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CONSTRUCTION OF RIEMANN SURFACES BY PARALLEL TRANSFORMATIONS

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1. INTRODUCTION

In this paper we introduce a new method of constructing once punctured Riemann surfaces. In our construction we use line segments in the complex plane $\mathbb{C}$ and parallel transformations: For a pair of disjoint parallel line segments with the same length in $\mathbb{C}$, we first cut $\mathbb{C}$ along the segments and paste each side of one segment and the opposite side of the other segment by a parallel transformation obtaining a once punctured elliptic curve. The puncture is at infinity. (See §2, Figure 1.) We shall call such a pair an *Igeta.* (Igeta is a Japanese word coming from a technical term "Igeta-kuzushi" used in a Japanese martial art.) Putting g disjoint pieces of Igeta on $\mathbb{C}$, we obtain a once punctured Riemann surface of genus g in the same way. We denote a set of g disjoint Igeta by Γ and the resulting once punctured Riemann surface by $(R(\Gamma), p_\infty)$. Moreover when we move the position of g Igeta, there appears a family of once punctured Riemann surfaces of genus g. All the possible configurations of g disjoint Igeta up to the affine automorphisms of $\mathbb{C}$ form a $3g-2$-dimensional complex V-manifold and this dimension is the same as the dimension of the moduli space $\mathcal{M}_{g,1}$ of once punctured Riemann surfaces of genus g. We thus expect to have a visual image of the moduli space by using this construction.

We first consider the Kodaira-Spencer maps of the family. Let $I_g\eta$ be the collection of Γ's, and let $I_g\eta_0$ be the subset of $I_g\eta$ consisting of those Γ having $[0,1]$ as one of its $2g$ line segments. $I_g\eta$ turns out to be a $3g$-dimensional complex manifold and $I_g\eta_0$ a $3g-2$-dimensional complex manifold. Our first main result is as follows:

Theorem 1 . *The Kodaira-Spencer map*

$$\rho_\Gamma[-3]\ : T(I_g\eta)_\Gamma \longrightarrow H^1(R(\Gamma);\Theta(-3p_\infty))$$

is an isomorphism for any $\Gamma\in I_g\eta$, *where* $T(I_g\eta)_\Gamma$ *is the holomorphic tangent space of* $I_g\eta$ *at* Γ *and* $\Theta(-3p_\infty)$ *is the sheaf of germs of holomorphic vector fields on* $R(\Gamma)$ *having zero at* p_∞ *of order at least* 3.

Corollary 1 . *The Kodaira-Spencer map*

$$\rho_{\Gamma,0}\ : T(I_g\eta_0)_\Gamma \longrightarrow H^1(R(\Gamma);\Theta(-p_\infty))$$

is an isomorphism for any $\Gamma\in I_g\eta_0$.

For a closed Riemann surface R of genus g we define a *Lagrangian sublattice* Λ of R to be a subgroup of $H_1(R;\mathbb{Z})$ which coincides its orthogonal complement with respect to the intersection form on $H_1(R;\mathbb{Z})$, i.e. a subgroup isomorphic to $\mathbb{Z}^g$ such that the quotient $H_1(R;\mathbb{Z})/\Lambda$ is also isomorphic to $\mathbb{Z}^g$ and the intersection number of any two elements in Λ equals zero. Moreover, for any once punctured Riemann surface (R,p) of genus g, a Lagrangian sublattice Λ of $H_1(R;\mathbb{Z})$ and the puncture p determine a certain Abelian differential ω_Λ of the second kind on the surface unique up to scalars. When we construct a once punctured Riemann surface $(R(\Gamma), p_\infty)$ from Γ, $R(\Gamma)$ has a natural Lagrangian sublattice Λ_Γ. On the other hand if we denote by ζ the standard coordinate of $\mathbb{C}$, $R(\Gamma)$ has a natural Abelian differential ω_Γ of the second kind induced by $d\zeta$. It turns out that ω_Γ is equal to ω_{Λ_Γ} up to scalars. We use ω_Γ to prove Theorem 1.

