数理解析研究所講究録
1022巻 1997年 53-57

ISOMONODROMIC DEFORMATION OF FUCHSIAN-TYPE PROJECTIVE CONNECTIONS ON ELLIPTIC CURVES

SHINGO KAWAI

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-01, JAPAN
(京大数理研　河井 真吾)

In this note we consider a natural mapping between the following two spaces:

E = { (certain) linear ordinary differential equations on (marked) Riemann surfaces s.t. the local monodromy representations around the singular points are as specified },

R = { representation classes of π_1(punctured surface) s.t. the local representations around the punctures are as specified }.

Let us denote by F the mapping that assigns the elements of E their monodromy representations and consider for instance a one-parameter family of differential equations lying in the fiber $F^{-1}(r)$ above a point $r \in R$. The characteristic feature of that family is of course that the corresponding monodromy of each element of the family is always the same r and therefore we call it an isomonodromic family. Our goal in this note will be to give an *infinitesimal* description of isomonodromic families in terms of a completely integrable system of partial differential equations on some local coordinate parameters of E. More specifically, that amounts to describing the tangential directions to the fibers $F^{-1}(r)$ and can be carried out in the following geometric manner. The key observation of our method is that there exists a natural *symplectic structure* ω on the space R of representations [2]. (A symplectic structure ω on R is, by definition, a closed nondegenerate 2-form on R.) By pulling back the 2-form ω onto E by the mapping F, we obtain a possibly degenerate closed 2-form on E; and that 2-form can then be used to describe the tangential directions to the fibers of F as follows: For a tangent vector ξ to E at a point $p \in E$, we have

$$\begin{aligned} & \xi \text{ is tangent to a fiber of } F \\ \Longleftrightarrow\ & d_pF(\xi) = 0 \\ \Longleftrightarrow\ & \omega(d_pF(\xi), \cdot) \equiv 0 \qquad \text{since } \omega \text{ is nondegenerate} \\ \Longleftrightarrow\ & F^*\omega(\xi, \cdot) \equiv 0 \qquad \text{where we assume } d_pF \text{ is surjective.} \end{aligned}$$

(The surjectivity condition will always be satisfied in our discussion below.) It thus follows that the problem of describing the tangential directions to the fibers of F can be reduced to determining precisely the vectors ξ such that $F^*\omega(\xi\,,\cdot\,) \equiv 0$ (we say that $F^*\omega$ is *degenerate* in the direction ξ if $F^*\omega(\xi\,,\cdot\,) \equiv 0$) and consequently we shall first write out the pulled-back 2-form $F^*\omega$ explicitly in terms of some local coordinates and then determine the directions making $F^*\omega$ degenerate. (The distribution $\{\,\xi \in TE\,;\, F^*\omega(\xi\,,\cdot\,) \equiv 0\,\}$ is obviously integrable (since the fibers of F are precisely the maximal integral manifolds) and therefore defines a foliation on E, which will be called the *null-foliation* of $F^*\omega$.)

Let X be a compact Riemann surface of genus one, and $H = \{\,\tau \in \mathbb{C}\,;\, \operatorname{Im}\tau > 0\,\}$ the upper half-plane. Selecting a suitable $\tau \in H$, one can represent X as the quotient $\mathbb{C}/\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau$; and then equations on X can be represented as equations on $\mathbb{C}$ with doubly periodic coefficients. Consider the Fuchsian equation

$$\frac{d^2y}{dz^2} = q(z)\,y\,, \tag{1}$$

$$\begin{aligned} q(z) = k &+ \sum_{i=0}^{m}\Big[\,H_i\,\zeta(z-t_i\,,\tau) + \frac{1}{4}\,(\theta_i^2-1)\,\wp(z-t_i\,,\tau)\Big] \\ &+ \sum_{\alpha=0}^{m}\Big[-\mu_\alpha\,\zeta(z-\lambda_\alpha\,,\tau) + \frac{3}{4}\,\wp(z-\lambda_\alpha\,,\tau)\Big]\,, \end{aligned}$$

