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On exterior problems in elasiticy

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§0 Introduction

Let Ω be a domain containing in $\mathbf{R}^n$ representing an elastic medium in the three dimensional case. A motion of elastic medium is described by the displacement vector $u(t,x) = {}^t(u_1(t,x), \cdots, u_n(t,x))$ which is defined as the displacement of a point $x \in \Omega$ in the elastic medium at time t. In elasticity theory, the displacement vector $u(t,x)$ should satisfy the following equation:

$$(0.1) \qquad (\partial_t^2 - A(\partial_x))u(t,x) = 0 \quad \text{in} \quad \mathbf{R} \times \Omega,$$

where the differential operator $A(\partial_x)$ is of the form

$$A(\partial_x)u = \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}\partial_{x_j}u).$$

Thus, the elastic wave equation is a typical example of the second order hyperbolic system.

The most famous example of the elastic waves appears in the waves in earthquakes. It is well known that there are two different type of waves named as P-waves (primary waves) and S-waves (secondary waves) in earthquakes. Since these waves propagate inside of the elastic medium, in this case, that is in earth, such waves are called as body waves.

In the case of the Neumann boundary problem of the equation (0.1), we have different types of waves which propagate along the boundary of the elastic media. Since these waves on the elastic media were found by Lord Rayleigh, we call these waves as the Rayleigh surface waves. The existence of the Rayleigh surface waves is a conspicuous feature of the elastic wave equation with the Neumann boundary condition. The purpose in this article is to consider how the existence of the Rayleigh surface waves affect on the properties of the solutions of the elastic wave equations with the Neumann boundary condition.

To formulate this problem precisely, we consider the case of exterior problems. In exterior problems for hyperbolic equations, we have many fruitful results in the framework of the scattering theory for the hyperbolic equations. In particular, for exterior problems of the usual wave equation for a scalar valued function, we have many works. In this case, trappness of singularities of the solution of the wave equation are closely related to the asymptotic behaviour of the local energy near the boundary and the location of the poles of the resolvent (that is, so called as "resonances").

In the case of the exterior problem in the isotropic elastic wave equation with the Neumann boundary condition, by the existence of the Rayleigh surface waves, we can see the trappness phenomenon in the sense of propagation of singularities (cf. §2). Thus, by the analogue in the case of the usual wave equaion, we can expect that even in the elastic case, the local energy decaying very slowly and the poles of the resolvent appear near the real axis although the causes for the trappness are quite different.

In the isotropic case, Ikehata and Nakamura [**3**] show that these analogues hold if the boundary is the unit sphere in $\mathbf{R}^3$. After obtaining this result, the studies from the point of view in these analogues developed. In the isotropic case, the existence of the poles approaching the real axis is obtained by Stefanov and Vodev [**16**], [**17**] and [**18**], even in the case of the general curved boundary. In their proof, they proposed a new approach to show the existence of the poles of the resolvent converging the real axis.

In this approach, we treat the stationary problem directly. It is quite different from the methods via the time dependent problem used to show the existence of the poles in the case of the usual wave equations (cf. §3, 3.2).

This new approach is applicable to the various types of problems since we essentially treat elliptic problems with large parameter which can be treated for very general class. Indeed, by this strategy, even in the quite general anisotropic case which is physically natural class, we can obtain the same results as in the isotropic case (cf. §3, 3.3).

For the other example, Popov and Vodev [**11**] treat the existence of the poles in the case of transmission problems of the usual wave equation with a strictly convex transmission boundary. Thus, this approach can be used to the various types of the problems. This largeness in terms of applicable classes is an advantage of this new method, however, by this approach, we can not know how and where the poles of the resolvent appear precisely. On the other hand, the approach via the time dependent problem gives more precise infomation though it need to trace every path on which singularities propagate. By this reason, the method treating the time dependent problem restricts the class to which this approach can be applied. This is a differences between both approachs.