Furthermore, using ω_Λ of (R, p, Λ) we obtain the following result:

Corollary 2 . *For an arbitrary once punctured Riemann surface with a Lagrangian sublattice (R, p, Λ), (R, ω_Λ) and $(\mathbb{C}P_1, d\zeta)$ are piecewise parallel.*

We call two Riemann surfaces (R, ω) and (R', ω') with Abelian differentials of the second kind *piecewise parallel* if after decomposing (R, ω) into small pieces having line-segment-boundaries we can obtain (R', ω') by pasting them together using parallel transformations in another way. This operation turns out to be reversible. (See §3.)

Corollary 2 indicates that any once punctured Riemann surface can be obtained from $\mathbb{C}$ by cutting along line segments and pasting by parallel transformations. We have to remark here that this corollary does not imply that any Riemann surface can be obtained by Igeta-construction. Nevertheless from this result we expect that any once punctured Riemann surface with a Lagrangian sublattice would appear in some natural extension of our family.

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2. Igeta-construction and the Kodaira-Spencer maps

The Gauss plane is the complex affine line $\mathbb{A}^1$ with a fixed global coordinate $\zeta : \mathbb{A}^1 \to \mathbb{C}$. Consider the set η consisting of unordered pairs (σ^+, σ^-) of disjoint line segments in the Gauss plane $(\mathbb{A}^1, \zeta)$ such that σ^+ and σ^- are parallel and equilateral. We denote by $I_g\eta$ the collection of unordered sets Γ of g elements of η where

$$\Gamma = ((\sigma_j^+, \sigma_j^-) \in \eta \ ; \ j = 1, \ldots, g)$$

such that $\sigma_j^\pm$ are pairwise disjoint. Let $\phi_j^\pm = \phi_j^\pm[\Gamma]$ be the affine map from the Gauss plane $(\mathbb{A}^1, \xi)$ to $(\mathbb{A}^1, \zeta)$ given by

$$\zeta = a_j \xi + b_j^\pm, \quad a_j \in \mathbb{C}^\times, \ b_j^\pm \in \mathbb{C}$$

such that $\sigma_j^\pm = \phi_j^\pm([-1, 1])$. The space $I_g\eta$ is a $3g$-dimensional open complex manifold with local coordinates $(a_j, b_j^\pm \ ; \ j = 1, \ldots, g)$ for a fixed order of line segments.

We construct a holomorphic family of once punctured Riemann surfaces of genus g over $I_g\eta$ as follows. Let B be an open and relatively compact subset of $I_g\eta$. Set

$$E_\Gamma^0 = \mathbb{A}^1 - \bigcup_{j=1}^{g} (\sigma_j^+ \cup \sigma_j^-)$$

for $\Gamma = (\sigma_j^\pm) \in I_g\eta$ and set

$$E^0 = \bigsqcup_{\Gamma \in I_g\eta} E_\Gamma^0 \subset I_g\eta \times \mathbb{A}^1,$$

$$E_B^0 = E^0 \cap (B \times \mathbb{A}^1).$$

Let U_∞ be the disk $\{w \in \mathbb{C} ; |w| < \epsilon\}$ and V_j $(j = 1, \ldots, g)$ copies of the annulus

$$\{z \in \mathbb{C} ; (1+\epsilon)^{-1} < |z| < 1+\epsilon\}$$

for $\epsilon > 0$ and let

$$V_j^+ = \{z \in V_j ; |z| > 1\}, \quad V_j^- = \{z \in V_j ; |z| < 1\}.$$

Note that the Joukowski transform

$$J(z) = \frac{1}{2}(z + z^{-1})$$

maps the unit circle in $\mathbb{C}$ onto the interval $[-1, 1]$. For sufficiently small $\epsilon > 0$, we paste the patches

$$E_B^0, \quad B \times U_\infty, \quad B \times V_j \ (j = 1, \ldots, g)$$

by the attaching maps

$$B \times (U_\infty - \{0\}) \ni (\Gamma, w) \longmapsto (\Gamma, w^{-1}) \in E_B^0,$$
$$B \times V_j^\pm \ni (\Gamma, z) \longmapsto (\Gamma, \phi_j^\pm[\Gamma] \circ J(z)) \in E_B^0$$

and obtain a complex manifold E_B, which is the total space of a holomorphic family of once punctured Riemann surfaces of genus g over B. As $I_g\eta$ is locally compact, we can construct the holomorphic family $\pi : E \to I_g\eta$ such that $\pi^{-1}(B) = E_B$ for any open and relatively compact subset B of $I_g\eta$. For a point Γ of $I_g\eta$ the Riemann surface $R(\Gamma) = \pi^{-1}(\Gamma)$ is constructed by pasting the patches

$$E_\Gamma^0, \quad U_\infty, \quad V_j \ (j = 1, \ldots, g)$$

through the attaching map

$$U_\infty \ni w \longmapsto \zeta = w^{-1} \in E_\Gamma^0,$$
$$V_j^\pm \ni z \longmapsto \zeta = \phi_j^\pm \circ J(z) \in E_\Gamma^0.$$

