for every $\varphi, \psi \in G(A)$.

(a) $\varphi^{**} = \varphi$.

(b) $(\varphi \cdot \psi)^* = \psi^* \cdot \varphi^*$.

Proof : We put $\psi(x) := (\varphi(x^{\perp}))^{\perp}$ for any $x \in A$ and show that $\psi = \varphi^{*\sharp}$.

(i) First we will show that ψ is monotone. Suppose that $x \leq y$ for $x, y \in A$. Then by the properties of the operation ${}^{\perp}$, we have $x^{\perp} \geq y^{\perp}$. Since φ is monotone, we have $\varphi(x^{\perp}) \geq \varphi(y^{\perp})$. Again by the properties of ${}^{\perp}$, we have $(\varphi(x^{\perp}))^{\perp} \leq (\varphi(y^{\perp}))^{\perp}$, which means $\psi(x) \leq \psi(y)$. Therefore ψ is monotone.

(ii) Next we will show that ψ is the residual map for φ. By the properties of the operation ${}^{\perp}$ and the properties of $\varphi^{\sharp}$, we can derive: $\psi \cdot \varphi^*(x) = \psi \cdot (\varphi^{\sharp}(x^{\perp}))^{\perp} = [\varphi(\varphi^{\sharp}(x^{\perp}))^{\perp\perp}]^{\perp} = [\varphi(\varphi^{\sharp}(x^{\perp}))]^{\perp} \geq x^{\perp\perp} = x$. So we have $\psi \cdot \varphi^*(x) \geq x$. Similarly we can derive: $\varphi^* \cdot \psi(x) = \varphi^* \cdot (\varphi(x^{\perp}))^{\perp} = [\varphi^{\sharp}(\varphi(x^{\perp}))^{\perp\perp}]^{\perp} = [\varphi^{\sharp}(\varphi(x^{\perp}))]^{\perp} \leq x^{\perp\perp} = x$. So we have $\varphi^* \cdot \psi(x) \leq x$.

Hence we can conclude that $\psi = \varphi^{*\sharp}$ since the residual map of φ^* is unique. By (i) and (ii) in the above, we have that $\varphi^* \in G(A)$. Thus $*$ is a unary operator on G(A). Now we will check the conditions (a) and (b). By the properties of the operation $\perp$, and the definition of φ^*, we calculate as follows: for any φ, ψ, and for any $x \in A$,

(a): $\varphi^{**}(x) = [\varphi^{*\sharp}(x^{\perp})]^{\perp} = [(\varphi(x^{\perp\perp})^{\perp}]^{\perp} = \varphi(x)$.

(b): $\psi^* \cdot \varphi^*(x) = \psi^*(\varphi^{\sharp}(x^{\perp}))^{\perp} = [\psi^{\sharp}(\varphi^{\sharp}(x^{\perp}))^{\perp\perp}]^{\perp} = [\psi^{\sharp} \cdot \varphi^{\sharp}(x^{\perp})]^{\perp} = [(\varphi \cdot \psi)^{\sharp}(x^{\perp})]^{\perp} = (\varphi \cdot \psi)^*(x)$.

Consequently this $*$ satisfies conditions for the operator $*$ in Rickart $*$ semigroups. □

From the above consideration, we can define the notions of projection, closed projection and right annihilator for an element in G(A). In order to get a Rickart $*$ semigroup from G(A), we must show that for any element $\varphi \in G(A)$, there exists some closed projection μ such that $\{\varphi\}^{(r)} := \{\psi \in G(A) \mid \varphi \cdot \psi = \theta\} = \mu \cdot G(A)$.

Lemma 3.5 Let $\mathcal{A} = \langle A, \leq, \sqcap, \sqcup, {}^{\perp}, \mathbf{1}, \mathbf{0}\rangle$ be an orthomodular lattice. For each $a \in A$, define a map γ_a by $\gamma_a(x) := (x \sqcup a^{\perp}) \sqcap a$ for every $x \in A$.

(i) γ_a is a projection in G(A) for any $a \in A$.