§1 Elastic wave equations

Let $\Omega \subset \mathbf{R}^n$ $(n \geq 3)$ be an exterior domain with smooth and compact boundary Γ. We consider the exterior mixed problem for the elastic wave equation with the Neumann boundary condition:

$$(1.1) \qquad \begin{cases} (\partial_t^2 - A(\partial_x))u(t,x) = 0 & \text{in } \mathbf{R} \times \Omega, \\ N(\partial_x)u(t,x) = 0 & \text{on } \mathbf{R} \times \Gamma, \\ u(0,x) = f_1(x),\ \partial_t u(0,x) = f_2(x) & \text{on } \Omega, \end{cases}$$

where the boundary operator $N(\partial_x)$ is the conormal derivative of the operator $A(\partial_x)$, that is, $N(\partial_x)$ is represented as $N(\partial_x)u = \sum_{i,j=1}^n \nu_i(x)a_{ij}\partial_{x_j}u|_\Gamma$ by using the unit outer normal $\nu(x) = {}^t(\nu_1(x), \cdots, \nu_n(x))$ of the boundary Γ. We denote by a_{ipjq} the (p,q)-component of the coefficient matrices a_{ij} of the operator $A(\partial_x)$. Throughout in this article, we assume each a_{ipjq} does not depend on the variables t and x. Further, we always assume the following physically natural assumptions:

(A.1)
$$a_{ipjq} = a_{jqip} = a_{pijq}$$
$$\text{for any } i,p,j,q = 1, \cdots, n,$$

(A.2) there is a constant $\delta > 0$ such that
$$\sum_{i,p,j,q=1}^n a_{ipjq}\epsilon_{jq}\bar{\epsilon}_{ip} \geq \delta \sum_{i,p=1}^n |\epsilon_{ip}|^2,$$
for any $n \times n$-symmetrix matrix (ϵ_{ip}).

These assumptions (A.1) and (A.2) are derived from some physical investigation in elasticity theory. They express some physical situations of elastic media.

We call that material is isotropic if and only if each coefficient is of the form $a_{ipjq} = \lambda_0\delta_{ip}\delta_{jq} + \mu_0(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp})$ by some constants λ_0, μ_0 called Lamé constants. In this case, we have $A(\partial_x)u = \nu_0\Delta u + (\lambda_0 + \mu_0)\text{grad}\,(\text{div}\, u)$. Note that in the isotropic case, (A.1) and (A.2) are satisfied if and only if $\lambda_0 + \frac{2}{n}\mu_0 > 0$, $\mu_0 > 0$ hold. In what follows, we call the material is anisotropic if and only if the elastic tensors a_{ipjq} satisfy (A.1) and (A.2) and it is not isotropic.

§2 Existence of the Rayleigh surface waves

Rayleigh [13] found special solutions of the isotropic elastic wave equation with the Neumann boundary condition in the half space in $\mathbf{R}^3$. They are considered as the surface waves, which are the prototype of the Rayleigh surface waves. At the present time, mathematically, we consider the Rayleigh surface waves as solutions having singularities in the elliptic region $\mathcal{E}$ of the operator $\partial_t^2 - A(\partial_x)$. They appear in the case that Lopatinski matrix is not invertible at some point in $\mathcal{E}$. Indeed, corresponding to that point, in the half space case, we can construct exact solutions which can be considered as the Rayleigh surface waves.

In the case of general curved boundary, the existence of the Rayleigh surface waves are treated in the sense of propagation of singularities by analysing the time dependent outgoing (resp. incoming) Neumann operator T^+ (resp. T^-). The Neumann operator $T^\pm$ are defined as

$$T^\pm f(t,x) = N(\partial_x)w^\pm(t,x)|_{\mathbf{R}\times\Gamma},$$

where $w^\pm(t,x)$ is the following problem:

$$\begin{cases} (\partial_t^2 - A(\partial_x))w^\pm(t,x) = 0 & \text{in } \mathbf{R}\times\Omega, \\ w^\pm(t,x) = f(t,x) & \text{on } \mathbf{R}\times\Gamma, \\ w^\pm(t,x) = 0 \quad \text{if } \pm t < 0 \text{ is sufficiently small.} & \end{cases}$$

Since the principal symbol $\sigma_p(T^\pm)$ of the Neumann operator coincides with the Lopatinski matrix, we can show the existence of the Rayleigh surface waves by showing $T^\pm$ is a pseudo-differential operator of real principal type in $\mathcal{E}$ (cf. Taylor [**20**] in the isotropic case, and Nakamura [**10**] in the anisotropic case).

