# The elementary Kleinian groups and three-manifolds

Hiroki SATO

佐藤　宏樹

Department of Mathematics

Shizuoka University

## §0. Introduction.

This paper has the following two aims: (1) To describe generators, fundamental regions and three-manifolds for the elementary kleinian groups based on the lecture due to Oikawa [2] and Ford [1]; (2) To give Jørgensen's number for the elementary Kleinian groups (Sato [3]).

In §1 we will classify the elementary Kleinian groups into seventeen groups by the number of limit points of the groups. In §2 we will consider the finite groups, that is, the polyhedral groups. In §3 we will consider the groups with one limit point and in §4 the groups with two limit points. In §5 we will give Jørgensen's numbers for the elementary Kleinian groups. In Appendix A we will draw the pictures of fundamental regions and the three-manifolds for the elementary Kleinian groups, and in Appendix B we will make a table of the groups.

## §1. Classification of the elementary Kleinian groups.

In this section we give the definition of an elementary group and classify the groups according to the number of limit points. We denote by Möb the set of all Möbius transformations.

DEFINITION 1.1. A subgroup $G$ of Möb is said to be *elementary* if the number of limit points of $G$ is finite.

REMARK. We easily see that $G$ ia an elementary group if and only if the number of limit points of $G$ is 0, 1 or 2.

THEOREM 1.2.  *The elementary Kleinian groups consist of the following seventeen groups and their conjugate groups in Möb, that is, six finite groups, seven groups with one limit point and four groups with two limit points.*

(i) $O$: *The finite groups (the polyhedral groups), that is, the groups containing only elliptic transformations.*

    (1) $O_1$: *The trivial group* $G = \{1\}$.

    (2) $O_2$: *The elliptic cyclic group of order* $n$ $(n \geq 2)$.

    (3) $O_3$: *The dihedral group of order* $2n$ $(n \geq 2)$.

    (4) $O_4$: *The tetrahedral group.*

    (5) $O_5$: *The octahedral group.*

    (6) $O_6$: *the icosahedral group.*

(ii) $I$: *The groups $G$ with one limit point, that is, the elementary Kleinian groups containing parabolic transformations.*

    (1) $I_1$: *A parabolic cyclic group (A simply periodic group).*

    (2) $I_2$: *A doubly periodic group.*

    (3) $I_3$: *The infinite dihedral group.*

    (4) $I_4$: $G = \langle z \mapsto z + \omega, z \mapsto z + \omega', z \mapsto -z \rangle$.

    (5) $I_5$: $G = \langle z \mapsto z + \omega, z \mapsto z + \omega', z \mapsto iz \rangle$.

    (6) $I_6$: $G = \langle z \mapsto z + \omega, z \mapsto z + \varepsilon\omega, z \mapsto \varepsilon z \rangle$ $(\varepsilon = e^{2\pi i/3})$.

    (7) $I_7$: $G = \langle z \mapsto z + \omega, z \mapsto z + \varepsilon\omega, z \mapsto \varepsilon z \rangle$ $(\varepsilon = e^{\pi i/3})$.

(iii) $II$: *The groups $G$ with two limit points, that is, the elementary Kleinian groups containing loxodromic (hyperbolic) transformations.*

    (1) $II_1$: *A loxodromic (hyperbolic) cyclic group.*

    (2) $II_2$: $G = \langle z \mapsto Kz, z \mapsto e^{2\pi i/n}z \rangle$ $(|K| \neq 1, n \geq 2)$.

    (3) $II_3$: $G = \langle z \mapsto Kz, z \mapsto 1/z \rangle$ $(|K| \neq 1)$.

    (4) $II_4$: $G = \langle z \mapsto Kz, z \mapsto e^{2\pi i/n}z, z \mapsto 1/z \rangle$ $(|K| \neq 1, n \geq 2)$.

## §2. Generators, fundamental regions and three-manifolds.

In this section we describe generators, fundamental regions and three-manifolds for the elementary Kleinian groups. Let $\mathbf{B}$ and $\bar{\mathbf{B}}$ be the unit ball and its closure, respectively. We denote by $F(G)$ and $M(G) = (\mathbf{B} \cup \Omega)/G$ a fundamental region and the three-manifold for an elementary Kleinian group $G$, respectively, where $\Omega$ is the region of discontinuty on $\partial\mathbf{B}$ of $G$.

