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POLYNOMIAL HULLS WITH NO ANALYTIC STRUCTURE

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0. Introduction. Let X be a compact set in $\mathbf{C}^N$ and $\hat{X}$ its polynomial hull:

$$\hat{X} := \{(z_1, ..., z_N) \in \mathbf{C}^N : |p(z_1, ..., z_N)| \leq ||p||_X \text{ for all polynomials } p\},$$

where $||p||_X$ denotes the supremum norm of p on X. If X contains the boundary of an H^∞ disk, i.e., if there exists a bounded, nonconstant holomorphic map $g = (g_1, ..., g_N)$ from the unit disk Δ in $\mathbf{C}$ into $\mathbf{C}^N$ with radial limit values $g^*(e^{i\theta})$ belonging to X for a.e. θ, then, by the maximum modulus principle, $\hat{X}$ contains the analytic disk $g(\Delta)$. In general, we say a set S has *analytic structure* if it contains an analytic disk $g(\Delta)$. In this note, we discuss well-known examples of Stolzenberg [S] and Wermer [W] and recent modifications which show that a compact set can have non-trivial hull (i.e., $\hat{X} \neq X$) with $\hat{X}$ (or at least $\hat{X} \setminus X$) containing no analytic structure. We remark that in both examples, the set $\hat{X}$ is constructed as a limit (in the Hausdorff metric) of compact subsets of analytic varieties in $\mathbf{C}^2$.

1. The Stolzenberg Example. Stolzenberg's set X is a subset of the topological boundary of the bidisk $\Delta \times \Delta$ in $\mathbf{C}^2$ such that the origin $(0,0)$ lies in $\hat{X}$. However, the projection of the hull in each coordinate plane contains no nonempty open set; hence $\hat{X}$ contains no analytic structure. The rough idea of the Stolzenberg construction is, first of all, to take a countable dense set of points $\{a_j\}$ in the punctured disk $\{t \in \mathbf{C} : 0 < |t| < 1\}$ and form the algebraic varieties $C_j := \{(z, w) \in \mathbf{C}^2 : (z - a_j)(w - a_j) = 0\}$. These varieties avoid $(0,0)$ and have the property that each of the coordinate projections π_z and π_w of the union $\cup_j (C_j \cap (\Delta \times \Delta))$ equals $\{a_j\}$. Then a decreasing sequence of compact subsets X_i of the topological boundary of the bidisk is constructed inductively so that $(0,0)$ lies in $\hat{X}_i$ for each i and $\hat{X}_i \cap (\cup_{j=1}^i C_j) = \emptyset$; i.e., the hulls $\hat{X}_i$ avoid more and more of the algebraic varieties C_j. The intersection $X := \cap X_i$ is the desired set.

Remarks. Although the coordinate projections of $\hat{X}$ are nowhere dense, they have positive Lebesgue measure (as subsets of $\mathbf{R}^2$). This can be seen as follows: first of all, despite the lack of analytic structure in $\hat{X}$, (holomorphic) polynomials are **not** dense in the continuous (complex-valued) functions on $\hat{X}$, or, in the standard notation of uniform algebras, $P(\hat{X}) \neq C(\hat{X})$. Indeed, for any $p \in P(\hat{X})$, $||p||_{\hat{X}} = ||p||_X$; thus if $f \in C(\hat{X})$ satisfies $|f(0,0)| > ||f||_X$ (such f clearly exist), $f \notin P(\hat{X})$. Now if the coordinate projections of $\hat{X}$ have positive Lebesgue measure, by the Hartogs-Rosenthal theorem, the functions $\overline{z}$ and $\overline{w}$ are in $P(\hat{X})$; then, using the Stone-Weierstrass theorem, we get that $P(\hat{X}) = C(\hat{X})$, a contradiction.

Further Examples. By choosing $\{a_j\}$ a bit more carefully (in particular, to avoid an entire interval $[a, b]$ instead of just the origin), and by slightly modifying the construction of the sets X_i, Fornaess and the author proved the following.

Theorem 1 ([FL]). *Let D be a bounded domain in $\mathbf{C}^2$ with $\hat{\overline{D}} = \overline{D}$ and such that both coordinate projections of D yield the unit disk. Let $0 < a < b < 1$. Then there exists a compact set $X \subset \partial D$ such that $\hat{X}$ contains no analytic structure but with $[a, b] \times [a, b] \subset \hat{X} \setminus X$.*

We remark that $[a, b] \times [a, b]$ is *non-pluripolar* in $\mathbf{C}^2$; i.e., if a plurisubharmonic function u is equal to $-\infty$ on $[a, b] \times [a, b]$, then $u \equiv -\infty$.

Abstracting the concrete ideas in [FL], Duval and the author generalized Theorem 1.

Theorem 2 ([DL]). *Let D be a bounded domain in $\mathbf{C}^N$ with $\hat{\overline{D}} = \overline{D}$. Given $K \subset D$ with $K = \hat{K}$ (or $K \subset \overline{D}$ with $K = \hat{K} = \widehat{K \cap \partial D}$), there exists $X \subset \partial D$ compact with $K \subset \hat{X}$ such that $\hat{X} \setminus K$ contains no analytic structure. In particular, if K contains no analytic structure, then $\hat{X}$ contains no analytic structure.*

As a corollary, by taking $K = \Gamma \times ... \times \Gamma$ (N times) where Γ is a Jordan arc in $\mathbf{C}$ with positive Lebesgue measure (in $\mathbf{R}^2$), we get a compact set X in ∂D whose hull $\hat{X}$ contains no analytic structure but such that $\hat{X} \setminus X$ has positive Lebesgue measure in $\mathbf{R}^{2N}$.