The anisotropic case, Stroh [**19**] and Barnett and Lothe [**1**] investigate the existence of the Rayleigh surface waves if the material consists of the half space in $\mathbf{R}^3$. In this case, the Rayleigh surface waves do not always appear. They give a sufficient and necessary condition of the existence of the Rayleigh surface waves by so called "the surface impedance tensor". Nakamura [**10**] define the surface impedance tensor in the case of general curved boundary. By using this, he gives the condition of the existence of the Rayleigh surface waves.

The surface impedance tensor $Z(\zeta,\tau)$ is a $n\times n$-Hermit matrix valued C^∞ function on $\mathcal{E}$. This can be extended on $\overline{\mathcal{E}} \subset T^*(\mathbf{R}\times\Gamma)$ continuously (cf. [**1**], [**10**], [**9**]). Here, we do not write t in the variable of Z since the surface impedance tensor $Z = Z(\zeta,\tau)$ does not depend on t of $(t,\tau,\zeta)\in\mathcal{E}\subset T^*(\mathbf{R}\times\Gamma)$. In what follows, for simplicity, we write $(\zeta,\tau)\in\mathcal{E}$ instead of $(t,\tau,\zeta)\in\mathcal{E}$.

By using the surface impedance tensor $Z(\zeta,\tau)$, the necessary and sufficient condition (ERW) of the existence of the Rayleigh surface waves is described as follows:

$$\text{(ERW)} \qquad \begin{cases} \text{There exists a point } (\zeta^0,\tau^0)\in\partial\mathcal{E} \text{ such that} \\ \text{the Hermit matrix } Z(\zeta^0,\tau^0) \text{ is not non-negative definite.} \end{cases}$$

Remark 2.1

In the three dimensional case, the condition (ERW) is the same as that in [1] and [10].

Between the Neumann operator $T^{\pm}$ and the surface impedance tensor $Z(\zeta,\tau)$, we have the following relation:

$$\sigma_p(T^{\pm})(\zeta,\tau) = \| \zeta \|_{\Gamma} Z(\zeta,\tau) \qquad \text{on } \mathcal{E} ,$$

where $\| \zeta \|_{\Gamma}$ is the fiber metric of $T^*(\Gamma)$ inducing by the ordinary Riemann metric of Γ. By this relation, the same argument as in [1], [10] implies that the condition (ERW) is equivalent to the fact that $T^{\pm}$ is a real principal type in some neighbourhood of a point on the boundary. We do not introduce the surface impedance tensor by only the historical reason why it is used to argue the existence of the Rayleigh surface waves in elasticity theory. In the anisotropic case, it is rather difficult to show the properties of the principal symbol of the Neumann operator from its form directly. We can know these properties via analysing the surface impedance tensor.

Remark 2.2

If the material is isotropic, we can compute the form of $\sigma_p(T^{\pm})$ and $Z(\zeta,\tau)$. In particular, we can show the Rayleigh surface waves appear from the points $(\zeta,\tau) \in \mathcal{E}$ satisfying $C_R \| \zeta \|_{\Gamma} - |\tau| = 0$, where the constant $C_R > 0$ is the phase speed of the Rayleigh surface waves. Hence, we can show how the Rayleigh surface waves propagate on the whole boundary Γ and the trappness in the sense of propagation of singularities. Definitely, in the isotropic case, the condition (ERW) always holds. On the other hand, in the anisotropic case, the condition (ERW) only ensure the local existence of the Rayleigh surface waves. Thus, note that we can not know the trappness of singularities from the condition (ERW).