$$\sum_{i=0}^{m} H_i - \sum_{\alpha=0}^{m}\mu_\alpha = 0\,, \tag{2}$$

where $\zeta(z,\,\tau)$ and $\wp(z,\,\tau)$ denote Weierstrass' ζ-function and $\wp$-function with fundamental periods 1, τ and t_0 will always be normalized so that $t_0 = 0$. It has its (regular) singularities at $[t_i]\,(\,i = 0,\,\ldots\,,m)$ and $[\lambda_\alpha]\,(\,\alpha = 0,\,\ldots\,,m)$ with characteristic exponents $\frac{1}{2}(1\pm\theta_i)$ and $\frac{1}{2}(1\pm 2)$ respectively ($[z]$ denotes the congruence class of a point $z \in \mathbb{C}$) and determines its monodromy representation

$$\rho\colon \pi_1(\,X \setminus \{\,[t_0],\,\cdots\,,[t_m],\,[\lambda_0],\,\cdots\,,[\lambda_m]\,\}\,) \to \mathrm{SL}(2,\mathbb{C})$$

up to conjugacy. If we assume here that (i) the parameters θ_i's are *not* integers, and that (ii) the singularities $[\lambda_\alpha]$'s are not logarithmic (i.e., *apparent*), then the local monodromies around the $[t_i]$'s and $[\lambda_\alpha]$'s become respectively (conjugate to)

$$\begin{pmatrix} -\exp(\pi\sqrt{-1}\,\theta_i) & 0 \\ 0 & -\exp(-\pi\sqrt{-1}\,\theta_i) \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Keeping this in mind and viewing the θ_i's as fixed (non-integral) constants, let us define our space E to be the set of equations having the form (1) and satisfying assumption (ii). Since it follows from (2) and assumption (ii) (under a generic condition) that the parameters k, H_i ($i = 0, \ldots, m$) are described as certain functions

$$
\begin{cases} k = k(\vec{t}, \tau, \vec{\lambda}, \vec{\mu}) \\ H_i = H_i(\vec{t}, \tau, \vec{\lambda}, \vec{\mu}) \quad (i = 0, \ldots, m) \end{cases} \tag{4}
$$

of the other parameters of equation (1), we find that the space E thus defined can be locally parametrized by the parameters $(\vec{t}, \tau, \vec{\lambda}, \vec{\mu})$. (We have introduced here the vector notation $\vec{t} = (t_1, \ldots, t_m)$ $(t_0 = 0)$, $\vec{\lambda} = (\lambda_0, \ldots, \lambda_m)$, $\vec{\mu} = (\mu_0, \ldots, \mu_m)$.) Having finished the (local) description of the space E of equations, we are now ready to write out the specific form of the pulled-back 2-form $F^*\omega$ (in terms of the coordinate parameters $(\vec{t}, \tau, \vec{\lambda}, \vec{\mu})$). (In view of the construction of E, the space R of representations will correspondingly be defined as

$$
R = \{ \rho \colon \pi_1(X \setminus \{2m+2 \text{ points}\}) \to \mathrm{SL}(2, \mathbb{C}) ; \begin{array}{l} \text{the local representations around} \\ \text{the punctures are as in (3) up to} \\ \text{conjugacy} \} / \sim , \end{array}
$$

where $/\sim$ means taking the quotient space by the conjugate action of the group $\mathrm{SL}(2, \mathbb{C})$ on R.)

Theorem 1 [3]. *In terms of the local parameters* $(\vec{t}, \tau, \vec{\lambda}, \vec{\mu})$ *of* E, *the pulled-back 2-form* $F^*\omega$ *takes the form*

$$
F^*\omega = -2 \left(\sum_{\alpha=0}^{m} d\mu_\alpha \wedge d\lambda_\alpha - \sum_{i=1}^{m} dH_i \wedge dt_i - dK \wedge d\tau \right), \tag{5}
$$

where

$$
K = \frac{1}{2\pi\sqrt{-1}} \left[k + \eta_1(\tau) \left(\sum_{\alpha=0}^{m} \lambda_\alpha \mu_\alpha - \sum_{i=1}^{m} t_i H_i \right) \right]
$$

and the term $\eta_1(\tau)$ *is defined by* $\eta_1(\tau) = \zeta(z+1, \tau) - \zeta(z, \tau)$.