(ii) For any $\varphi \in G(A)$, if we put $a := \varphi^{\sharp}(\mathbf{0})$, then $\{\varphi\}^{(r)} = \gamma_a \cdot G(A)$ holds.

Proof : By our assumption, the following orthomodular law holds. For $a, b, c \in \mathrm{A}$,
(1) $a \leq b$ implies $b = (b \sqcap a^{\perp}) \sqcup a$. (2) $c \leq a$ implies $c = (c \sqcup a^{\perp}) \sqcap a$.
It is easy to see that (2) follows from (1) and vice versa.

(i) First we will show that $\gamma_a \in \mathrm{G(A)}$. It is obvious that γ_a is monotone. We put $\psi(x) := (x \sqcap a) \sqcup a^{\perp}$ for any x in A. Clearly ψ is also monotone. Moreover, as shown below, it is the residual map for γ_a.

$$\begin{aligned} \gamma_a \cdot \psi(x) &= [((x \sqcap a) \sqcup a^{\perp}) \sqcup a^{\perp}] \sqcap a \\ &= [(x \sqcap a) \sqcup a^{\perp}] \sqcap a \\ &= x \sqcap a \leq x. \end{aligned}$$

In the last equation in the above, we used (2) since $x \sqcap a \leq a$.

$$\begin{aligned} \psi \cdot \gamma_a(x) &= [((x \sqcup a^{\perp}) \sqcap a) \sqcap a] \sqcup a^{\perp} \\ &= [(x \sqcup a^{\perp}) \sqcap a] \sqcup a^{\perp} \\ &= x \sqcup a^{\perp} \geq x \end{aligned}$$

Also, we used (1) since $x \sqcup a^{\perp} \geq a^{\perp}$.
Therefore $\gamma_a{}^{\sharp}(x) = \psi(x) = (x \sqcap a) \sqcup a^{\perp}$. So $\gamma_a \in \mathrm{G(A)}$.

Next we will show that γ_a satisfies the conditions for projections.

$$\begin{aligned} \gamma_a{}^{*} = (\gamma_a{}^{\sharp}(x^{\perp}))^{\perp} &= [(x^{\perp} \sqcap a) \sqcup a^{\perp}]^{\perp} \\ &= (x^{\perp} \sqcap a)^{\perp} \sqcap a^{\perp\perp} \\ &= (x \sqcup a^{\perp}) \sqcap a = \gamma_a(x) \end{aligned}$$

$$\begin{aligned} \gamma_a \cdot \gamma_a(x) &= [\{(x \sqcup a^{\perp}) \sqcap a\} \sqcup a^{\perp}] \sqcap a \\ &= (x \sqcup a^{\perp}) \sqcap a = \gamma_a(x) \end{aligned}$$

Since $(x \sqcup a^{\perp}) \sqcap a \leq a$, we used (2) in the above calculation. Thus γ_a is a projection.