§3 Properties of the solutions

For exterior problems of the usual wave equations for scalar valued function, there are many works investigating precise properties of the solutions from the point of view in the scattering theory. In general, there are solutions whose energy mainly propagate along paths on which singularities propagate (that is, the rays of geometrical optics). Thus, if the obstacle is trapping, that is, if singularities never escape from near the boundary, the local energy of solutions hardly go out from that neighbourhood of the boundary (cf. [12]). It clarify this property to consider the analytic continuation of the stationary problem with respect to the spectrum parameter. Indeed, intuitively, it is known that the locations of the poles of this analytic continuation make much influence on the speed of the decay of the local energy decay for the solution of the time dependent problem. It is considered that these locations represent the strongness of the trappness (cf. [**2**], [**23**] and their references).

On the other hand, in the case of the exterior Neumann problem for the elastic wave equation, the solutions have singularities on the boundary which can be regarded as the Rayleigh surface waves. In the isotropic case, by Remark 2.2, we can see the obstacle is trapping in the sense of propagations of singularities. Thus, we can expect that the solutions have the properties which reflect on this trappness. In this direction, the isotropic case were mainly studied.

In the anisotropic case, as is in Remark 2.2, we can only know the local existence of the Rayleigh surface waves. Nevertheless, we can show the same properties hold even in the anisotropic case.

3.1 Asymptotic behaviour of the local energy

For a domain $D \subset \mathbf{R}^n$, we define the local energy $E(u, D, t)$ at time t of the solution $u(t,x)$ of the problem (1.1) as

$$E(u,D,t) = \frac{1}{2}\int_{D\cap\Omega}\{\sum_{i,p,j,q=1}^{n} a_{ipjq}\partial_{x_j}u_q(t,x)\overline{\partial_{x_i}u_p(t,x)} + |\,\partial_t u(t,x)\,|^2\}\,dx.$$

In the elastic case, although the Rayleigh surface waves appear, the local energy of the solution tends to go out near the boundary.

THEOREM 3.1 (Shibata and Soga [**14**]). *Under the assumptions* (A.1) *and* (A.2), *the local energy decays, that is, for any bounded domain* $D \subset \mathbf{R}^n$, *we have* $\lim_{t\to\infty} E(u,D,t) = 0$.

From Theorem 3.1, to clarify the influence of the Rayleigh surface waves, we need to see more precise properties of the local energy. Thus, we introduce the uniformity of the decay of the local energy. For $a > 0$, we set $B_a = \{x \in \mathbf{R}^n |\, |\,x\,| < a\}$.

DEFINITION 3.1 (the uniform decay rate). *For non-negative integer* m *and a positive constant* $a > 0$ *satisfying* $\Gamma \subset B_a$, *we define the uniform decay rate* $p_{m,a}(t)$ *as*

$$p_{m,a}(t) = \sup\left\{\frac{E(u,\Omega\cap B_a,t)}{\|\,\nabla_x f_1\,\|^2_{H^m(\Omega)} + \|\,f_2\,\|^2_{H^m(\Omega)}} | 0 \neq (f_1,f_2)\in C_0^\infty(\overline{\Omega}\cap B_a)\right\},$$

where $u(t,x)$ is the solution of (1.1) *with the initial data* (f_1, f_2).

For the uniform decay rate $p_{m,a}(t)$, we can obtain some results which can be considered as the local energy tends to remain near the boundary.

THEOREM 3.2 (Ikehata and Nakamura [3]). *If the material is isotropic and the boundary Γ is the unit sphere in $\mathbf{R}^3$, for any $\gamma > 0$ and $m \in \mathbf{N} \cup \{0\}$, we have $\lim_{t\to\infty} e^{\gamma t} p_{m,a}(t) = \infty$, that is, we can not have the estimate of the form*

$$p_{m,a}(t) \leq Ce^{-\gamma t} \qquad \text{for any } t > 0.$$

In the author's knowledge, Theorem 3.2 is the first work showing the influence of the Rayleigh surface waves in exterior domains. There proof also suggest the existence of the poles of the resolvent approaching the real axis.