Denote by p_∞ the puncture on $R(\Gamma)$ corresponding to $0 \in U_\infty$.

We call such a pair (σ^+, σ^-) *Igeta* and the construction mentioned above *Igeta-construction.* Roughly speaking, the Igeta-construction is to cut the Gauss plane along the g pairs of line segments $(\sigma_j^\pm ; j = 1, \ldots, g)$ and to paste each side of σ_j^+ and the opposite side of σ_j^- by a parallel transformation. (Figure 1. The numbers (1), ... ,(6) in Figure 1 indicate where to paste.)

We investigate the infinitesimal deformation for this family $\pi : E \to I_g\eta$. We differentiate at a point Γ of $I_g\eta$ the coordinate transformation for the Riemann surface $R(\Gamma)$ by the parameters of $I_g\eta$, and obtain the *Kodaira-Spencer map* ([K]). We consider the Kodaira-Spencer map with respect to the deformation fixing p_∞ to order n:

$$\rho_\Gamma[-n] : T(I_g\eta)_\Gamma \longrightarrow H^1(R(\Gamma), \Theta(-np_\infty))$$

where Θ denotes the sheaf of holomorphic vector fields on $R(\Gamma)$. Now we state the following:

Theorem 1 . *The Kodaira-Spencer map*

$$\rho_\Gamma[-3] : \ T(I_g\eta)_\Gamma \longrightarrow H^1(R(\Gamma), \Theta(-3p_\infty))$$

is an isomorphism.