(ii) First we will prove that $\gamma_a \cdot \mathrm{G(A)} \subseteq \{\varphi\}^{(\mathrm{r})}$. Take any $\psi \in \gamma_a \cdot \mathrm{G(A)}$. Then there exists some element $\lambda \in \mathrm{G(A)}$ such that $\psi = \gamma_a \cdot \lambda$. For any $x \in \mathrm{A}$, $\gamma_a(x) = (x \sqcup a^{\perp}) \sqcap a \leq a = \varphi^{\sharp}(\mathbf{0})$. So by the monotonicity of φ, we have that $\varphi \cdot \gamma_a(x) \leq \varphi \cdot \varphi^{\sharp}(\mathbf{0}) \leq \mathbf{0}$. This means that $\varphi \cdot \gamma_a = \theta$. Then $\varphi \cdot \psi = \varphi \cdot \gamma_a \cdot \lambda = \theta$, that is $\psi \in \{\varphi\}^{(\mathrm{r})}$.
Thus we conclude that $\gamma_a \cdot \mathrm{G(A)} \subseteq \{\varphi\}^{(\mathrm{r})}$. Next we will show that $\{\varphi\}^{(\mathrm{r})} \subseteq \gamma_a \cdot \mathrm{G(A)}$. Take any $\psi \in \{\varphi\}^{(\mathrm{r})}$.
Then ψ satisfies that $\varphi \cdot \psi = \theta$, which means that for any $x \in \mathrm{A}$, we have that $\varphi \cdot \psi(x) = \mathbf{0}$. Taking $\mathbf{1}$ for x, we have $\varphi \cdot \psi(\mathbf{1}) = \mathbf{0}$, and hence $a = \varphi^{\sharp}(\mathbf{0}) = \varphi^{\sharp} \cdot \varphi \cdot \psi(\mathbf{1}) \geq \psi(\mathbf{1})$. Therefore we have that for any $x \in \mathrm{A}$, $\psi(x) \leq \psi(\mathbf{1}) \leq a$. By combining this result with the orthomodular law (2), we have that $\gamma_a \cdot \psi(x) = (\psi(x) \sqcup a^{\perp}) \sqcap a = \psi(x)$. Consequently $\psi = \gamma_a \cdot \psi \in \gamma_a \cdot \mathrm{G(A)}$.
Thus we have proved $\{\varphi\}^{(\mathrm{r})} = \gamma_a \cdot \mathrm{G(A)}$.

□

Moreover, we can show the following lemma on the set of maps γ_a.

Lemma 3.6 For any orthomodular lattice $\mathcal{A} = \langle A, \leq, \sqcap, \sqcup, \perp, \mathbf{1}, \mathbf{0} \rangle$, the relation $P_c(G(A)) = \{\gamma_a \mid a \in A\}$ holds.

Proof : Take any $\lambda \in P_c(G(A))$. Then there exists some $\mu \in G(A)$ such that $\{\mu\}^{(r)} = \lambda \cdot G(A)$. Now putting $b := \mu^\sharp(\mathbf{0})$, we have $\{\mu\}^{(r)} = \gamma_b \cdot G(A)$ by Lemma 3.5 (ii). So the uniqueness of the right annihilating projection gives us that $\lambda = \gamma_b \in \{\gamma_a \mid a \in A\}$.
Conversely, consider γ_a for $a \in A$. Since γ_a is a projection, $\gamma_a = \gamma_a \cdot \gamma_a = \gamma_a{}^*$ holds. We have that $\gamma_a \cdot \gamma_a{}^r = \theta$. So by operating $*$ to this equation, we get $\gamma_a{}^r \cdot \gamma_a = \theta$. Then of course, $\gamma_a{}^r \cdot \gamma_a \cdot \lambda = \theta$ for any $\lambda \in G(A)$ holds. Therefore we get $\{\gamma_a{}^r\}^{(r)} = \gamma_a \cdot G(A)$. Thus $\gamma_a \in P_c(G(A))$.
Consequently we have proved that $P_c(G(A)) = \{\gamma_a \mid a \in A\}$. □

By all the lemmas 3.2, 3.3, 3.4 and 3.5, we can prove the following theorem.

Theorem 3.7 Let $\mathcal{A} = \langle A, \leq, \sqcap, \sqcup, \perp, \mathbf{1}, \mathbf{0} \rangle$ be an orthomodular lattice. Then $\mathcal{G}(\mathcal{A}) = \langle G(A), \cdot, * \rangle$ is a Rickart * semigroup, where $\cdot$ is a composition operator of maps and * is a unary operator defined in Lemma 3.3 .

4 Corresponding model and Completeness Theorem

Now we have prepared all the notions for constructing the corresponding model for orthomodular logic.