In the case of the scalar-valued wave equation, Ralston [12] shows the result explaining why trappness of the singularities prevent the uniform decay of the local energy. In the elastic case, although the cause making trappness is quite different, we also have the same result.

THEOREM 3.3 ([6], [7]). *If the material is isotropic and* (A.2) *holds, then $p_{0,a}(t)$ never goes to 0 as $t \to \infty$.*

For $m \geq 1$, the same argument as in Walker [22], we can show $\lim_{t\to\infty} p_{m,a}(t) = 0$, since in Walker's argument, he only use the Rellich compactness theorem and the local energy decay property in Theorem 3.1. In the elastic case, however, the speed of the decay of $p_{m,a}(t)$ can be expected rather slow than not so fast. Indeed, we have the following results:

THEOREM 3.4. *We assume that the material is isotropic and* (A.2) *holds.*

(1) *For any $\gamma > 0$ and $m \in \mathbf{N} \cup \{0\}$, we have $\lim_{t\to\infty} t^{\gamma} p_{m,a}(t) = \infty$, that is, we can not have the estimate of the form*

$$p_{m,a}(t) \leq Ct^{-\gamma} \qquad \text{for any } t > 0.$$

(2) *Further, if we assume that the boundary Γ is real analytic and n is odd, then for any $m \in \mathbf{N}$, we have $\overline{\lim_{t\to\infty}} (\log t)^m p_{m,a}(t) > 0$.*

The fact (1) in Theorem 3.4 are treated in [8] with an additional assumption which means the other waves than the Rayleigh surface waves behave like satisfying the non-trapping condition. This additional assumption is removed by combining the argument in Stefanov and Vodev [18] showing the existence of the poles. Vodev [21] obtains (2) in Theorem 3.4. In the proof, he use the fact that approximations of the Neumann operator for the stationary problem in the elliptic region can be extended as a pseudo-differential operator with a large parameter whose symbol is defined in the whole cotangent bundle of Γ. This is one of the main parts in Sjöstrand and Vodev [15] which count the number of the poles corresponding to the Rayleigh surface waves described in the sub-section 3.4.

3.2 The poles of the resolvent

For the spectral parameter $z \in \mathbf{C}$, consider the following stationary problem:

$$(3.1) \qquad \begin{cases} (A(\partial_x) + z^2)v(x:z) = f(x) & \text{in } \Omega, \\ N(\partial_x)v(x:z) = 0 & \text{on } \Gamma. \end{cases}$$

In the case of Im $z < 0$, for any $f \in L^2(\Omega)$, there exists the unique solution $v(x:z) \in H^2(\Omega)$ of the problem (3.1). We define the operator valued function $R(z)$ as $R(z)f(x) = v(x:z)$. We can continue the operator $R(z)$ as $B(L^2_a(\Omega), H^2(\Omega \cap B_a))$-valued meromorphic function in $\widetilde{\mathbf{C}}_+$ (cf. [5], [4], [9]). In the above, $L^2_a(\Omega) = \{f \in L^2(\Omega) \,|\, f(x) = 0 \text{ in } |x| > 0\}$, $\widetilde{\mathbf{C}}_+ = \mathbf{C}$ (if n is odd) and $\widetilde{\mathbf{C}}_+ = \{z \in \mathbf{C} \setminus \{0\} | -\frac{3}{2}\pi < \arg z < \frac{1}{2}\pi\}$, (if n is even). We call the operator valued function $R(z)$ the outgoing resolvent.

