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The realizations of the amalgamated free products of 3-orbifold fundamental groups

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Abstract.
We introduce a kind of generalized orbifolds called "orbifold compositions," and study on their topology, and extensions and deformations of maps between them. As the main goal, we show the theorem which yields the geometric realizations of amalgamated free products 3-orbifold fundamental groups. For the details or the HNN extension case, see [T-Y 3].

§1. Preliminaries on orbifolds

A covering $p : \tilde{M} \to M$ is called a *manifold covering* if $\Sigma\tilde{M} = \phi$. An orbifold M is *good* if the universal covering of M is a manifold covering and *bad* otherwise.

Let M be a 3-orbifold and F a connected 2-suborbifold which is either properly embedded in M or contained in ∂M. We say that F is *compressible* in M if one of the following conditions is satisfied.

(i) F is a spherical orbifold which bounds a ballic orbifold in M, or

(ii) F is a discal orbifold and either $F \subset \partial M$ or there is a discal 2-suborbifold $G \subset \partial M$ and a ballic 3-suborbifold $B \subset M$ s.t. $F \cap G = \partial F = \partial G$ and $\partial B = F \cup G$, or

(iii) there is a discal orbifold $D \subset M$ with $D \cap F = \partial D$ and ∂D does not bound any discal orbifolds in F.

Otherwise, F is *incompressible*. By [K-S] and [M-Y 1], the ballic orbifold bounded by F in (i) is the cone on F. Hence, a locally orientable 3-orbifold does not include compressible $\mathbf{R}P^2$'s. A 3-orbifold M is *irreducible* if there are no incompressible spherical 2-suborbifolds in M.

Throughout this paper, all orbifolds are connected unless otherwise stated. At first, we review three theorems in [T-Y 2].

Theorem 1.1. (The Loop Theorem [T-Y 2, 6.4]) *Let M be a good 3-orbifold with boundaries. Let F be a connected 2-suborbifold in ∂M. If* $\mathrm{Ker}(\pi_1(F) \to \pi_1(M)) \neq 1$, *then there exists a discal 2-suborbifold D properly* *embedded in M s.t. $\partial D \subset F$ and ∂D does not bound any discal 2-suborbifold in F.*

Theorem 1.2. (Dehn's Lemma [T-Y 2, 6.5]) *Let M be a good 3-orbifold with boundaries. Let γ be a simple closed curve in $\partial M - \Sigma M$ s.t. the order of $[\gamma]$ is n in $\pi_1(M)$. Then there exists a discal suborbifold $D^2(n)$ properly embedded in M with $\partial D^2(n) = \gamma$.*

Theorem 1.3. (The Sphere Theorem [T-Y 2, 6.7]) *Let M be a good 3-orbifold. Let $p : \tilde{M} \to M$ be the universal cover of M. If $\pi_2(\tilde{M}) \neq 0$, then there exists a spherical suborbifold S in M s.t. $[\tilde{S}] \neq 0$ in $\pi_2(\tilde{M})$, where $\tilde{S}$ is any component of $p^{-1}(S)$.*

The next corollary is derived directly from 1.3.

Corollary 1.4. *Let M be a good 3-orbifold. If M is irreducible, then for any manifold covering $\tilde{M}$ of M, $\pi_2(\tilde{M}) = 0$.*

§3. Orbifold compositions

From now on, we assume that all orbifolds are good, connected, and locally orientable, unless otherwise stated.

Definition 3.1. Let I, J be countable sets, X_i ($i \in I$) n-orbifolds, Y_j ($j \in J$) (n-1)-orbifolds. Let $f_j^\varepsilon : Y_j \times \varepsilon \to X_{i(j,\varepsilon)}$ be orbi-maps s.t. $(f_j^\varepsilon)_*$ are monic where $j \in J$, $i(j,\varepsilon) \in I$, $\varepsilon = 0, 1$. Then we call $X = (X_i, Y_j \times [0,1], f_j^\varepsilon)_{i\in I, j\in J, \varepsilon=0,1}$ an *n-dimensional orbifold composition*. The maps f_j^ε are called the *attaching maps* of X. Each X_i or $Y_j \times [0,1]$ is called a *component of* X. The equivalence relation $\sim$ in $\amalg_{i\in I, j\in J}(|X_i| \cup |Y_j| \times [0,1])$ is defined to be generated by

$$(y, \varepsilon) \sim \overline{f}_j^\varepsilon(y), \quad \varepsilon = 0, 1, \quad y \in |Y_j|, \quad j \in J.$$

We call the identified space $\amalg_{i\in I, j\in J}(|X_i| \cup |Y_j| \times [0,1])/\sim$ the *underlying space of* X, denoted by $|X|$, and call the identified space $\{(\cup_{i\in I}\Sigma X_i) \cup (\cup_{j\in J}\Sigma(Y_j \times [0,1]))\}/\sim$ the *singular set of* X, denoted by ΣX.