In particular, it follows from this result that if we consider the space E_0 of differential equations (again having the form (1) and satisfying assumption (ii)) on the *fixed* elliptic curve X (we therefore regard the parameter τ as a fixed constant), then the resulting 2-form $F^*\omega$ becomes

$$
-2 \left(\sum_{\alpha=0}^{m} d\mu_\alpha \wedge d\lambda_\alpha - \sum_{i=1}^{m} dH_i \wedge dt_i \right);
$$

and that formula can indeed be viewed as a special instance of Iwasaki's result [2].

As explained earlier, we turn next to describing the null-foliation of the 2-form $F^*\omega$, which is given by integrating the distribution $\{ \xi \in TE ; F^*\omega(\xi, \cdot) \equiv 0 \}$. For this we first note that the first term $-2 \left(\sum_{\alpha=0}^{m} d\mu_\alpha \wedge d\lambda_\alpha \right)$ of the right-hand side of (5) is *nondegenerate* on the (locally defined) $(\vec{\lambda}, \vec{\mu})$-space. From this simple but useful observation it follows that the leaves of the null-foliation of $F^*\omega$ are *transversal* to the $(\vec{\lambda}, \vec{\mu})$-directions, or equivalently that any tangent vector ξ satisfying $F^*\omega(\xi, \cdot) \equiv 0$ must be a linear combination of the vectors having the form

$$\begin{cases} \mathcal{H}_i = \dfrac{\partial}{\partial t_i} + \displaystyle\sum_{\alpha=0}^{m} \left(A^i_\alpha \dfrac{\partial}{\partial \lambda_\alpha} + B^i_\alpha \dfrac{\partial}{\partial \mu_\alpha} \right) & (i = 1, \dots, m) \\ \mathcal{H}_\tau = \dfrac{\partial}{\partial \tau} + \displaystyle\sum_{\alpha=0}^{m} \left(C_\alpha \dfrac{\partial}{\partial \lambda_\alpha} + D_\alpha \dfrac{\partial}{\partial \mu_\alpha} \right), \end{cases}$$

where $A^i_\alpha, B^i_\alpha, C_\alpha, D_\alpha$ are some complex numbers. Moreover a simple calculation shows that the vectors $\mathcal{H}_i$'s and $\mathcal{H}_\tau$ above make $F^*\omega$ degenerate precisely when

$$\text{(6)} \quad \begin{cases} A^i_\alpha = \dfrac{\partial H_i}{\partial \mu_\alpha} \\ B^i_\alpha = -\dfrac{\partial H_i}{\partial \lambda_\alpha} \\ \displaystyle\sum_{\alpha=0}^{m} \left(\dfrac{\partial H_j}{\partial \mu_\alpha} \dfrac{\partial H_i}{\partial \lambda_\alpha} - \dfrac{\partial H_j}{\partial \lambda_\alpha} \dfrac{\partial H_i}{\partial \mu_\alpha} \right) = \dfrac{\partial H_j}{\partial t_i} - \dfrac{\partial H_i}{\partial t_j} & (j = 1, \dots, m) \end{cases}$$

and

$$\text{(7)} \quad \begin{cases} C_\alpha = \dfrac{\partial K}{\partial \mu_\alpha} \\ D_\alpha = -\dfrac{\partial K}{\partial \lambda_\alpha} \\ \displaystyle\sum_{\alpha=0}^{m} \left(\dfrac{\partial H_j}{\partial \mu_\alpha} \dfrac{\partial K}{\partial \lambda_\alpha} - \dfrac{\partial H_j}{\partial \lambda_\alpha} \dfrac{\partial K}{\partial \mu_\alpha} \right) = \dfrac{\partial H_j}{\partial \tau} - \dfrac{\partial K}{\partial t_j} & (j = 1, \dots, m) \end{cases}$$