From Theorems 3.2 and 3.3, we can expect that the poles of the reslovent appear near the real axis. This is justified by Stefanov and Vodev [16] first. They consider the isotropic materials with the unit sphere boundary in $\mathbf{R}^3$. By developing the argument in Ikehata and Nakamura [3], they show the follwing fact:

THEOREM 3.5([16]). *Under the same assumption as in Theorem 3.2, there exist constants C_0, $C_1 > 0$ such that we have only one sequence $\{z_m\}_{m=1,2,\cdots}$ of the poles of the resolvent in the region $0 < \mathrm{Im}\, z \le C_0 \,|\, z\,|^{1/3} - C_1$, $\mathrm{Re}\, z > 0$. Moreover, there exists constants d_0, d_1, γ, $C > 0$ such that*

$$\begin{cases} z_m = d_0 m + d_1 + O(m^{-1}), & (\text{as } m \to \infty), \\ 0 < \mathrm{Im}\, z_m \le C e^{-\gamma|\mathrm{Re}\, z_m|} & (\text{for any } m \in \mathbf{N}). \end{cases}$$

hold.

Theorem 3.5 says in general the trappness arizing the existence of the Rayleigh surface waves is very strong. For general curved boundary, we can expect that there are infinite many poles in the below of some exponential curve like as in Theorem 3.5. In this direction, Stefanov and Vodev [**17**] and [**18**] gives an essential development. In what follows, we use the following notation.

DEFINITION 3.2. *We call that the property* (PC) *holds if and only if there exists a sequence* $\{z_j\}_{j=1,2,\cdots}$ *of the poles of the outgoing resolvent* $R(z)$ *satifsfying* $\lim\limits_{j\to\infty}$ Re $z_j = \infty$ *and for any* $N > 0$ *there exists a constant* $C_N > 0$ *such that the poles* $\{z_j\}_{j=1,2,\cdots}$ *can be estimated as*

$$0 < \mathrm{Im}\ z_j \leq C_N\,|\,\mathrm{Re}\ z_j\,|^{-N} \qquad (j = 1, 2, 3, \cdots).$$

Now, we state the result obtained by Stefanov and Vodev [**18**].

THEOREM 3.6([**18**]). *If the isotropic material in* $\mathbf{R}^3$ *satifies* (A.2), *the property* (PC) *holds.*

Thus, for general C^∞ boundary, we also have the poles approaching the real axis rapidly, although we do not know whether that speed is exponentially fast. Theorem 3.6 is first given by [**17**] in the case of strictly convex boundary. Note that the property (PC) also holds in the odd dimensional case.

In general, by the Laplace transform, a decay estimate of the uniform decay rate $p_{m,a}(t)$ implies holomorphicity of the resolvent in some region concluding the real axis which is determined by the speed of the decay estimate (cf. [**8**] and [**21**]). Further, we also have an estimate of the resolvent in this region. This estimate eventually contradicts the existence of the Rayleigh surface waves. By this procedure, Theorem 3.4 is shown. It means that it is one of the essential part to get an a priori estimate of the resolvent.

By [**17**] and [**18**], it is established how to obtain an a priori estimate of the resolvent $R(z)$ from the assumption that it is holomorphic in some region concluding the real axis (cf. Proposition 5.2 in [**17**] and Proposition 1 in [**18**]). For example, suppose that Theorem 3.6 is not ture. It means that the resolvent is holomorphic in $|\,\mathrm{Im}\ z\,| \leq C\,|\,\mathrm{Re}\ z\,|^{-N}$ for some constants C and $N > 0$. From this, we have an estimate of the resolvent $R(z)$, which eventually contradicts the existence of the Rayleigh surface waves. Thus, it is essential to show the a priori estimate of $R(z)$. To obtain this estiamte, the assumption that n is odd is necessary since we have to use the fact that the outgoing resolvent for the free space problem (that is the one in the case that $\Omega = \mathbf{R}^n$), is entire function as an operator valued function.

In the case that the boundary is analytic, we can improve Theorem 3.6, which gives affirmative answer for the existence of the poles closely to the real axis exponentially fast.