(1) $O_2$: $G$ is the elliptic cyclic group of order $n$.

Set $A(z) = e^{2\pi i/n}z$ $(n \geq 2)$. Then $G = \langle 1, A, A^2, \ldots, A^{n-1} \rangle$, that is, $G = \langle A | A^n = 1 \rangle$. A fundamental region and three-manifold are as follows: $F(G) = \{z \in \mathbf{C} | 0 < \arg z < 2\pi/n\}$ and $M(G) = \bar{\mathbf{B}}$.

(2) $O_3$: $G$ is the dihedral group of order $2n$ $(n \geq 2)$.

Let $S$ be the rotation about the axis joining the North Pole and the South Pole through angle of $2\pi/n$. Let $T_k$ $(k = 1, 2, \ldots, n)$ be the rotation about the axis through the point $z = e^{2\pi(k-1)i/n}$ and the origin. Then

$$G = \{1, S, S^2, \ldots, S^{n-1}, T_1, T_2, \ldots, T_n\}.$$

If we set $G_0 = \{1, S, S^2, \ldots, S^{n-1}\}$, then $G = G_0 + T_1 G_0$. If we set $T = T_1$, then we have $G_0 = \langle S | S^n = 1 \rangle$ and $G = \langle S, T | S^n = 1, T^2 = 1 \rangle$. By the stereographic projection, we can consider $S$ and $T$ as the following Möbius transformations of the complex plane:

$$S(z) = e^{2\pi i/n} z \quad \text{and} \quad T(z) = 1/z.$$

A fundamental region $F(G)$ and the three-manifold $M(G)$ of $G$ are as follows: $F(G) = \{z \in \mathbf{C} | |z| < 1, 0 < \arg z < 2\pi/n\}$ and $M(G) = \bar{\mathbf{B}}$.

(3) $O_4$: $G$ is the tetrahedral group.

$S_j$: the rotation about the axis joining $v_j$ with the center of the opposite face to $v_j$ through $2\pi/3$ $(j = 1, 2, 3, 4)$.

$V_{ij}$: the rotation about the axis joining the midpoint of the side $v_i v_j$ with the midpoint of the side $v_k v_\ell (k, \ell \neq i, j)$. $b_{ij}$ through $\pi$ $(V_{12} = V_{34}, V_{13} = V_{24}, V_{14} = V_{23})$.
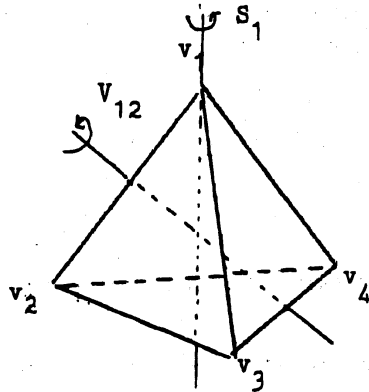


Then we have

$$G = \{1, S_i, S_i^2 \ (i = 1, 2, 3, 4), V_{12}, V_{13}, V_{14}\}$$

and $\mathrm{ord}(G) = 12$. Noting $V_{13} = V_{14} V_{12}$, $S_2 = V_{12} S_1 V_{12}^{-1}$, $S_3 = V_{13} S_1 V_{13}^{-1}$, $S_4 = V_{14} S_1 V_{14}^{-1}$ and setting $S = S_1, V = V_{12}, W = V_{14}$, we have

$$G = \langle S, V, W | S^3 = 1, V^2 = 1, W^2 = 1 \rangle$$

and

$$G = \begin{pmatrix} 1 & V & W & WV \\ S & SV & SW & SWV \\ S^2 & S^2V & S^2W & S^2WV \end{pmatrix}.$$

Furthermore, since $W = S^2VS$, we have

$$G = \langle S, V | S^3 = 1, V^2 = 1, (SV)^3 = 1 \rangle$$

and

$$G = \begin{pmatrix} 1 & V & S^2VS & S^2VSV \\ S & SV & VS & VSV \\ S^2 & S^2V & SVS & SVSV \end{pmatrix}.$$

By the stereographic projection we can represent $S$ and $V$ as follows:

$$S(z) = i(z+1)/(z-1), \qquad V(z) = 1/z.$$

A fundamental region $F(G)$ and the three-mnifold $M(G)$ are as follows: $F(G) = \{z \in \mathbf{C} | |z| < 1, -\pi/3 < \arg z < \pi/3\}$ and $M(G) = \bar{\mathbf{B}}$.