From now on, we assume that the underlying space $|X|$ is connected. Note that $|X_i|$ and $|Y_j \times (0,1)|$ are embedded in $|X|$.

Definition 3.2. Let $X = (X_i, Y_j \times [0,1], f_j^\varepsilon)_{i\in I, j\in J, \varepsilon=0,1}$ be an orbifold composition. Define the identified space $\mathcal{C}(X)$ by $|X|/\approx$ where

$$x \approx y \Leftrightarrow \begin{cases} \exists i \in I & \text{s.t. } x,\ y \in |X_i|/\sim, \text{ or} \\ \exists j \in J, \exists t \in [0,1] & \text{s.t. } x,\ y \in |Y_j \times t|/\sim. \end{cases}$$

We call $\mathcal{C}(X)$, each X_i, each $Y_j \times [0,1]$, and each $Y_j \times \frac{1}{2}$, the ***associated 1-complex***, a ***vertex orbifold***, an ***edge orbifold of*** X, and the core of $Y_j \times [0,1]$, respectively.

An isomorphism of orbifold compositions is a map which is componentwise isomorphism and commutes with attaching maps.

Definition 3.4. Let $X = (X_k, Y_\ell \times [0,1], f_\ell^\varepsilon)_{k\in K, \ell\in L, \varepsilon=0,1}$ and $X' = (X'_i, Y'_j \times [0,1], f'^\varepsilon_j)_{i\in I, j\in J, \varepsilon=0,1}$ be orbifold compositions. We say that X' is a *covering of* X if there exist a set of maps $\{\varphi_i, \psi_j\}_{i\in I, j\in J}$ s.t. after changing the orientations of $[0,1]$'s if necessary, the following (1) $\sim$ (3) hold.

(1) Each φ_i is a covering map (of orbifolds) from X'_i to X_{k_i}, where $k_i \in K$. And each ψ_j is a covering map (of orbifolds) from $Y'_j \times [0,1]$ to $Y_{\ell_j} \times [0,1]$, where $\ell_j \in L$.

(2) For $\forall j$ and $\epsilon = 0,1$, $\varphi_{i(j,\varepsilon)} \circ f'^\varepsilon_j = f^\varepsilon_{\ell_j} \circ (\psi_j \mid Y'_j \times \varepsilon)$.

(3) The continuous map $p : |X'| \to |X|$ which is naturally induced by $\{\varphi_i, \psi_j\}_{i\in I, j\in J}$ is onto and induces the usual covering map from $|X'| - p^{-1}(\Sigma X)$ to $|X| - \Sigma X$.

We call the above map p a *covering map from* X' *to* X.

Remark 3.5. In the above definition, if each component X'_i is the universal cover of a component X_{k_i}, then for some base point $x_0 \in |X| - \Sigma X$, any path ℓ with the base point x_0 s.t. $\mathrm{Int}\ell \cap \Sigma X = \phi$, and any point $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lift of ℓ with the base point $\tilde{x}_0$. This holds because $(f_\ell^\varepsilon)_*$ are monic.

Definition 3.6. Let X be an orbifold composition, $x_0 \in |X| - \Sigma X$ a base point, ℓ a path with the base point x_0 s.t. $\mathrm{Int}\ell \cap \Sigma X = \phi$, and $p : \tilde{X} \to X$ any covering. Fix any point $\tilde{x}_0 \in p^{-1}(x_0)$. Suppose there is a covering $\hat{p} : \hat{X} \to \tilde{X}$ s.t. each component of $\hat{X}$ is the universal cover of a component of $\tilde{X}$. Fix any point $\hat{x}_0 \in \hat{p}^{-1}(\tilde{x}_0)$. By Remark 3.5 there exists a unique lift $\hat{\ell}$ to $\hat{X}$ of ℓ with the base point $\hat{x}_0$. Then we can determine a lift $\tilde{\ell}$ of ℓ uniquely by putting $\tilde{\ell} = \hat{p} \circ \hat{\ell}$, which is called the *canonical lift of ℓ with the base point $\tilde{x}_0$*.

Definition 3.7. Let X', X be orbifold compositions, and $p : X' \to X$ a covering. We define the *deck transformation group* $\mathrm{Aut}(X', p)$ *of* p by

$$\mathrm{Aut}(X', p) = \{h : X' \to X' \mid h \text{ is an isomorphism s.t. } p \circ h = p\}.$$

Definition 3.8. Let $\tilde{X}$, X be orbifold compositions, and $p : \tilde{X} \to X$ a covering. We say that p is a *universal covering* if for any covering $p' : X' \to X$, there exists a covering $q : \tilde{X} \to X'$ s.t. $p = p' \circ q$.

Lemma 3.9. *For any orbifold composition X, there exists a unique universal covering $p : \tilde{X} \to X$.*

Proof. See [T-Y 3].