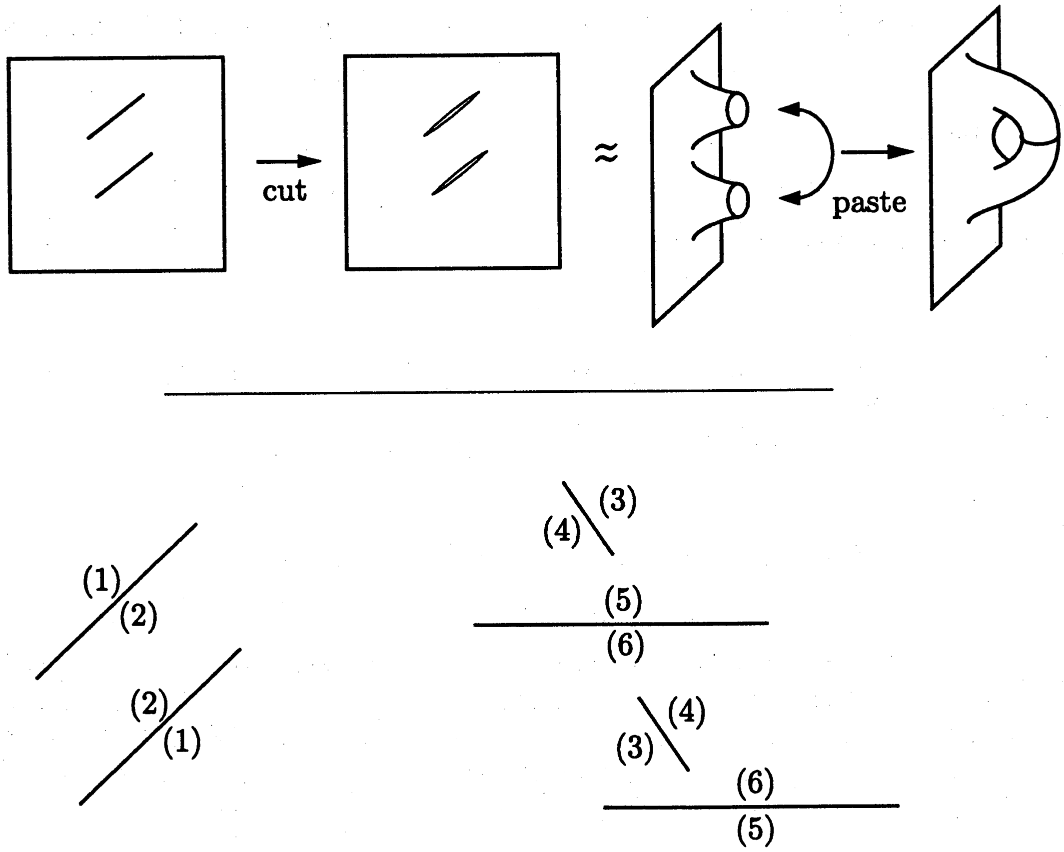

FIGURE 1. Igeta-construction

We first calculate the dimension of $H^1(\Theta(-3p_\infty))$. (Note that we omit $R(\Gamma)$ in $H^*(R(\Gamma), *)$.) By the Riemann-Roch formula,

$$\dim H^0(\Theta(-3p_\infty)) - \dim H^1(\Theta(-3p_\infty)) = 1 - g + c_1(\Theta(-3p_\infty)) = -3g.$$

As $c_1(\Theta(-3p_\infty)) = -(2g+1)$ is negative, $H^0(\Theta(-3p_\infty)) = 0$. So $\dim H^1(\Theta(-3p_\infty)) = 3g$, which coincides with the dimension of $I_g\eta$. Thus we show the surjectivity of $\rho = \rho_\Gamma[-3]$.

The pairing

$$\langle\ ,\ \rangle\ :\ H^0(\Omega^1 \otimes \Omega^1(3p_\infty)) \times H^1(\Theta(-3p_\infty)) \to H^1(\Omega^1) \cong \mathbb{C}$$

is nondegenerate because of the Serre duality. Note that the 1-form $d\zeta$ on E^0_Γ extends to a meromorphic 1-form $\omega = \omega_\Gamma$ on $R(\Gamma)$, which has a 2-pole at p_∞ and $2g$ zeros at $q_j^\pm$ $(j = 1, \ldots, g)$ corresponding to $\pm 1 \in V_j$. The multiplication by ω induces the isomorphism

$$N := H^0(\Omega^1(p_\infty + \sum_{j=1}^{g}(q_j^+ + q_j^-))) \to H^0(\Omega^1 \otimes \Omega^1(3p_\infty)).$$

Hence it is sufficient to show the following :

$$\chi \in N \text{ vanishes if } \langle \chi\omega,\ \rho(v)\rangle = 0 \text{ for any } v \in T(I_g\eta)_\Gamma.$$

The calculation of the pairing is based on the following lemma :

Lemma 1. *Let $R = \bigcup_{U \in S} U$ be a Riemann surface with an open covering. A holomorphic 1-form α on $U_1 \cap U_2$ for $U_1, U_2 \in S$ induces an element $[\alpha]$ of $H^1(R, \Omega^1)$ (the cocycle* *vanishes on the other intersections of the coordinate neighborhoods). If $U_1 \cap U_2$ is an annulus bounded by two circles $C_1 \subset U_2$, $C_2 \subset U_1$, then the evaluation of $[\alpha]$ is given by $\langle\alpha\rangle = \pm\int_{C_1}\alpha$.*

Proof. We introduce a cut-off function ψ on $U_1 \cap U_2$ such that

$$\text{supp}\ \psi \subset U_2, \quad \text{supp}\ (1-\psi) \subset U_1.$$

Then the (1,1)-form $\bar{\partial}(\psi\alpha)$ extends by 0 outside $U_1 \cap U_2$ and the evaluation of $[\alpha]$ is given by

$$\int_R \bar{\partial}(\psi\alpha) = \int_R d(\psi\alpha) = \int_{C_1+C_2} \psi\alpha = \int_{C_1} \alpha$$

up to sign. □

Now we differentiate the coordinate transformations of $R(\Gamma)$. The map

$$U_\infty - \{0\} \ni w \longmapsto \zeta = w^{-1} \in E_\Gamma^0$$

is independent of the parameters of $I_g\eta$, and

$$V_j^\pm \ni z \longmapsto \zeta = \phi_j^\pm \circ J(z) \in E_\Gamma^0$$

depends only on $(a_j, b_j^\pm)$ and

$$\frac{\partial}{\partial a_j}(\phi_j^\pm \circ J(z)) = \frac{1}{2}(z+z^{-1}),$$
$$\frac{\partial}{\partial b_j^\pm}(\phi_j^\pm \circ J(z)) = 1.$$