(4) $O_5$: $G$ is the octahedral group.

$S_j$: the rotation about the axis $u_j v_j$ through $2\pi/3$.

$T_{kj}$: the rotation about the axis joining the midpoint of $v_k v_j$ and the center of the opposite face to $v_k v_j$ through $\pi/4$.

$U_{kj}$: the rotation about the midpoint of the side $u_k v_j$ and the midpoint of the opposite side through $\pi$.
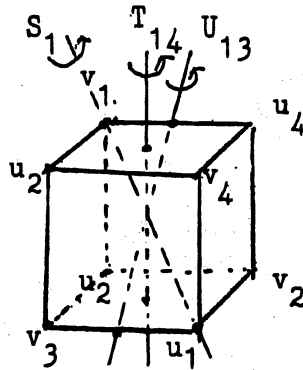


Then

$$G = \{1, S_j, S_j^2 \ (j = 1, 2, 3, 4), T_{12}^j, T_{13}^j, T_{23}^j \ (j = 1, 2, 3), U_{12}, U_{13}, U_{14}, U_{23}, U_{24}, U_{34}\}$$

and ord$(G) = 24$.

If we set $S = S_1, V = V_{12} = T_{12}^2$ and $U = U_{12}$, then we have

$$G = \begin{pmatrix} 1 & U & S^2US & S^2USU \\ S & SU & US & USU \\ S^2 & S^2U & SUS & SUSU \\ T & TU & TS^2US & TS^2USU \\ TS & TSU & TUS & TUSU \\ TS^2 & TS^2U & TSUS & TSUSU \end{pmatrix}.$$

By the stereographic projection we can represent $S, U, T$ and $V$ as follows:

$$S(z) = i(z+1)/(z-1), \quad U(z) = 1/z, \quad T(z) = iz, \quad V(z) = -z.$$

If we set $R(z) = (z+1)/(z-1)$, then we have

$$G = \langle R, T | T^4 = 1, R^2 = 1, (TR)^3 = 1 \rangle.$$

A fundamental region $F(G)$ and three-manifold $M(G)$ are as follows: $F(G) = \{z \in C | |z| < 1, -\pi/4 < \arg z < \pi/4\}$ and $M(G) = \overset{\circ}{B}$.

(5) $O_6$: $G$ is the icosahedral group.

$S_i$: the rotation about the axis joining the center of a face with the center of the opposite face through $2\pi/5$ for $i = 1, 2, 3, 4, 5, 6$.

$T_j$: the rotation about the axes joining the midpoint of a side with the midpoint of the opposite side through $\pi$ for $j = 1, 2, \ldots, 15$.

$U_k$: the rotation about the axis joining a vertex with the opposite vertex through $2\pi/3$ for $k = 1, 2, \ldots, 10$.
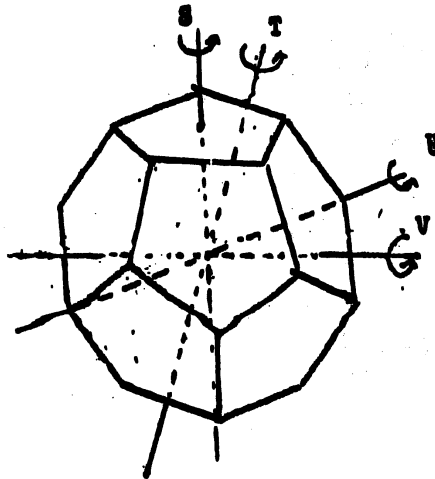


Then

$$G = \{1, S_i, S_i^2, S_i^3, S_i^4 \ (i = 1, 2, \ldots, 6), T_j \ (j = 1, 2, \ldots, 15), U_k, U_k^2 \ (k = 1, 2, \ldots, 10)\}$$

and $\text{ord}(G) = 60$.