We sometimes denote an orbifold composition or a good orbifold X by $(\tilde{X}, p, |X|)$ where $p : \tilde{X} \to X$ is the universal covering, and $|X|$ is the underlying space of X. A good orbifold is considered as a special case of an orbifold composition.

Proposition 3.10. *Let $\tilde{X}$, X be orbifold compositions and $p : \tilde{X} \to X$ a covering. If the restriction of p to each component of $\tilde{X}$ is universal and $C(\tilde{X})$ is a tree, then the covering $p : \tilde{X} \to X$ is universal.*

Proof. See [T-Y 3].

Definition 3.11. Let $X = (\tilde{X}, p, |X|)$ be an orbifold composition with the base point $x_0 \in |X| - \Sigma X$. Put

$$\Omega(\tilde{X}, x_0) = \{\tilde{\alpha} \mid \text{a continuous map } \tilde{\alpha} : [0,1] \to \tilde{X} \text{ with } p(\tilde{\alpha}(0)) = p(\tilde{\alpha}(1)) = x_0\}.$$

For any two elements $\tilde{\alpha}$, $\tilde{\beta} \in \Omega(\tilde{X}, x_0)$, $\tilde{\alpha}$ is *equivalent to* $\tilde{\beta}$, denoted by $\tilde{\alpha} \sim \tilde{\beta}$, if there exists an element $\tau \in \mathrm{Aut}(\tilde{X}, p)$ s.t. $\tilde{\alpha}(0) = \tau(\tilde{\beta}(0))$ and $\tilde{\alpha}(1) = \tau(\tilde{\beta}(1))$. The relation $\sim$ is an equivalence relation and $\Omega(\tilde{X}, x_0)/\sim$ is a group with the product defined by

$$[\tilde{\alpha}] \cdot [\tilde{\beta}] = [\tilde{\alpha} \cdot \rho(\tilde{\beta})]$$

where $\rho \in \mathrm{Aut}(\tilde{X},p)$ is the element s.t. $\rho(\tilde{\beta}(0)) = \tilde{\alpha}(1)$. The group $\Omega(\tilde{X},x_0)/\sim$ is called the *fundamental group of* X and denoted by $\pi_1(X,x_0)$. Note that the fundamental group $\pi_1(X,x_0)$ is isomorphic to the deck transformation group $\mathrm{Aut}(\tilde{X},p)$. By the symbol σ_A, we mean the element of $\mathrm{Aut}(\tilde{X},p)$ which is corresponding to $\sigma \in \pi_1(X,x_0)$.

Definition 3.12. Let $X = (\tilde{X},p,|X|)$ and $Y = (\tilde{Y},q,|Y|)$ be orbifold compositions (or orbifolds). By an *orbi-map* $f : X \to Y$, we mean the pair $(\bar{f},\tilde{f})$ of continuous maps $\bar{f} : |X| \to |Y|$ and $\tilde{f} : \tilde{X} \to \tilde{Y}$ satisfying

(i) $\bar{f} \circ p = q \circ \tilde{f}$,

(ii) for each $\sigma \in \mathrm{Aut}(\tilde{X},p)$, there exists $\tau \in \mathrm{Aut}(\tilde{Y},q)$ s.t. $\tilde{f} \circ \sigma = \tau \circ \tilde{f}$,

(iii) there exists $x \in |X| - \Sigma X$ s.t. $\bar{f}(x) \in |Y| - \Sigma Y$.

Definition 3.13. Let $X = (\tilde{X},p,|X|)$ and $Y = (\tilde{Y},q,|Y|)$ be orbifold compositions, and $f = (\bar{f},\tilde{f}) : X \to Y$ an orbi-map. By the definition of an orbi-map, there exists a point $x \in |X| - \Sigma X$ s.t. $\bar{f}(x) \in |Y| - \Sigma Y$. Then the induced homomorphism $f_* : \pi_1(X,x) \to \pi_1(Y,\bar{f}(x))$ of f is naturally defined by $f_*([\tilde{\alpha}]) = [\tilde{f} \circ \tilde{\alpha}]$.

For an orbi-map and a covering between orbifold compositions we can define the notions of C-equivalence, orbi-homotopy, and lifting as well as those for an orbi-map and a covering between orbifolds. We derive the relations among fundamental groups, coverings, and liftings similar to those for orbifolds. See [Ta 2] for the orbifold case.

The next proposition can be shown in a way similar to one in [Prop. 2.2 of Ta 2].