Thus

$$\begin{aligned}\langle\chi\omega,\ \rho(\frac{\partial}{\partial b_j^\pm})\rangle &= \langle\chi\omega,\ [\frac{\partial}{\partial\zeta}\ \text{ on }\ \phi_j^\pm \circ J(V_j^\pm)]\rangle \\ &= \langle\chi \text{ on } \phi_j^\pm \circ J(V_j^\pm)\rangle \\ &= \int_{C_j^\pm}\chi\end{aligned}$$

where $C_j^\pm = \{z \in V_j\ ;\ |z| = (1+\frac{\epsilon}{2})^{\pm 1}\}$ oriented appropriately. Since the left-hand side vanishes by the assumption,

$$\int_{C_j^\pm}\chi = 0.$$

Hence $\text{Res}_{q_j^+}\chi + \text{Res}_{q_j^-}\chi = \int_{C_j^+ + C_j^-}\chi = 0$. Further,

$$\begin{aligned}\langle\chi\omega,\ \rho(\frac{\partial}{\partial a_j})\rangle &= \langle\chi\omega,\ [\frac{1}{2}(z+z^{-1})\frac{\partial}{\partial\zeta}\ \text{ on }\ \phi_j^+ \circ J(V_j^+) \cup \phi_j^- \circ J(V_j^-)]\rangle \\ &= \langle\frac{1}{2}(z+z^{-1})\chi\ \text{ on }\ \phi_j^+ \circ J(V_j^+) \cup \phi_j^- \circ J(V_j^-)\rangle \\ &= \int_{C_j^+ + C_j^-}\frac{1}{2}(z+z^{-1})\chi \\ &= \text{Res}_{q_j^+}\chi - \text{Res}_{q_j^-}\chi\end{aligned}$$

and it vanishes, so we get

$$\mathrm{Res}_{q_j^\pm}\chi = 0$$

and

$$\chi \in H^0(\Omega^1(p_\infty)) = H^0(\Omega^1)$$

(by the residue theorem). Finally

$$\int_{C_j^+} \chi = 0, \quad j = 1, \ldots, g$$

yields $\chi = 0$ by the bilinear relations of Riemann. □

The group Aut($\mathbb{C}$) of automorphisms of $\mathbb{C}$ acts on $I_g\eta$ as

$$(a_j, b_j^\pm) \mapsto (aa_j, b_j^\pm + b) \quad a \in \mathbb{C}^\times, b \in \mathbb{C}$$

preserving the complex structure of any once punctured Riemann surface $(R(\Gamma), p_\infty)$. Hence we obtain Corollary 1 below, which implies that the family of once punctured Riemann surfaces of genus g by Igeta-construction is complete and effectively parametrized at any point for each g.

Corollary 1 . *The Kodaira-Spencer map*

$$\rho_{\Gamma,0} \ : T(I_g\eta_0)_\Gamma \longrightarrow H^1(R(\Gamma); \Theta(-p_\infty))$$

is an isomorphism where

$$I_g\eta_0 = \{\Gamma \in I_g\eta \ ; \Gamma \textit{ has } [0,1] \textit{ as one of its } 2g \textit{ line segments.}\}.$$

Note that the submanifold $I_g\eta_0$ gives a local manifold cover of the V-manifold $I_g\eta/\mathrm{Aut}(\mathbb{C})$ at any point.

3. Cutting and pasting of Riemann polygons

In the previous section, given a once punctured Riemann surface $(R(\Gamma), p_\infty)$ constructed from Igeta Γ, we used the Abelian differential ω_Γ of the second kind on $R(\Gamma)$ in order to prove Theorem 1. Let Λ_Γ be the subgroup of $H_1(R(\Gamma); \mathbb{Z})$ generated by C_j^+ $(j = 1, \ldots, g)$. The integral of ω_Γ on any element λ of Λ_Γ vanishes, and the orthogonal complement of Λ_Γ with respect to the intersection form coincides with Λ_Γ itself.