Definition 4.1 **(Corresponding model)** Let $\mathcal{A} = \langle A, \leq, \sqcap, \sqcup, \perp, \mathbf{1}, \mathbf{0} \rangle$ be an orthomodular lattice, and $v : \Phi \to A$ an orthomodular valuation. The corresponding model to $\mathcal{A}$ and v is the structure $\mathcal{M}_{\mathcal{A}} = \langle G(A), \cdot, *, u_{\mathcal{A}} \rangle$, where

(i) G(A) is the set of all residuated monotone maps on A,

(ii) $\cdot$ is the composition operator of maps on A,

(iii) $*$ is the unary operator on G(A) defined in Lemma 3.4, that is,
for any $\varphi \in G(A)$, $\varphi^*(x) := (\varphi^\sharp(x^\perp))^\perp$ for all $x \in A$,

(iv) $u_{\mathcal{A}}$ is a function assigning to each propositional variable p_i an element of the set $\{\gamma_a \mid a \in A\}$, such that, $u_{\mathcal{A}}(p_i) := \gamma_{v(p_i)}$.

▮

Lemma 4.2 Let $\mathcal{A}$ be an orthomodular lattice and v an orthomodular valuation. Then the corresponding model $\mathcal{M}_{\mathcal{A}} = \langle G(A), \cdot, *, u_{\mathcal{A}} \rangle$ is an orthomodular model.

Proof : This is obvious from Lemma 3.6 and Lemma 3.7 . □

Since $\mathcal{M}_{\mathcal{A}}$ is an orthomodular model, the notion of truth in $\mathcal{M}_{\mathcal{A}}$ can be defined similarly in Definition 2.1 as follows Let α, β be formulas, φ, ψ elements in G(A). Then:

(i) $(\mathcal{M}_{\mathcal{A}}, \varphi) \models p_i$ iff $p_i \in \mathrm{u}(p_i) \cdot \mathrm{G(A)}$.

(ii) $(\mathcal{M}_{\mathcal{A}}, \varphi) \models \alpha \wedge \beta$ iff $(\mathcal{M}_{\mathcal{A}}, \varphi) \models \alpha$ and $(\mathcal{M}_{\mathcal{A}}, \varphi) \models \beta$.

(iii) $(\mathcal{M}_{\mathcal{A}}, \varphi) \models \neg\alpha$ iff $\forall\psi \in \mathrm{G(A)}$, $[\ (\mathcal{M}_{\mathcal{A}}, \varphi) \models \alpha$ only if $\psi^* \cdot \varphi = 0\ ]$.

By denoting $\|\alpha\|^{\mathcal{M}_{\mathcal{A}}} := \{\varphi \in \mathrm{G(A)} \mid (\mathcal{M}_{\mathcal{A}}, \varphi) \models \alpha\}$, we can restate the above conditions in the following way.

(i) $\|p_i\|^{\mathcal{M}_{\mathcal{A}}} = \mathrm{u}(p_i) \cdot \mathrm{G(A)}$.

(ii) $\|\alpha \wedge \beta\|^{\mathcal{M}_{\mathcal{A}}} = \|\alpha\|^{\mathcal{M}_{\mathcal{A}}} \cap \|\beta\|^{\mathcal{M}_{\mathcal{A}}}$.

(iii) $\|\neg\alpha\|^{\mathcal{M}_{\mathcal{A}}} = \{\varphi \in \mathrm{G(A)} \mid \forall\psi \in \|\alpha\|^{\mathcal{M}_{\mathcal{A}}}\ (\psi^* \cdot \varphi = 0)\ \}$.

Here we will make a comment about the order on $\mathrm{P}_c(\mathrm{G(A)})$, where A is an orthomodular lattice. Because $\gamma_a \in \mathrm{P}_c(\mathrm{G})$ is a projection, the order on the set $\{\gamma_a \mid a \in \mathrm{A}\}$ is defined as in Definition 1.4, that is,

$$\text{For } a, b \in \mathrm{A}, \qquad \gamma_a \leq \gamma_b \qquad \text{iff} \qquad \gamma_a \cdot \gamma_b = \gamma_a$$

By Lemma 1.3, we have that $\gamma_a \leq \gamma_b$ is equivalent to $\gamma_a \cdot \mathrm{G(A)} \subseteq \gamma_b \cdot \mathrm{G(A)}$.
We can show the following lemma on this order relation.