Proposition 3.14. *Let $X = (\tilde{X},p,|X|)$, $Y = (\tilde{Y},q,|Y|)$ be orbifold compositions, and $f = (\bar{f},\tilde{f}) : X \to Y$ an orbi-map. Then for $\forall[\tilde{\alpha}] \in \pi_1(X,x)$, we have that*

$$\tilde{f} \circ [\tilde{\alpha}]_A = (f_*([\tilde{\alpha}]))_A \circ \tilde{f}.$$

§4. The tree constructions of the universal coverings

Let X be an orbifold composition and $Y \times [0,1]$ one of edge orbifold components of X. Suppose that $X - Y \times (0,1)$ are two disjoint orbifold compositions X^0 and X^1, and attaching orbi-maps from $Y \times \varepsilon$ are mapped into X^ε and described as $f^\varepsilon : Y \times \varepsilon \to X^\varepsilon$, $\varepsilon = 0,1$. We construct the universal covering of an orbifold composition X by the "tree construction", and show that the fundamental group $\pi_1(X)$ of X is the free product of $\pi_1(X^0)$ and $\pi_1(X^1)$ with the amalgamated subgroups $f_*^\varepsilon \pi_1(Y \times \varepsilon)$, $\varepsilon = 0,1$.

Let $p^\varepsilon : \tilde{X}^\varepsilon \to X^\varepsilon$, $\varepsilon = 0,1$, and $q : \tilde{Y} \times [0,1] \to Y \times [0,1]$ be the universal coverings. Put $H^\varepsilon = f_*^\varepsilon \pi_1(Y \times \varepsilon)$ and $A^\varepsilon =$ (a left coset representative system of $\pi_1(X^\varepsilon)$ by H^ε, which includes the identity e), $\varepsilon = 0,1$. A group G is defined as the free product of $\pi_1(X^0)$ and $\pi_1(X^1)$ with the amalgamated subgroups H^0 and H^1, under the map $f_*^1 \circ (f_*^0)^{-1}$, denoted by

$$G = \langle \pi_1(X^0) * \pi_1(X^1) \mid H^0 = H^1,\ f_*^1 \circ (f_*^0)^{-1} \rangle.$$

And three subsets K, K^0, K^1 of G are defined by

$$\begin{aligned} K &= \{e, a_1 a_2 \cdots a_m \mid a_i \neq e, a_i \in A^0 \cup A^1, \\ &\qquad\qquad a_i, a_{i+1} \text{ are not both in } A^0 \text{ or both in } A^1.\} \\ K^0 &= \{e, a_1 a_2 \cdots a_m \in K \mid a_m \in A^1\} \\ K^1 &= \{e, a_1 a_2 \cdots a_m \in K \mid a_m \in A^0\}. \end{aligned}$$

For each $k \in K^\varepsilon$, prepare a copy $\tilde{X}_k^\varepsilon$ of $\tilde{X}^\varepsilon$, and the identity map $id_k^\varepsilon : \tilde{X}_k^\varepsilon \to \tilde{X}^\varepsilon$. Note that there are $\#A^\varepsilon$ equivalent classes of $\mathrm{Aut}(\tilde{X}^\varepsilon, p^\varepsilon)\tilde{f}^\varepsilon(\tilde{Y} \times \varepsilon)$ mod $(H^\varepsilon)_A$, $\varepsilon = 0,1$. And for each $(k,a) \in K^0 \times A^0$, prepare a copy $\tilde{Y}_{(k,a)} \times [0,1]$ of $\tilde{Y} \times [0,1]$, and the identity map $id_{(k,a)} : \tilde{Y}_{(k,a)} \times [0,1] \to \tilde{Y} \times [0,1]$. Let $\tilde{f}^\varepsilon : \tilde{Y} \times \varepsilon \to \tilde{X}^\varepsilon$ be structure maps of f^ε, $\varepsilon = 0,1$. Then we can define structure maps $\tilde{f}^\varepsilon_{(k,a)} : \tilde{Y}_{(k,a)} \times \varepsilon \to \tilde{X}_h^\varepsilon$ naturally. Put $\tilde{X} = (\tilde{X}_k^0, \tilde{X}_\ell^1, \tilde{Y}_{(k,a)} \times [0,1], \tilde{f}^0_{(k,a)}, \tilde{f}^1_{(k,a)})_{k \in K^0, \ell \in K^1, a \in A^0}$. Define the projections $p_k^\varepsilon : \tilde{X}_k^\varepsilon \to X^\varepsilon$ and $q_{(h,a)} : \tilde{Y}_{(h,a)} \times [0,1] \to Y \times [0,1]$ by $p_k^\varepsilon = p^\varepsilon \circ id_k^\varepsilon$ and $q_{(h,a)} = q \circ id_{(h,a)}$, $k \in K^\varepsilon$, $\varepsilon = 0,1$, $(h,a) \in K^0 \times A^0$, respectively. Note that p_k^ε and $q_{(h,a)}$ are the universal coverings. Furthermore, it is easy to see that $C(\tilde{X})$ is a tree. Hence by 3.10, $p = \cup_{k \in K^\varepsilon, \varepsilon = 0,1, (h,a) \in K^0 \times A^0} (p_k^\varepsilon \cup q_{(h,a)}) : \tilde{X} \to X$ is the universal covering.