We first give a definition of *Lagrangian sublattice*, which is deduced from the properties of Λ_Γ, for any closed Riemann surface as follows:

Definition 3.1. Let R be a closed Riemann surface. A *Lagrangian sublattice* is defined to be a subgroup Λ of $H_1(R; \mathbb{Z})$ satisfying $\Lambda = \Lambda^\perp$, where $\Lambda^\perp$ denotes the orthogonal complement with respect to the intersection form on $H_1(R; \mathbb{Z})$.

Let (R, p) be a once punctured Riemann surface of genus g and Λ a Lagrangian sublattice of $H_1(R; \mathbb{Z})$. The kernel Z_Λ of the homomorphism given by Abelian integrals

$$H^0(R; \Omega^1(2p)) \longrightarrow \mathrm{Hom}(\Lambda, \mathbb{C}) \ (\cong \mathbb{C}^g)$$

is always one-dimensional because it holds that

$$\dim H^0(R; \Omega^1(2p)) = g + 1$$

from the Riemann-Roch formula and the surjectivity is implied by the bilinear relations of Riemann. Accordingly, a Lagrangian sublattice and a point on the surface determine an Abelian differential up to scalars. (Note that $\Lambda \cong \mathbb{Z}^g$.)

Each Igeta Γ is associated to a once punctured Riemann surface $(R(\Gamma), p_\infty)$ together with the Lagrangian sublattice Λ_Γ. In the case of $(R, p, \Lambda) = (R(\Gamma), p_\infty, \Lambda_\Gamma)$, the kernel Z_{Λ_Γ} is generated by ω_Γ. Now the problem is the following:

Problem . *Is it possible to construct any (R, p, Λ) by cutting and pasting the Gauss plane using line segments and parallel transformations as in the Igeta-construction?*

We next introduce a concept "Riemann polygon" so that we can consider cutting and pasting of Riemann surfaces with Abelian differentials using "line segments" and "parallel transformations".

Let R be a Riemann surface and ω an Abelian differential on it. We call a simple path or simple loop $\gamma\ : [0,1] \to R$ *ω-line-segment* if its image contains no poles of ω and the integral

$$\int_{\gamma(0)}^{\gamma(t)} \omega$$

depends on $t \in [0,1]$ linearly. We also call its image *ω-line-segment.* The 2-form $\frac{i}{2}\omega \wedge \bar{\omega}$ induces a metric g_ω on R which has conical singularities at the zeros of ω. ω-line-segments are geodesics for this metric g_ω. Let us cut R along ω-line-segments and separate into finitely many pieces. We call a collection of such pieces *Riemann polygon:*

Definition 3.2. A *Riemann polygon* (F, ω) is defined to be a pair consisting of a compact, not necessarily connected Riemann surface F and an Abelian differential ω on F such that the boundary of F, if not empty, consists of ω-line-segments.

A Riemann surface with boundary in our understanding is a 2-dimensional topological manifold F with boundary which is embedded in a Riemann surface R and whose interior inherits its complex structure from R; in addition an Abelian differential on F is a restriction of some Abelian differential on R. For example, any polygon P in the real plane $\mathbb{R}^2$ is considered as a Riemann polygon $(P, d\zeta|_P)$ when $\mathbb{R}^2$ is identified with $\mathbb{C}$. If we compactify $E^0_\Gamma \cup U_\infty$ in §2 by attaching line segments to both sides of $\sigma_j^\pm$, the pair consisting of the compactification $\overline{E^0_\Gamma}$ and the 1-form $d\zeta$ is a Riemann polygon. We can also consider a closed Riemann surface with an Abelian differential as a Riemann polygon with empty boundary.

We can easily generalize the concept of "the translation scissors congruence" (see [Mo], [S]) to the case of Riemann polygons. In order to do so, we introduce two operations "P-cutting" and "P-pasting" for getting a Riemann polygon from another Riemann polygon.