Let $V$ be the rotation about the horizontal cross-line through $\pi$. Since $V = (STS^3)T(STS^3)^{-1}$ we have

$$G = \begin{pmatrix} 1 & S & S^2 & S^3 & S^4 & V & VS & VS^2 & VS^3 & VS^4 \\ T & ST & S^2T & S^3T & S^4T & VT & VST & VS^2T & VS^3T & VS^4T \\ TS & STS & S^2TS & S^3TS & S^4TS & VTS & VSTS & VS^2TS & VS^3TS & VS^4TS \\ TS^2 & STS^2 & S^2TS^2 & S^3TS^2 & S^4TS^2 & VST^2 & VSTS^2 & VS^2TS^2 & VS^3TS^2 & VS^4TS^2 \\ TS^3 & STS^3 & S^2TS^3 & S^3TS^3 & S^4TS^3 & VTS^3 & VST^3 & VS^2TS^3 & VS^3TS^3 & VS^3TS^4 \\ TS^4 & STS^4 & S^2TS^4 & S^3TS^4 & S^4TS4 & VTS^4 & VSTS^4 & VS^2TS^4 & VS^3TS^4 & VS^4TS^4 \end{pmatrix}$$

and

$$G = \langle S, T | S^5 = 1, T^2 = 1, (TS)^3 = 1 \rangle.$$

By the stereographic projection we have

$$S(z) = e^{2\pi i/5}z \quad \text{and} \quad T(z) = \frac{(\sqrt{5}+1)z + 2}{2z - (\sqrt{5}+1)}.$$

A fundamental region $F(G)$ and $M(G)$ are as follows: $F(G) = \{z \in \mathbf{C} | |z| < 1, -\pi/5 < \arg z < \pi/5\}$ and $M(G) = \bar{\mathbf{B}}$.

## §3. The groups with one limit point.

In this section we consider groups with one limit point, that is, elementary Kleinian groups containing parabolic transformations.

(1) $I_1$: $G$ is a parabolic cyclic group (a simply periodic group).

$G = \langle A \rangle$, where $A(z) = z + \omega$ $(\omega \neq 0)$. Then $M(G) = \bar{\mathbf{B}} \setminus \{L\}$, where $L$ is the axis joining the North Pole with the South Pole. For a fundamental region $F(G)$ see Appendix A.

(2) $I_2$: $G$ is a doubly periodic group.

$G = \langle A, B \rangle$, where $A(z) = z + \omega, B(z) = z + \omega'$ $(\text{Im}(\omega'/\omega) \neq 0, \omega \neq 0, \omega' \neq 0)$. $M(G) = $ Thesolidtorus $- \{L\}$, where $L$ is the central axis in the solid torus. For a fundamental region $F(G)$ see Appendix A.

(3) $I_3$: $G$ is the infinite dihedral group.

$G = \langle A, C | C^2 = 1 \rangle$, where $A(z) = z + \omega$ $(\omega \neq 0), C(z) = -z$. $M(G) = \bar{\mathbf{B}} - \{R\}$, where $R$ is the radius joining the North Pole with the origin. For a fundamental region $F(G)$ see Appendix A.

(4) $I_4$: $G = \langle A, B, C | C^2 = 1 \rangle$, where $A(z) = z + \omega$ $(\omega \neq 0), B(z) = z + \omega'$ $(\omega' \neq 0, \text{Im}(\omega'/\omega) \neq 0), C(z) = -z$. Then $M(G) = \bar{\mathbf{B}} - \{O\}$, where O is the center of the ball $\mathbf{B}$.

For a fundamental region $F(G)$ see Appendix A. The Riemann surface $\Omega/G$ is the sphere with four branched points of order two.

(5) $I_5$: $G = \langle A, B, C | C^4 = 1 \rangle$, where $A(z) = z + \omega$ $(\omega \neq 0)$, $B(z) = z + i\omega$, $C(z) = iz$. If we set $S = C$ and $T = AC$, then $G = \langle S, T | S^4 = 1, T^4 = 1, (TS)^2 = 1 \rangle$. Then $M(G) = \bar{\mathbf{B}} - \{O\}$, where O is the center of the ball $\mathbf{B}$. For a fundamental region $F(G)$ see Appendix A. The Riemann surface $\Omega/G$ is the sphere with three branched points of orders 2, 4, 4.

(6) $I_6$: $G = \langle A, B, C | C^3 = 1 \rangle$, where $A(z) = z + \omega$, $B(z) = z + \varepsilon\omega$ $(\varepsilon = e^{2\pi i/3})$, $C(z) = \varepsilon z$. If we set $S = C$ and $T = AC$ , then $G = \langle S, T | S^3 = 1, T^3 = 1, (TS)^3 = 1 \rangle$. Then $M(G) = b\bar{f}B - \{O\}$, where O is the center of the ball $\mathbf{B}$. For a fundamental region $F(G)$ see Appendix A. The Riemann surfaces $\Omega/G$ is the sphere with three branched points of order three.