Lemma 4.1. $\pi_1(X, x_0) \cong G$. *Proof.* See [T-Y 3].

§5. Extensions and constructions of orbi-maps

Definition 5.1. Let X be an orbifold composition. Define

$$\begin{aligned} O_1(X) &= \{f : \partial D \to X \mid D \text{ is a discal 2-orbifold, } f \text{ is an orbi-map}\},\\ O_2(X) &= \{f : S \to X \mid S \text{ is a spherical 2-orbifold, } f \text{ is an orbi-map}\},\\ O_3(X) &= \{f : \mathcal{D}B \to X \mid \mathcal{D}B \text{ is the double of a ballic 3-orbifold } B,\\ &\qquad f \text{ is an orbi-map}\}. \end{aligned}$$

We call $f : \partial D \to X \in O_1(X)$ *trivial* if there exists an orbi-map $g : D \to X$ s.t. $g|\partial D = f$, and call $O_1(X)$ *trivial* if any element of $O_1(X)$ is trivial. We call $f : S \to X \in O_2(X)$ *trivial* if there exists an orbi-map $g : c * S \to X$ s.t. $g|S = f$, where $c * S$ is the cone on S, and call $O_2(X)$ *trivial* if any element of $O_2(X)$ is trivial. We define the trivialities of $O_3(X)$ similarly.

Note that if $O_i(X)$ is trivial, then any covering $\tilde{X}$ of X inherits the triviality.

Proposition 5.2. *Let F be a compact 2-orbifold and X be an orbifold composition. If $O_1(X)$ is trivial, then for any homomorphism $\varphi : \pi_1(F, y) \to \pi_1(X, x)$, there exists an orbi-map $f : (F, y) \to (X, x)$ s.t. $f_* = \varphi$.*

Proof. Let $F_0 = F - \mathrm{Int}\, U(\Sigma F)$, where $U(\Sigma F)$ is the small regular neighborhood of ΣF. We construct an (orbi-) map from F_0 to X associated with φ. Since $O_1(X)$ is trivial, it is extendable to the desired orbi-map. (Q.E.D.)

The following propositions 5.3 and 5.4 are proved similarly.

Proposition 5.3. *Let M be a compact 3-orbifold and X an orbifold composition s.t. $O_1(X)$ and $O_2(X)$ are trivial. Then for any homomorphism $\varphi : \pi_1(M, x) \to \pi_1(X, y)$, there exists an orbi-map $f : (M, x) \to (X, y)$ s.t. $f_* = \varphi$.*

Proposition 5.4. *Let M be a 3-orbifold and X be an orbifold composition s.t. $O_3(X)$ is trivial. If f, $g : M \to X$ are C-equivalent orbi-maps, then f and g are orbi-homotopic.*

The following lemmas 5.5, 5.6, and 5.7 give sufficient conditions which enable us to extend certain orbi-maps.

Lemma 5.5. *Let X be an orbifold composition, D a discal 2-orbifold, and*

$f : \partial D \to X$ an orbi-map. If $\mathrm{Fix}([f]_A) \neq \phi$, *then f is extendable to an orbi-map from D to X.*

Proof. Let $q : D^2 \to D$ be the universal covering. Take a point $x \in \mathrm{Fix}([f]_A)$. We can construct the structure map of the desired orbi-map by mapping the cone point of D^2 to x and performing the skeletonwise and equivariant extension. (Q.E.D.)

Let S be a spherical 2-orbifold and $q : \tilde{S} \to S$ the universal covering. Let τ be an element of $\pi_1(S)$ and x_τ the point of ΣS s.t. $[\ell]^k = \tau$, where ℓ is the normal loop around x_τ and k is an integer. By the symbol $\mu(\ell)$, we mean the local normal loop around x_τ s.t. $\ell = m^{-1} \cdot \mu(\ell) \cdot m$, where m is a path. Let $\tilde{x}_\tau$ be the point of $q^{-1}(\Sigma S)$ s.t. the lft of $\mu(\ell)$ following after the lift of m^{-1} is a path around $\tilde{x}_\tau$.

Lemma 5.6. *Let X be an orbifold composition, S a spherical 2-orbifold, and $f : S \to X$ an orbi-map. Suppose that there is a point $\tilde{d} \in \mathrm{Fix}(f_*\pi_1(S))_A$ and for any $\tau \in \pi_1(S)$, there is an interval ℓ_σ including $\tilde{d}$ and $\tilde{f}(\tilde{x}_\tau)$ which is fixed by σ_A, where $\sigma = f_*(\tau)$. If π_2 of the universal cover $\tilde{X}$ of X is 0, then f is extendable to an orbi-map from the cone on S to X.*

Proof. See [T-Y 3].

