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PSL(2,$\mathbb{C}$) and its Cayley transforms

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Let $\mathbb{H}^n$ be the upper half space model of the hyperbolic n-space and $\mathbb{H} = \langle h=h_0+ih_1+jh_2+kh_3 ;\ h_m \in \mathbb{R}\ (m=0,\cdots,3)\rangle$ be the set of quaternions. It is well-known (see [B], for example) that, under the identification of $\mathbb{H}^3$ with $\mathbb{C}\times j\cdot\mathbb{R}_{>0}$ and via the embedding $\mathbb{H}\ni h\mapsto \begin{bmatrix}h\\1\end{bmatrix}\in \mathbb{HP}^1$, the orientation preserving isometry group PSL(2,$\mathbb{C}$)($\subset$ PGL(2,$\mathbb{H}$)) acts on $\mathbb{H}^3$ as fractional linear transformations (here $\mathbb{H}^\times$ acts on $\mathbb{H}^2\setminus\langle 0\rangle$ from right and PGL(2,$\mathbb{H}$) acts on $\mathbb{HP}^1 = (\mathbb{H}^2\setminus\langle 0\rangle)/\mathbb{H}^\times$ from left). We will construct a family of "Cayley transformations" for this pair (PSL(2,$\mathbb{C}$), $\mathbb{H}^3$).

Suppose that $h \notin (-\mathbb{H}^3)\cup\langle 0\rangle$. Then $T_h(x) = (x-h)(x+h)^{-1}$ is well-defined for $x \in \mathbb{H}^3$ and has its inverse $T_h^{-1} : y \longmapsto (1-y)^{-1}(1+y)h$; thus T_h is a diffeomorphism onto its image. We will describe the image of $\mathbb{H}^3$ under T_h:

THEOREM. For $h\in \mathbb{H}\setminus((-\mathbb{H}^3)\cup\langle 0\rangle)$ and the map $T_h : \mathbb{H}^3 \ni x \mapsto (x-h)(x+h)^{-1}$, we have

(1). The image $T_h(\mathbb{H}^3)$ is given by $\langle y\in\mathbb{H}$; y satisfies $2y_3h_0 + (1-|y|^2)h_3 + 2y_1h_2 - 2y_2h_1 = 0$ and $2y_2h_0 + (1-|y|^2)h_2 + 2y_3h_1 - 2y_1h_3 > 0\rangle$.

(2). The pull-back of the Poincaré metric $g_P = \sum_{m=0}^{2}(dx_m)^2/x_2^2$ by T_h^{-1} is written as

$$(T_h^{-1})^*g_P = \iota^*\Big(4|h|^2\sum_{m=0}^{3}(dy_m)^2/(2y_2h_0+(1-|y|^2)h_2+2y_3h_1-2y_1h_3)^2\Big),$$

where $\iota : T_h(\mathbb{H}^3)\hookrightarrow \mathbb{H}$ is the inclusion.

(3). The orientation preserving isometry group $T_h\circ PSL(2,\mathbb{C})\circ T_h^{-1}$ on $T_h(\mathbb{H}^3)$ is given by

$$\Big\langle \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\in PGL(2,\mathbb{H})\ ;\ \frac{1}{2}\begin{bmatrix}\alpha+\beta+\gamma+\delta & (-\alpha+\beta-\gamma+\delta)h\\ h^{-1}(-\alpha-\beta+\gamma+\delta) & h^{-1}(\alpha-\beta-\gamma+\delta)h\end{bmatrix}\in PSL(2,\mathbb{C})\Big\rangle.$$

Moreover if $h \in j\cdot\mathbb{R} \times k\cdot\mathbb{R}$ or $h \in \mathbb{C}$, this group has quite simple description:

(3.1). For $h = q \in (j\cdot\mathbb{R} \times k\cdot\mathbb{R}) \cap (\mathbb{H} \setminus ((-\mathbb{H}^3) \cup \langle 0 \rangle))$, we have

$$T_q \, PSL(2,C) \, T_q^{-1} = \langle \begin{bmatrix} \alpha_1 + q\alpha_2 & \beta_1 + q\beta_2 \\ \beta_1 - q\beta_2 & \alpha_1 - q\alpha_2 \end{bmatrix} \in PGL(2,\mathbb{H}) \; ; \alpha_n, \beta_n \in \mathbb{C} \; (n=1,2),$$

$$(\alpha_1+\beta_1)(\overline{\alpha}_1-\overline{\beta}_1) + |q|^2(\alpha_2+\beta_2)(\overline{\alpha}_2-\overline{\beta}_2) = 1 \rangle.$$