(7) $I_7$: $G = \langle A, B, C | C^6 = 1 \rangle$, where $A(z) = z + \omega$ $(\omega \neq 0)$, $B(z) = z + \varepsilon\omega$ $(\varepsilon = e^{\pi i/3})$, $C(z) = \varepsilon z$. If we set $S = C$ and $T = A^{-1}BC^3$, then $G = \langle S, T | S^6 = 1, T^2 = 1, (TS)^3 = 1 \rangle$. Then $M(G) = \bar{\mathbf{B}} - \{O\}$, where O is the center of the ball $\mathbf{B}$. For a fundamental region $F(G)$ see Appendix A. The Riemann surface $\Omega/G$ is the sphere with three branched points of orders 2, 3, 6.

## §4. The groups with two limit points.

In this section we consider groups with two limit points, that is, elementary Kleinian groups containing loxodromic (hyperbolic) transformations. We denote by $H^3$ the upper half space.

(1) $II_1$: $G$ is a loxodromic (hyperbolic) cyclic group.
$G = \langle A \rangle$, where $A(z) = Kz$ $(|K| \neq 1)$. $M(G)$ is the solid torus and $H^3 \cup \Omega \mapsto M(G)$ is an unbranched covering map.
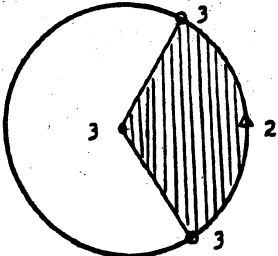
(2) $II_2$: $G = \langle A, B \rangle$, where $A(z) = Kz$ $(|K| \neq 1)$ and $B(z) = e^{2\pi i/n}z$ $(n \geq 2)$. $M(G)$ is the solid torus and $H^3 \cup \Omega \mapsto M(G)$ is a branched covering map and the mapping is locally $n$ to 1 on the central axis in the solid torus.

(3) $II_3$: $G = \langle A, C | C^2 = 1 \rangle$, where $A(z) = Kz$ $(|K| \neq 1)$ and $C(z) = 1/z$. $M(G) = \bar{\mathbf{B}}$ and $H^3 \cup \Omega \mapsto M(G)$ is a branched covering map. The Riemann surface $\Omega/G$ is the sphere with four branched points of order 2.

(4) $II_4$: $G = \langle A, B, C | B^n = 1, C^2 = 1 \rangle$, where $A(z) = Kz$ $(|K| \neq 1), B(z) = e^{2\pi i/n}z$ $(n \geq 2), C(z) = 1/z$. $M(G) = \bar{\mathbf{B}}$ and $H^3 \cup \Omega \mapsto M(G)$ is a branched covering map. The Riemann surface $\Omega/G$ is the sphere with four branched points of order two.

## §5. Jørgensen's number.

In this section we consider Jørgensen's number for the elementary Kleinian groups.

**DEFINITION 5.1.** Let $G = \langle A, B \rangle$ be a marked two-generator subgroup of Möb. We call

$$J(\langle A, B \rangle) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|$$

*Jørgensen's number* for $G = \langle A, B \rangle$. The Jørgensen's number $\parallel J(G) \parallel$ for a sugroup $G$ of Möb is defined as follows:

$$\parallel J(G) \parallel := \inf\{J(\langle A, B \rangle) | \langle A, B \rangle \subset G, A^m \neq B^m \ (m, n \in \mathbf{Z})\}.$$

Then we have the following.

**THEOREM 5.1.** (i) O: *the Finite groups.*

(1) $O_3$: $G = \langle S, T \rangle$ *is the dihedral group, where* $S(z) = e^{2\pi i/n}z$ *and* $T(z) = 1/z$. *Then* $\parallel J(G) \parallel = 8 \sin^2 \pi/n$.

(2) $O_4$: $G$ *is the tetrahedral group. Then* $\parallel J(G) \parallel = 5$.

(3) $O_5$: $G$ *is the octahedral group. Then* $\parallel J(G) \parallel = 3$.

(4) $O_6$: $G$ *is the icosahedral group. Then* $\parallel J(G) \parallel = 4 - \sqrt{5}$.