respectively (see [4, pp.10-11]). To describe the null-foliation of $F^*\omega$, it thus remains to prove (or disprove) the third formulas of (6) and (7). Although they can be shown directly by substituting into them the specific forms of the functions (4), the calculation needed is quite complicated and lengthy. Instead, following Iwasaki [1], we have proved them via the residue calculus of certain meromorphic differentials on Riemann surfaces. (The third formula of (6) has already been shown by Okamoto [5] and Iwasaki [1].) In summary then, one concludes that the distribution in question has as a local basis the vector fields

$$\begin{cases} \mathcal{H}_i = \dfrac{\partial}{\partial t_i} + \displaystyle\sum_{\alpha=0}^{m} \left(\dfrac{\partial H_i}{\partial \mu_\alpha} \dfrac{\partial}{\partial \lambda_\alpha} - \dfrac{\partial H_i}{\partial \lambda_\alpha} \dfrac{\partial}{\partial \mu_\alpha} \right) & (i = 1, \dots, m) \\ \mathcal{H}_\tau = \dfrac{\partial}{\partial \tau} + \displaystyle\sum_{\alpha=0}^{m} \left(\dfrac{\partial K}{\partial \mu_\alpha} \dfrac{\partial}{\partial \lambda_\alpha} - \dfrac{\partial K}{\partial \lambda_\alpha} \dfrac{\partial}{\partial \mu_\alpha} \right). \end{cases}$$

However, since these vector fields just describe the *time-evolution directions* of the parameters λ_α's and μ_α's with respect to the time-parameters t_i's and τ with Hamiltonians H_i's and K (see [4, pp.10-11]), we finally obtain the following result.

Theorem 2. *The null-foliation of the pulled-back 2-form $F^*\omega$ is locally described by the completely integrable Hamiltonian system*

$$\begin{cases} d\lambda_\alpha &= \displaystyle\sum_{i=1}^{m} \frac{\partial H_i}{\partial \mu_\alpha} dt_i + \frac{\partial K}{\partial \mu_\alpha} d\tau \\ d\mu_\alpha &= -\displaystyle\sum_{i=1}^{m} \frac{\partial H_i}{\partial \lambda_\alpha} dt_i - \frac{\partial K}{\partial \lambda_\alpha} d\tau . \end{cases} \qquad (\alpha = 0, \dots, m)$$

Just as before, if we regard the parameter τ as a fixed constant, the resulting Hamiltonian system becomes

$$\begin{cases} d\lambda_\alpha &= \displaystyle\sum_{i=1}^{m} \frac{\partial H_i}{\partial \mu_\alpha} dt_i \\ d\mu_\alpha &= -\displaystyle\sum_{i=1}^{m} \frac{\partial H_i}{\partial \lambda_\alpha} dt_i ; \end{cases} \qquad (\alpha = 0, \dots, m)$$

and that system has been obtained by Okamoto [5] and Iwasaki [1].

References

[1] K. Iwasaki, *Moduli and deformation for Fuchsian projective connections on a Riemann surface*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **38** (1991), 431–531.

[2] ———, *Fuchsian moduli on Riemann surfaces — its Poisson structure and the Poincaré-Lefschetz duality*, Pacific J. Math. **155** (1992), 319–340.

[3] S. Kawai, *Deformation of complex structures on a torus and monodromy preserving deformation*, preprint.

[4] Yu. I. Manin, *Sixth Painlevé equation, universal elliptic curve, and mirror of* $\mathbb{P}^2$, alg-geom/9605010.

[5] K. Okamoto, *On the holonomic deformation of linear ordinary differential equations on an elliptic curve*, Kyushu J. Math. **49** (1995), 281–308.