(3.2). For $h = p \in \mathbb{C}\setminus\langle 0 \rangle$, we have $T_p \circ PSL(2,\mathbb{C}) \circ T_p^{-1} = PSL(2,\mathbb{C})$. □

EXAMPLES. (1). For $h = j$, we have $T_j(\mathbb{H}^3) = \langle y = y_0 + jy_2 + ky_3 \; ; \; |y| < 1 \rangle$ (which we denote by $\mathbb{D}^3$), $(T_j^{-1})^* g_P = 4((dy_0)^2 + (dy_2)^2 + (dy_3)^2)/(1-|y|^2)^2$ and $\mathrm{Isom}_+(\mathbb{D}^3) =$

$$\langle \begin{bmatrix} \alpha_1 + j\alpha_2 & \beta_1 + j\beta_2 \\ \beta_1 - j\beta_2 & \alpha_1 - j\alpha_2 \end{bmatrix} \in PGL(2,\mathbb{H}) \; ; \alpha_n, \beta_n \in \mathbb{C} \; (n=1,2),$$
$$(\alpha_1+\beta_1)(\overline{\alpha}_1-\overline{\beta}_1) + (\alpha_2+\beta_2)(\overline{\alpha}_2-\overline{\beta}_2) = 1 \rangle$$

(Poincaré disk model). (Remark: This case, together with more higher dimensional ones, is already known. See [B], [W].)

(2). For $h = k$, we have $T_k(\mathbb{H}^3) = \langle y \in \mathbb{H} \; ; \; |y| = 1, \; y_1 < 0 \rangle$ (the lower unit hemi-sphere with respect to the i-axis) and $(T_k^{-1})^* g_P = \iota^* (\sum_{m=0}^{3} (dy_m)^2 / y_1^2)$ (the restriction of the hyperbolic metric on $\langle y \in \mathbb{H} \; ; \; y_1 < 0 \rangle$ ($\cong \mathbb{H}^4$) to the totally geodesic hemi-sphere $T_k(\mathbb{H}^3)$).

(3). For $h = i$, we have $T_i(\mathbb{H}^3) = \mathbb{C} \times k\cdot\mathbb{R}_{>0}$ and $(T_i^{-1})^* g_P = ((dy_0)^2 + (dy_1)^2 + (dy_3)^2)/y_3^2$.

Thus we obtain a "natural" homotopy of models of the hyperbolic 3-space, connecting the above three well-known models.

Let us focus on the case "h=j". Suppose that $\mathbb{H}$ is oriented by the ordered set $\langle 1,i,j,k \rangle = \langle j,k,1,i \rangle$ of generators over $\mathbb{R}$, and introduce an orientation on $\mathbb{D}^3$ by $\langle 1,j,k \rangle = \langle j,k,1 \rangle$.

EXAMPLES of elements of $Isom_+(\mathbb{D}^3)$. (0). $eh(\ell) = \begin{bmatrix} ch(\ell/2) & sh(\ell/2) \\ sh(\ell/2) & ch(\ell/2) \end{bmatrix}$ $(\ell \in \mathbb{R})$ (hyperbolic displacement of length ℓ along the real axis).

(1). $e_i(\varphi) = \begin{bmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{bmatrix}$ $(\varphi \in \mathbb{R}/2\pi\mathbb{Z})$ (rotation of angle φ on the oriented plane $\langle j,k\rangle_{\mathbb{R}}$).

(2). $e_j(\theta) = \begin{bmatrix} e^{j\theta/2} & 0 \\ 0 & e^{-j\theta/2} \end{bmatrix}$ $(\theta \in \mathbb{R}/2\pi\mathbb{Z})$ (rotation of angle θ on $\langle 1,j\rangle_{\mathbb{R}}$).

(3). $e_k(\psi) = \begin{bmatrix} e^{-k\psi/2} & 0 \\ 0 & e^{k\psi/2} \end{bmatrix}$ $(\psi \in \mathbb{R}/2\pi\mathbb{Z})$ (rotation of angle ψ on $\langle k,1\rangle_{\mathbb{R}}$).

Now a combination of the method [O, Lemma] of reading the holonomy representations of Riemann surfaces (the PSU(1,1)-case) and the well-known argument of Euler's angles for SO(3) yields the following

PROPOSITION. $Isom_+(\mathbb{D}^3)$ is generated by these $eh(\ell)$, $e_i(\varphi)$, $e_j(\theta)$ and $e_k(\psi)$. □

References.

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