(ii) I: *Elementary Kleinian groups with one limit point.*
*For all groups $G$ in this type* $\parallel J(G) \parallel = 0$.

(iii) II: *Elementary Kleinian groups with two limit points.*
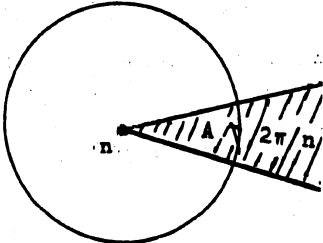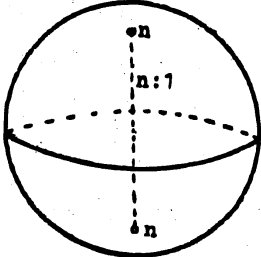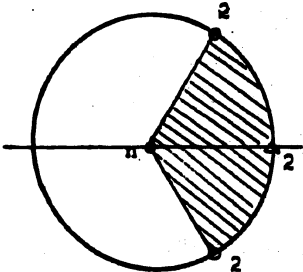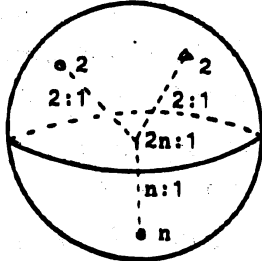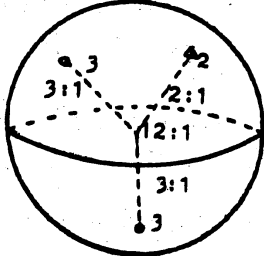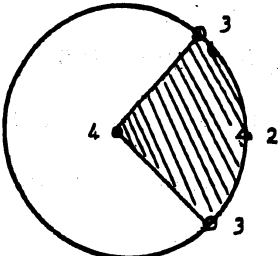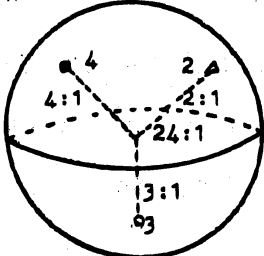
(1) $II_2$: $G = \langle A, B \rangle$, *where* $A(z) = Kz$ $(|K| \neq 1)$ *and* $B(z) = e^{2\pi i/n}z$ $(n \geq 2)$. *Then* $\parallel J(G) \parallel = \min\{4 \sin^2 \pi/n, |K^{p/2}e^{k\pi i/n} - K^{-p/2}e^{-k\pi i/n}|^2 \ (1 \leq p \leq 2\log 3/\log|K|, p \in \mathbf{Z}; k = 0, 1, 2, \ldots, n-1))\}$.

(2) $II_3$: $G = \langle A, C \rangle$, *where* $A(z) = Kz$ $(|K| \neq 1)$ *and* $C(z) = 1/z$. *Then* $\parallel J(G) \parallel = \min\{4 + |K^{p/2} - K^{-p/2}|^2, 2|K^{p/2} - K^{-p/2}|^2 \ (1 \leq p \leq 2\log 3/\log|K|, p \in \mathbf{Z})\}$.