Lemma 4.3 Let $\mathcal{A} = \langle \mathrm{A}, \leq, \sqcap, \sqcup, ^{\perp}, \mathbf{1}, \mathbf{0} \rangle$ be an orthomodular lattice. Then the following two conditions are equivalent.

(i) $a \leq b$ on A.

(ii) $\gamma_a \leq \gamma_b$ on $\mathrm{P}_c(\mathrm{G(A)})$.

Proof : ((i)$\Rightarrow$(ii)): Suppose that $a \leq b$. Then, for all $x \in \mathrm{A}$ the following holds:

$$\begin{aligned} \gamma_b \cdot \gamma_a(x) &= [\{(x \sqcup a^{\perp}) \sqcap a\} \sqcup b^{\perp}] \sqcap b \\ &= (x \sqcup a^{\perp}) \sqcap a = \gamma_a(x) \end{aligned}$$

Since we have $(x \sqcup a^{\perp}) \sqcap a \leq a \leq b$, we used the orthomodular law (2) in the proof of Lemma 3.5 . Thus we conclude that $\gamma_a \leq \gamma_b$.
((i)$\Leftarrow$(ii)): Suppose that $\gamma_a \leq \gamma_b$. This means that $\gamma_a \cdot \gamma_b = \gamma_b \cdot \gamma_a = \gamma_a$. Since $\gamma_a(\mathbf{1}) \leq \mathbf{1}$, $\gamma_a(\mathbf{1}) = \gamma_b \cdot \gamma_a(\mathbf{1}) = \gamma_b(\gamma_a(\mathbf{1})) \leq \gamma_b(\mathbf{1})$. Recall here that $\gamma_a(x) := (x \sqcup a^{\perp}) \sqcap a$ for any $x \in \mathrm{A}$, then we have that $a = \gamma_a(\mathbf{1}) \leq \gamma_b(\mathbf{1}) = b$. □

As in Lemma 3.3, we can also extend the domain of valuation function $\mathrm{u}_{\mathcal{A}}$ from the set of propositional variables to the set of all formulas Φ.

Lemma 4.4 Let $\mathcal{A} = \langle \mathrm{A}, \leq, \sqcap, \sqcup, {}^{\perp}, \mathbf{1}, \mathbf{0} \rangle$ be an orthomodular lattice and v an orthomodular valuation. Let $\mathcal{M}_{\mathcal{A}}$ be the canonical orthomodular model corresponding to $\mathcal{A}$. Then for any formula α, $\|\alpha\|^{\mathcal{M}_{\mathcal{A}}} = \gamma_{v(\alpha)} \cdot \mathrm{G(A)}$.

Proof : Induction on the construction of the formula α. □

We have now reached the following Completeness Theorem.

Theorem 4.5 **(Completeness theorem)** For given formulas α and β, let (S) and (T) be the same statements in Theorem 2.4 . That is,
(S): for any orthomodular lattice $\mathcal{A}$ and any orthomodular valuation $v : \Phi \to \mathrm{A}$, $v(\alpha) \leq v(\beta)$.
(T): for any orthomodular model $\mathcal{M}$, $\mathcal{M} : \alpha \models \beta$.
Then (T) implies (S). □

5 Relation between two types of models

Theorem 5.1 Let $\mathcal{M} = \langle \mathcal{G}, \mathrm{u} \rangle = \langle \mathrm{G}, \cdot, *, \mathrm{u} \rangle$ be an orthomodular model. Then $\mathcal{Q} = \langle \mathrm{G}', \mathrm{R}, \zeta, \mathrm{V} \rangle$ is a quantum model, where,

- $\mathrm{G}' := \mathrm{G} \setminus \{0\}$,
- $\zeta := \{e \cdot \mathrm{G}' \mid e \in \mathrm{P}_c(\mathrm{G}')\}$,
- R is a binary relation on G′ defined as the following:
  for $x, y \in \mathrm{G}'$, $x\mathrm{R}y \Leftrightarrow x^* \cdot y = 0$,
- V is a function assigning to each p_i an element $\mathrm{V}(p_i)$ of ζ.

Proof : Check the conditions for quantum model in Definition 0.4 . □

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