Lemma 5.7. *Let X be an orbifold composition, B a ballic 3-orbifold, and $f : \mathcal{D}B \to X$ an orbi-map. Suppose that there is a point $\tilde{d} \in \mathrm{Fix}(f_*\pi_1(\partial B))_A$ and for $\forall \tau \in \pi_1(\partial B)$, there is an interval ℓ_σ including $\tilde{d}$ and $\tilde{f}(\tilde{x}_\tau)$ which is fixed by σ_A, where $\sigma = f_*(\tau)$. If π_2 and π_3 of the universal cover $\tilde{X}$ of X is 0, then f is extendable to an orbi-map from the cone on $\mathcal{D}B$ to X.*

Proof. Similar to 5.6. (Q.E.D.)

Lemma 5.8. *Let M be an irreducible 3-orbifold. Let $p : \hat{M} \to M$ be the universal covering and $\sigma \in \mathrm{Aut}(\hat{M}, p)$ be an orientation preserving element of finite order. Suppose that $\hat{M}$ is non-compact, then the following* (i), (ii) *hold:*

(i) $\mathrm{Fix}(\sigma) \neq \phi$ *and is homeomorphic to an interval* (*i.e. homeomorphic to either* $[0,1]$, $[0,1)$, *or* $(0,1)$).

(ii) *If M is orientable, then $O_1(M)$ is trivial.*

Proof. (ii) It is obtained by (i) and 5.5. (i) See [T-Y 3].

Lemma 5.9. *Let M be an irreducible 3-orbifold, and $p : \hat{M} \to M$ the universal covering. Let G be any subgroup of* $\mathrm{Aut}(\hat{M}, p)$, *which is isomorphic to the orbifold fundamental group of a spherical 2-orbifold S and all elements of G preserve the orientation of $\hat{M}$. Suppose that $\hat{M}$ is non-compact, then the following* (i), (ii) *hold:*

(i) $\mathrm{Fix}(G) \neq \phi$.

(ii) *If M is orientable, then $O_i(M)$'s are trivial, $i = 1, 2, 3$.*

Proof. (ii) It is obtained by (i), 5.5, 5.6, 5.7, and 5.8. (i) See [T-Y 3].

Proposition 5.10. *Let $X = (X^\varepsilon, Y \times [0,1], f^\varepsilon)_{\varepsilon=0,1}$ be an orbifold composition, where each X^ε is an orientable, irreducible 3-orbifold, and Y is an orientable 2-orbifold. If the universal coverings of X^ε and Y are all non-compact, then $O_i(X)$ are trivial, $i = 1, 2, 3$.*

Proof. See [T-Y 3].

Let X be an orbifold composition, and E a core of an edge orbifold $Y \times [0,1]$ of X. When we consider each piece (or its closure) of $|X| - |F|$, it naturally admits the orbifold composition structure by restricting the structure of X. We denote it by $X - F$, etc. In this situation, a component of type $Y \times [\varepsilon, \frac{1}{2}]$ (resp. $Y \times [\varepsilon, \frac{1}{2})$), $\varepsilon = 0, 1$, appears, and is called a closed (resp. open) half-edge orbifold of the orbifold composition. Iterating this process, we can consider an orbifold composition with several half-edge orbifolds. About new types of orbifold compositions described above, the same arguments and statements hold as those in Sect. 3~5.

§6. More on orbifold compositions

Let X be an orbifold composition. An orbifold Y belongs to the set δX if Y satisfies the following (i) or (ii):

(i) Y is a boundary component of a vertex orbifold of X s.t. Y is disjoint from any images of attaching maps of X.

(ii) Y is the core of a closed half-edge of X s.t. $\partial Y = \phi$.

Theorem 6.1. (Transversality theorem) *Let M be a compact and ori-*

entable 3-orbifold, and X a 3-orbifold composition with trivial $O_i(X)$'s, $i = 2, 3$. Suppose that there is an edge orbifold whose core is an orientable and non-spherical 2-orbifold F s.t. $O_i(X - F)$ are trivial, $i = 2, 3$. Then, for any orbi-map $f : M \to X$, there is an orbi-map $g : M \to X$ s.t.

(i) *g is orbi-homotopic to f,*

(ii) *each component of $g^{-1}(F)$ is a compact, properly embedded, 2-sided, incompressible 2-suborbifold in M, and*

(iii) *for properly chosen product neighborhoods $F \times [-1, 1]$ of $F = F \times 0$ in X and $g^{-1}(F) \times [-1, 1]$ of $g^{-1}(F) = g^{-1}(F) \times 0$ in M, $\bar{g}$ maps each fiber $x \times |[-1, 1]|$ homeomorphically to the fiber $\bar{g}(x) \times |[-1, 1]|$ for each $x \in |g^{-1}(F)|$ where $\bar{g} : |M| \to |X|$ is the underlying map of g.*

Proof. See [T-Y 3].