P-cutting: Let (F, ω) be a Riemann polygon, and let γ be an ω-line-segment on it such that the image $\gamma((0,1))$ is in the interior of F. We first remove the ω-line-segment $\gamma([0,1])$ from F, and then compactify by attaching one copy of $\gamma([0,1])$ to each side of $\gamma([0,1])$ obtaining a new Riemann polygon (F', ω'), where ω' is induced by ω naturally. (If $\gamma(0)$ (resp. $\gamma(1)$) is in the interior of F and is different from $\gamma(1)$ (resp. $\gamma(0)$), $\gamma(0)$ (resp. $\gamma(1)$) in one of the copies of $\gamma([0,1])$ should be identified with $\gamma(0)$ (resp. $\gamma(1)$) in the other.)

P-pasting: Let (F, ω) be a Riemann polygon. If there are ω-line-segments γ and γ' on the boundary ∂F such that

$$\int_{\gamma(0)}^{\gamma(t)} \omega = \int_{\gamma'(0)}^{\gamma'(t)} \omega$$

for any $t \in [0,1]$ and the interior of F sits on opposite sides of the paths, then we can paste $\gamma([0,1])$ and $\gamma'([0,1])$ by identifying $\gamma(t)$ with $\gamma'(t)$ and obtain a new Riemann polygon (F', ω'), where ω' is induced by ω naturally.

For Riemann polygons in $\mathbb{C}$, P-cutting indicates cutting along line segments, and P-pasting indicates pasting by parallel transformations. Igeta-construction is a special way of P-cutting and P-pasting.

It is obvious that P-pasting is the inverse operation of P-cutting. So these operators give rise to an equivalence relation between Riemann polygons: We call Riemann polygons (F, ω) and (F', ω') *piecewise parallel*, if (F, ω) is obtained from (F', ω') by finitely many P-cuttings and P-pastings.

We shall consider the special case where F is a closed Riemann surface of genus g. Let R be a closed Riemann surface of genus g. For an Abelian differential ω of the second kind on R, we define a sequence $PT(\omega)$ and a real number $S(\omega)$ as follows:

Definition 3.3.

$PT(\omega) := \{n_i\}_{i \in \mathbb{Z}_+}$ (n_i is the number of poles of order i),

$$S(\omega) := \mathrm{Im}(\sum_{j=1}^{g} \int_{\alpha_j} \bar{\omega} \int_{\beta_j} \omega), \qquad ((\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g) \text{ is a symplectic basis of } H_1(R, \mathbb{Z})).$$

The value of S will be shown not to depend on the choice of the basis $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ in the proof of Theorem 2 below.

Theorem 2 . *Let ω and ω' be Abelian differentials of the second kind on closed Riemann surfaces R and R' such that $PT(\omega) = PT(\omega')$. Then the Riemann polygons (R, ω) and (R', ω') are piecewise parallel if and only if $S(\omega) = S(\omega')$.*

Before proving Theorem 2 we recall a result of Hadwiger-Glur about polygons in $\mathbb{R}^2$.

Let M be a finite set of polygons in $\mathbb{R}^2$, and let v be a unit vector in $\mathbb{R}^2$. (By our convention, M is a Riemann polygon.) Assume that the boundary of each polygon in M is oriented counterclockwise; we consider each boundary segment as a vector, and call them "boundary vectors". We denote by $A(M)$ the sum of the area of polygons in M, and define $J_v(M)$ to be the algebraic sum of the boundary vectors of M which are parallel to v.

Hadwiger-Glur's Theorem [Mo], [S] **.** *Let M and M' be finite sets of polygons in $\mathbb{R}^2$ such that $A(M) = A(M')$. M and M' are piecewise parallel if and only if $J_v(M) = J_v(M')$ for any unit vector v.*

Obviously, the invariant $J_v(M)$ can be extended as an invariant $J_v(R, \omega)$ of any Riemann polygon (R, ω) with respect to P-cuttings and P-pastings.

Proof of Theorem 2. It is obvious that $J_v(R, \omega) = J_v(R', \omega') = 0$ for any unit vector v because both R and R' are closed Riemann surfaces.