(3) $II_4$: $G = \langle A, B, C \rangle$, *where* $A(z) = Kz$ $(|K| \neq 1)$, $B(z) = e^{2\pi i/n}z$ $(n \geq 2)$ *and* $C(z) = 1/z$. *Then* $\parallel J(G) \parallel = \min\{4 \sin^2 \pi/n, |K^{p/2}e^{k\pi i/n} - K^{-p/2}e^{-k\pi i/n}|^2 \ (1 \leq p \leq 2\log 3/\log|K|, p \in \mathbf{Z}; k = 0, 1, 2, \ldots, n-1))\}$.
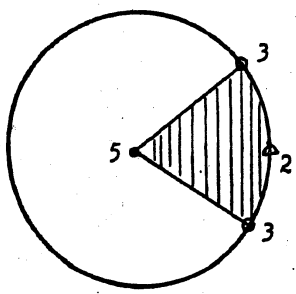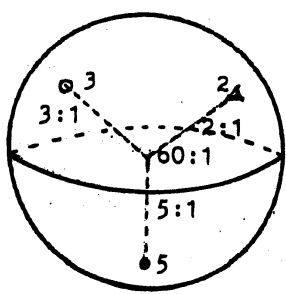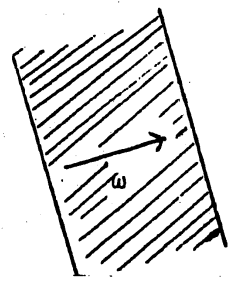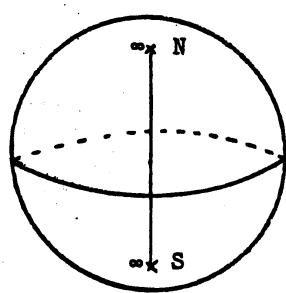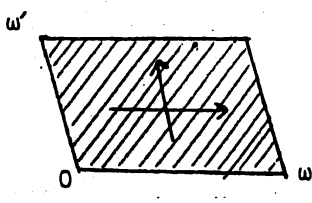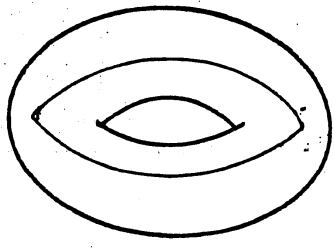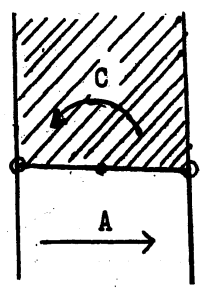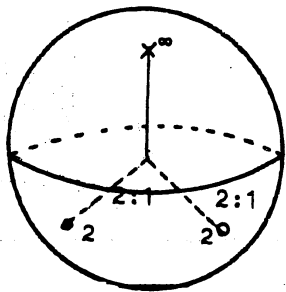
Our proof of this theorem will appear elsewhere.

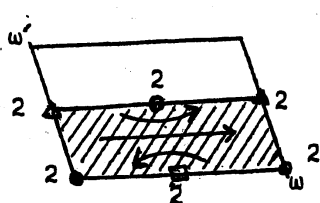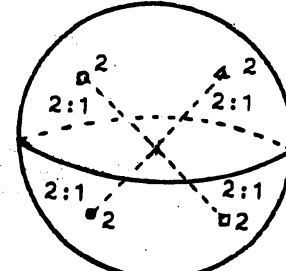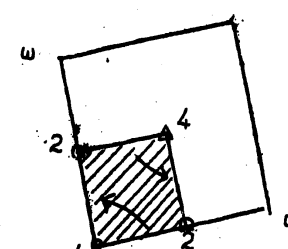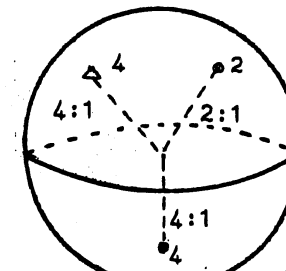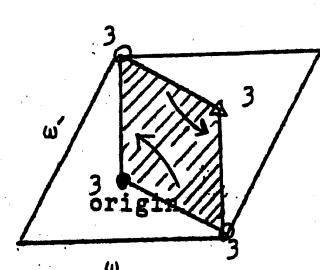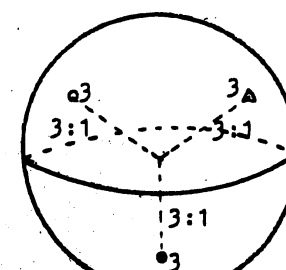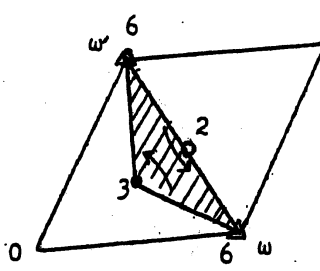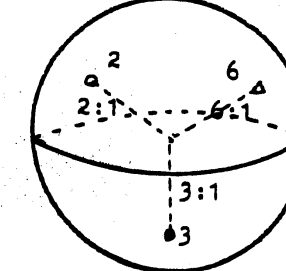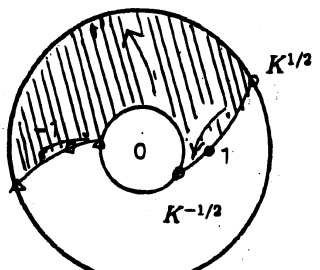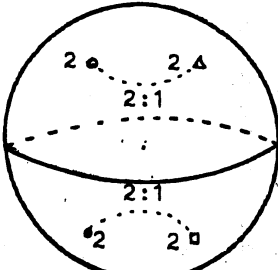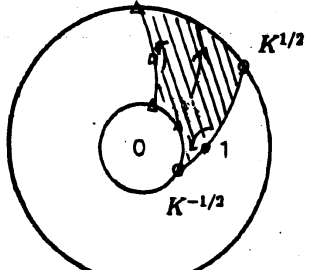# References
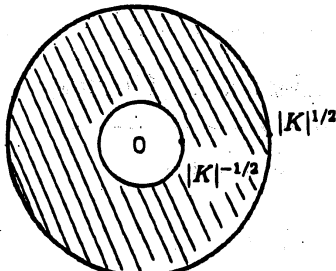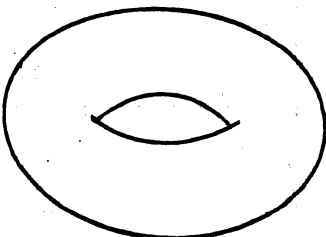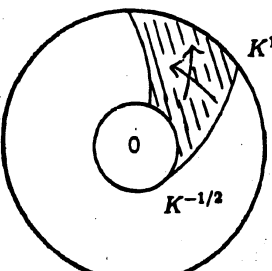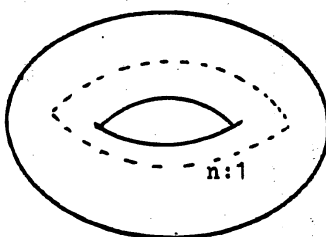
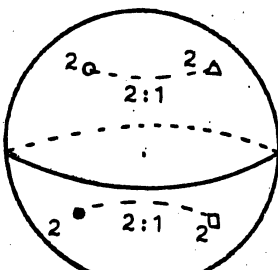[1] L. R. Ford, Automorphic Functions, Chelsea, New York, 1951.