Theorem 6.2. (I-bundle theorem) *Let M be a compact, orientable and irreducible 3-orbifold with boundaries, and X be a 3-orbifold composition. Let $f : (M, \partial M) \to (X, \delta X)$ be an orbi-map s.t. f_* is monic. Suppose there is a path $\alpha : (I, \partial I) \to (|M| - \Sigma M, |\partial M|)$, incompressible components B_0, B_1 of ∂M, and a component C of δX which satisfy the following* (i)~(iv);

(i) $\alpha(0) \neq \alpha(1)$.

(ii) $\bar{f}(\alpha(0)) = \bar{f}(\alpha(1)) \in |\delta X| - \Sigma X$.

(iii) $[\tilde{f} \circ \hat{\alpha}] = 1$ *in* $\pi_1(X)$,

where $\hat{\alpha}$ is a lift of α to the universal cover $\tilde{M}$ of M and $f = (\bar{f}, \tilde{f})$.

(iv) *B_i (resp. C) includes $\alpha(i)$ (resp. $\bar{f}(\alpha(0))$),* $\mathrm{Ker}(\pi_1(C) \to \pi_1(X)) = 1$, *and $(f|B_i) : B_i \to C$ is a covering, $i = 0, 1$ (possibly $B_0 = B_1$).*

Then M is an I-bundle over a closed 2-orbifold.

Proof. See [T-Y 3].

Theorem 6.3. (Retraction theorem) *Let M be an orientable 3-orbifold which is orbi-isomorphic to an I-bundle over a closed 2-orbifold F. Let X be a 3-orbifold composition with trivial $O_i(X)$'s, $i = 2, 3$. Let $f : (M, \partial M) \to (X, \delta X)$ be an orbi-map s.t. $f|\partial M$ is not an orbi-embedding and s.t. there is a component C of δX, for each component B of ∂M, $f(B) \subset C$ and $(f|B) : B \to C$ is an orbi-covering.*

If there is a point $x \in |F| - \Sigma F$ s.t. $f|(\varphi^{-1}(x))$ is orbi-homotopic to a path in C rel. $\{x\} \times \partial I \ldots (6.3.1)$, where $\varphi : M \to F$ is the fibration, then there is an orbi-homotopy $f_t : M \to X$ s.t. $f_0 = f, f_1(M) \subset \delta X$, and $f_t|\partial M = f|\partial M$.

Proof. See [T-Y 3].

Remark 6.4. *In 6.3, if $f_* : \pi_1(M) \to \pi_1(X)$ is an isomorphism and C is orientable, then the condition (6.3.1) stands. Furthermore, M is orbi-isomorphic to the product I-bundle over B_0, and B_0 is orbi-isomorphic to C.*

§7. Main Theorem

In this section, we assume that all free products with amalgamations are non-trivial.

Definition 7.1. Let M be a 3-orbifold with trivial $O_1(M)$. Let S be a closed, orientable, non-spherical 2-orbifold. Suppose $\pi_1(M) = \langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$ and there is an isomorphism $\psi : \pi_1(S) \to H_1$. Let $p_i : X_i \to M$ be the orbi-covering associated with A_i, $i = 1, 2$. Note that $O_1(X_i)$ are trivial, $i = 1, 2$. Put $\tilde{H}_i = p_{i*}^{-1}(H_i)$, $i = 1, 2$. Note that $(p_{1*}|\tilde{H}_1)^{-1} \circ \psi$ (resp. $(p_{2*}|\tilde{H}_2)^{-1} \circ \varphi \circ \psi$) is an isomorphism from $\pi_1(S)$ to $\tilde{H}_1$ (resp. $\tilde{H}_2$). By 5.2, we can construct orbi-maps $h_1 : S \to X_1$ and $h_2 : S \to X_2$ s.t. $h_{1*} = (p_{1*}|\tilde{H}_1)^{-1} \circ \psi$ and $h_{2*} = (p_{2*}|\tilde{H}_2)^{-1} \circ \varphi \circ \psi$. We call the orbifold composition $X = (X_1, X_2, S\times[0,1], h_1, h_2)$ the *orbifold composition associated with* $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$. We also define the *orbifold composition associated with* $\langle A, t \mid t^{-1}H_1 t = H_2, \varphi\rangle$ similarly.

From 4.1 (resp. 4.2), it holds that $\pi_1(X) = \langle \pi_1(X_1) * \pi_1(X_2) \mid h_{1*}\pi_1(S) = h_{2*}\pi_1(S), h_{2*} \circ h_{1*}^{-1}\rangle$ (resp. $\langle \pi_1(X'), t \mid t^{-1}h_{1*}\pi_1(S)t = h_{2*}\pi_1(S), h_{2*} \circ h_{1*}^{-1}\rangle$). Furthermore, we have the following proposition.