Fix a one-to-one correspondence between the poles of ω and the ones of ω' such that the orders of corresponding poles are equal, and fix for each pole p_j a local biholomorphism h_j which maps a neighborhood of p_j onto a neighborhood of the corresponding pole p'_j transforming ω into ω'. Let C_j be a small simple loop consisting of ω-line-segments around p_j. (We shall call such a loop simple polygonal loop.) Then $h_j(C_j)$ is a polygonal loop around p'_j. Cut off from R those components of $R - \sqcup_j C_j$ (resp. $R' - \sqcup_j h_j(C_j)$) which contain the poles. We then obtain a Riemann polygon (R_0, ω) (resp. (R'_0, ω')) with no poles. Furthermore, fix a symplectic basis $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ and $2g$ simple polygonal loops $(a_1, b_1, \ldots, a_g, b_g)$ representing them such that their intersection is only one point on R, and that they have no intersection with C_j's. Let $(\tilde{R}_0, \omega)$ be a Riemann polygon obtained from (R_0, ω) by cutting along $a_1, b_1, \ldots, a_g, b_g$. We do the same with (R'_0, ω') and denote by $(\tilde{R}'_0, \omega')$ the resulting Riemann polygon. It is sufficient for proving Theorem 2 to show that $(\tilde{R}_0, \omega)$ and $(\tilde{R}'_0, \omega')$ are piecewise parallel.

Now we can define a holomorphic function h on $\tilde{R}_0$ such that $dh = \omega$ because ω has no periods on $\tilde{R}_0$. Let g_ω be the metric on $\tilde{R}_0$ induced by the 2-form $\frac{i}{2}\omega \wedge \bar{\omega}$. The map h between $(\tilde{R}_0, g_\omega)$ and $(\mathbb{C}, g)$ is a local isometry at any point except zeros of ω, where g is the standard metric on $\mathbb{C}$. We deduce from Stokes formula

$$\int_{\tilde{R}_0} \frac{i}{2}\omega \wedge \bar{\omega} = S(\omega) + \int_{\sum_j C_j} \frac{i}{2} h\bar{\omega}$$

where C_j's are oriented counterclockwise.

The left-hand side of the equation above is the area of $\tilde{R}_0$ with respect to g_ω and the second term of the right hand side depends only on the behavior of the differential around its poles. So $S(\omega)$ is independent of the choice of the symplectic basis. We can decompose $\tilde{R}_0$ into small pieces each of which can be mapped to some polygon in $\mathbb{C}$ isometrically. Hence the conclusion follows from the theorem of Hadwiger-Glur. □

Now we return to the case of once punctured Riemann surfaces with Lagrangian sublattices. For the standard global coordinate ζ of $\mathbb{C}$, the differential $d\zeta$ uniquely extends to the complex projective line $\mathbb{C}P_1$ as an Abelian differential, which has a pole of order 2 at ∞. We also denote it by $d\zeta$.

Corollary 2 . *Let (R, p) be a once punctured Riemann surface and Λ be a Lagrangian sublattice of $H_1(R; \mathbb{Z})$. For a nonzero element $\omega \in Z_\Lambda$ the Riemann polygon (R, ω) is piecewise parallel to $(\mathbb{C}P_1, d\zeta)$.*

Proof. The assumption $\omega \in Z_\Lambda$ implies that $S(\omega) = 0$ and $PT(\omega) = PT(d\zeta)$. □

Corollary 2 indicates that any once punctured Riemann surface can be obtained from $\mathbb{C}$ by cutting along line segments and pasting by parallel transformations; the triple (R, p, Λ) is represented by a set of line segments on the complex plane plus pasting-data.

Remark 1. When we consider Riemann surfaces together with Abelian differentials of the first kind or holomorphic 1-forms, a result similar to Corollary 2 holds; any closed Riemann surface can be obtained from a fixed elliptic curve by cutting along line segments and pasting by parallel transformations. We first fix an elliptic curve and an Abelian differential on it. For instance, let E be the quotient $\mathbb{C}/L$ where L is the lattice generated by 1 and i, and we consider the standard Abelian differential $d\zeta$ of the first kind on E where ζ is the coordinate of $\mathbb{C}$. We next choose an Abelian differential ω of the first kind on any closed Riemann surface R so that $S(\omega) = S(d\zeta)(= 1)$. (It is easy to find such Abelian differentials.) We can show in the same way that (R, ω) and $(E, d\zeta)$ are piecewise parallel.

Remark 2. Igeta-construction leads us to consider the moduli space of once punctured Riemann surfaces with Lagrangian sublattices. In [H-O1] and [H-O2], we considered the case of genus 1, and described the moduli space using a natural extension of Igeta-construction, that is, we made a complete list of once puctured elliptic curves with Lagrangian sublattices.

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