[2] K. Oikawa, The Elementary Groups, Lecture Notes, Shizuoka Univ, 1989.

[3] H. Sato, Jørgensen's number for the elementary Kleinian groups, in preparation.

**Appendix A. Fundamental regions and three-manifolds.**

| $F(G)$ | $M(G)$ |
|---|---|

$O_2$:

$O_3$:

$O_4$:

$O_5$:

| $F(G)$ | $M(G)$ |
|---|---|
| $O_6$: | |
| $I_1$: | |
| $I_2$: | |
| $I_3$: | |

| $F(G)$ | $M(G)$ |
|---|---|
| $I_4$:  |  |
| $I_5$:  |  |
| $I_6$:  |  |
| $I_7$:  |  |

| $F(G)$ | $M(G)$ |
|---|---|

$II_1$:



$II_2$:



$II_3$:



$II_4$:

## Appendices B.   Signature of the elementary groups.

$$2g - 2 + \sum_{j=1}^{n}(1 - \frac{1}{\nu_j}) \leq 0,$$

where $g$ is the genus of the Riemann surface $R = \Omega/G$ and $\nu_j$ is the order of a branch point on $R$.

| $g$ | $n$ | $\nu_1$ | $\nu_2$ | $\nu_3$ | $\nu_4$ | Elementary Groups |
|---|---|---|---|---|---|---|
| 1 | 0 | – | – | – | – | $I_2, II_1, II_2$ |
| 0 | 4 | 2 | 2 | 2 | 2 | $I_4, II_3, II_4$ |
| 0 | 3 | 3 | 3 | 3 | – | $I_6$ |
| 0 | 3 | 2 | 4 | 4 | – | $I_5$ |
| 0 | 3 | 2 | 3 | 6 | – | $I_7$ |
| 0 | 3 | 2 | 3 | 5 | – | $O_6$ |
| 0 | 3 | 2 | 3 | 4 | – | $O_5$ |
| 0 | 3 | 2 | 3 | 3 | – | $O_4$ |
| 0 | 3 | 2 | 2 | $\infty$ | – | $I_3$ |
| 0 | 2 | 2 | 2 | $n$ | – | $O_3$ |
| 0 | 2 | $\infty$ | $\infty$ | – | – | $I_1$ |
| 0 | 2 | $n$ | $n$ | – | – | $O_2$ |
| 0 | 0 | – | – | – | – | $O_1$ |