Proposition 7.2. *Let M be a 3-orbifold with trivial $O_1(M)$. Let S be a closed, orientable, and non-spherical 2-orbifold. Suppose $\pi_1(M) = \langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$ (resp. $\langle A, t \mid t^{-1}H_1 t = H_2, \varphi\rangle$) and there is an isomorphism $\psi : \pi_1(S) \to H_1$. Let X be the orbifold composition associated with $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$ (resp. $\langle A, t \mid t^{-1}H_1 t = H_2, \varphi\rangle$). Then there is an isomorphism $\Psi : \pi_1(X) \to \pi_1(M)$ s.t.*

(i) $\Psi(\pi_1(X_i)) = A_i$, *$i = 1, 2$ (resp.* $\Psi(\pi_1(X')) = A$*).*

(ii) $\Psi(\tilde{H}_i) = H_i$, *$i = 1, 2$ (note that* $h_{i*}\pi_i(S) = \tilde{H}_i$*).*

(iii) $\Psi \circ (h_{2*} \circ h_{1*}^{-1}) = \varphi \circ \Psi$.

Definition 7.3. Let M be a 3-orbifold, and S be a closed, orientable, and non-spherical 2-orbifold. We say that S *algebraically splits* $\pi_1(M)$ *as an amal-*

gamated free product if $\pi_1(M)$ is expressed as a free product with an amalgamation, $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$, and there is an isomorphism $\Psi : H_1 \to \pi_1(S)$.

We say that the splitting above *respects the peripheral structure* of M if for each component G of ∂M, some conjugate of $\eta_* \pi_1(G)$ is contained in either A_1 or A_2, where η is the inclusion orbi-map $G \to M$.

Proposition 7.4. *Let M be a compact, orientable, and irreducible 3-orbifold. Let S be a closed, orientable, and non-spherical 2-orbifold. Suppose S algebraically splits $\pi_1(M)$ as an amalgamated free product $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$ and this splitting respects the peripheral structure of M. Let X be the orbifold composition associated with $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$. Then there is an orbi-map $f : M \to X$ s.t. f_* is an isomorphism and $f(\partial M) \cap (S \times (0,1)) = \phi$.*

Proof. See [T-Y 3].

Definition 7.5. Let F be a closed, properly embedded, 2-sided, incompressible, and separating 2-suborbifold in M. Let M_1, M_2 be the orbifolds derived from M by cutting open along F and $\eta_i : F \to M_i$, i =1, 2 be the inclusion orbi-maps. Note that $\pi_1(M)$ is expressed as the amalgamated free product $\langle \pi_1(M_1) * \pi_1(M_2) \mid \eta_{1*}\pi_1(F) = \eta_{2*}\pi_1(F), \eta_{2*} \circ \eta_{1*}^{-1}\rangle$. We say that F *geometrically realizes the algebraic splitting* $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$ of $\pi_1(M)$ if there is an isomorphism $\Psi : \pi_1(M) \to \pi_1(M)$ s.t.

(i) $\Psi(\pi_1(M_i)) = A_i$, i =1, 2.

(ii) $\Psi(\eta_{i*}\pi_1(F \times i)) = H_i$, i =1, 2.

(iii) $\Psi \circ (\eta_{2*} \circ \eta_{1*}^{-1}) = \varphi \circ \Psi$.

Theorem 7.6. *Let M be a compact, orientable, and irreducible 3-orbifold. Let S be a closed, orientable, and non-spherical 2-orbifold. Suppose S algebraically splits $\pi_1(M)$ as an amalgamated free product $\langle A_1 * A_2 \mid H_1 = H_2, \varphi\rangle$ and this splitting respects the peripheral structure of M. Then there exists a geometric splitting realizing the algebraic splitting above.*

Let us take an overview of the proof of the main theorem, to see how effectively our preparations are used:

(i) Recall that the fundamental group $\pi_1(M)$ of a 3-orbifold M is decomposed as $\langle A_1 * A_2 \mid H_1 = H_2, \phi\rangle$. First we take $S \times I$ and the orbi-covering M_i associated with A_i and construct an orbifold composition X by attaching them together where Section 4 and 5.2 essentially contribute to. This newly constructed space X plays a role like as an Eilenberg-MacLane space.

(ii) Make an orbi-map $f : M \to X$ which induces an isomorphism from $\pi_1(M)$ to $\pi_1(X)$. At this time, we need theorems prepared in Sections 4 and 5.

(iii) Each component of the inverse image of S by f is an incompressible 2-suborbifold by 6.1. We decrease the numbers of these components using 6.2 and 6.3 repeatedly. At last the inverse image turns to be only one component F which actually realizes the decomposition of $\pi_1(M)$.

For the details or the HNN extension case, see [T-Y 3].

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