THEOREM 3.7 (Vodev [**21**]). *Assume that the material is isotropic and satisfies* (A.2). *If the boundary is analytic and n is odd, then there exists a positive constant γ such that in the region $0 < |\operatorname{Im} z| \leq \exp(-\gamma |\operatorname{Re} z|)$, there are infinite many poles of the resolvent $R(z)$.*

3.3 Anisotropic elasticity

In the anisotropic case, we also have the resolvent $R(z)$. Since we consider the case that each coefficient of the operator $A(\partial_x)$ is constant, from (A.1) and (A.2), the resolvent is holomorphic on the real axis in the odd dimensional case and on the real axis except the origine in the even dimensional case. In this case, we also have the same result as in Theorems 3.3, 3.6 and (1) of Theorem 3.4.

THEOREM 3.8 (Kawashita and Nakamura [**9**]). *If we assume that the condition* (ERW) *is satisfied, then the following statements hold:*
(1) *We never have* $\lim_{t\to\infty} p_{0,a}(t) = 0$.
(2) *For any* $\gamma > 0$ *and* $m \in \mathrm{N} \cup \{0\}$, *we have* $\lim_{t\to\infty} t^{\gamma} p_{m,a}(t) = \infty$.
(3) *In the case of the odd dimension, the property* (PC) *holds.*

As is in the isotropic case, we can say one of the main point in the proof of Theorem 3.8 is based on obtaining an a priori estimate of the resolvent. In the anisotropic case, however, we have to modify the argument in the isotropic case, since this argument in the isotropic case requires smoothness of the eigen-values of the characteristic matrix $A(\xi) = \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j$. Note that this property of the eigen-values does not generally follow from the assumptions (A.1) and (A.2).

In the isotropic case, as is in Remark 2.2, we can get the information of singularities corresponding to the Rayleigh surface waves in the whole boundary. Since this fact says the existence of trapping singularities, it seems to be quite natural to obtain Theorems 3.2 $\sim$ 3.6.

In the anisotropic case, however, we can only know the local existence of the Rayleigh surface waves. From only such restricted infomations, we can show the almost all properties as in the isotropic case which are regarded as the reflection on the global properties like as trapping phenomenon. Thus, from Theorem 3.8, we can say, to obtain the assertions in Theorem 3.8 stating some global properties of the solutions of the problem (1.1), global informations for singularities by the Rayleigh surface waves do not essentially required.

3.4 Other topics

In Theorem 3.5, all other poles than those by the Rayleigh surface waves near the real axis do not appear in the below of the cubic curve. Thus, we have a region free from the poles. We can explain why such phenomenon occurs generally.

Consider the case that all other waves than the Rayleigh surface waves behave like satisfying the non-trapping condition. In this case, there is a logarithmic curve such that in the below of this curve, the resolvent is holomorphic in the outside of the region $0 < |\mathrm{Im}\, z| \leq C_N |\mathrm{Re}\, z|^{-N}$ for any $N \geq 0$ (in the anisotropic case, cf. [**8**], in the isotropic case, Stefanov and Vodev [**17**] also obtain the same result). Note that the isotropic material case with a strictly convex boundary is a typical example of the non-trapping case in this sense.

In the isotropic case, under some non-trapping condition which is stated by the solutions of stationary problems, Sjöstrand and Vodev [**15**] obtain the asymptotic behaviour of the number $N(\lambda)$ of the poles whose absolute value is less than λ corresponding to the Rayleigh surface waves. This behaviour as $\lambda \to \infty$ is of the form

$$N(\lambda) = \tau_n C_R^{-n+1} \mathrm{Vol}\,(\Gamma)\lambda^{n-1} + O(\lambda^{n-2}),$$

where $\tau_n = (2\pi)^{-n+1}\mathrm{Vol}\,(\{x \in \mathbf{R}^{n-1} \mid |x| \leq 1\})$. This is the same as the number of the negative eigen-value of the Weyl pseudo-differential operator with the principal symbol of the form $C_R \|\zeta\|_\Gamma - \lambda$. Note that the set of all zero points of this function consists of the characteistic set of the principal symbol of the Neumann operator, from which the Rayleigh surface waves appear.

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