Remarks. Intuitively, one might expect that if $\hat{X} \setminus X$ is nonempty but contains no analytic structure, then $\hat{X} \setminus X$ should still be "small" in some sense. The previous two theorems show that $\hat{X} \setminus X$ can still be quite "large" in certain cases. The next result, due independently to Alexander and Sibony, shows that $\hat{X} \setminus X$ is *always* "large" when $\hat{X} \setminus X$ is nonempty but contains no analytic structure. Below, $h_2(S)$ denotes the Hausdorff 2−measure of a set S.

Theorem 3 (Alexander [A1], Sibony [Si]). *Let $X \subset \mathbf{C}^N$ be compact and let $q \in \hat{X} \setminus X$. If there exists a neighborhood U of q in $\mathbf{C}^N$ with $h_2(\hat{X} \cap U) < +\infty$, then $\hat{X} \cap U$ is a one-dimensional analytic subvariety of U.*

As a corollary, if $\hat{X} \setminus X \neq \emptyset$ and $\hat{X} \setminus X$ contains no analytic structure, then $h_2(\hat{X} \setminus X) = +\infty$.

2. The Wermer Example. In 1982, Wermer [W] constructed a compact set X in $\partial\Delta \times \mathbf{C} \subset \mathbf{C}^2$; i.e., $\pi_z(X) = \partial\Delta$ (recall π_z denotes the projection onto the first coordinate), with $\pi_z(\hat{X}) = \overline{\Delta}$ and such that $\hat{X} \setminus X \subset \Delta \times \mathbf{C}$ does not contain any *topological* disk; i.e., there is no *continuous* nonconstant $g : \Delta \to \mathbf{C}^2$ with $g(\Delta) \subset \hat{X} \setminus X$. Clearly since $\pi_z(\hat{X} \setminus X) = \Delta$, the reason $\hat{X} \setminus X$ contains no analytic structure is not because of "small" coordinate projections as in the Stolzenberg example. Here, $\hat{X}$ is constructed as a limit (in the Hausdorff metric) of Riemann surfaces Σ_n over $\overline{\Delta}$ which branch over more and more points. Starting with a countable dense set of points $\{a_j\}$ in $\overline{\Delta}$, one chooses a sequence $\{c_j\}$ of positive numbers decreasing rapidly to 0 so that the graphs of the 2^n−valued functions

$$g_n(z) := c_1\sqrt{z - a_1} + c_2(z - a_1)\sqrt{z - a_2} + ... + c_n(z - a_1)\cdots(z - a_{n-1})\sqrt{z - a_n}$$

over $\overline{\Delta}$ form the desired Riemann surfaces Σ_n. To be precise, the actual construction done in [W] takes place over the disk of radius one-half centered at the origin in the z−plane; this yields the estimate $|a - b| < 1$ for $|a|, |b| < 1/2$.

Remarks. Although $\hat{X} \setminus X$ contains no analytic structure, there remains some semblance of analyticity in this set. A result of Goldmann [G] shows that functions in the uniform algebra $P(X)$ behave like analytic functions in the sense that if $f \in P(X)$ vanishes on an open set U (relative to $\hat{X}$), then f vanishes identically. Such a uniform algebra is called an *analytic algebra.*

Further Examples. One can choose the parameters in the Wermer construction so that the intersection of $\hat{X} \setminus X$ with any analytic disk is "small".

Theorem 4 ([L]). *There exist X compact in $\partial\Delta \times \mathbf{C}$ with $\pi_z(\hat{X}) = \overline{\Delta}$ and such that $g(\Delta) \cap (\hat{X} \setminus X)$ is polar in $g(\Delta)$ for all H^∞ disks g.*

Note that in the Wermer example, we have no analytic structure in $\hat{X} \setminus X$; however, the set X itself can contain lots of analytic disks. Indeed, we have the following "fattening lemma" of Alexander.

Theorem 5 (Alexander [A2]). *There exists a Wermer-type set X (X compact in $\partial\Delta \times \mathbf{C}$ with $\pi_z(\hat{X}) = \overline{\Delta}$ and such that $\hat{X} \setminus X \subset \Delta \times \mathbf{C}$ contains no analytic structure) such that for all proper, closed subsets α of $\partial\Delta$ and all $M > 0$, setting*

$$Z := X \cup \{(z, w) : z \in \alpha,\ |w| \leq M\},$$

we have $\hat{Z} \setminus Z = \hat{X} \setminus X$.

Remarks. One can also construct the Wermer set $\hat{X}$ as a decreasing intersection of the generalized lemniscates

$$X_n := \{(z, w) : |z| \leq 1/2,\ |p_n(z, w)| \leq \epsilon_n\}$$

where $\{p_n\}$ are polynomials in (z, w) which satisfy

1. $\Sigma_n = \{(z, w) : |z| \leq 1/2,\ p_n(z, w) = 0\}$;
2. $p_n(z, w) = c_n^{2^n} z^{m_n} + R_n(z, w)$ where $\deg R_n < m_n := \deg p_n$;
3. $\{c_n\}$, $\{\epsilon_n\}$ tend to 0 rapidly enough so that $X_{n+1} \subset X_n$ for all n and $\hat{X} = \cap_n X_n$

(cf., [W]). Thus, from results in [LT], if

$$\lim_{n\to\infty} \left(\frac{\epsilon_n}{c_n^{2^n}}\right)^{1/m_n} = 0,$$

the set $\hat{X} \setminus X$ is pluripolar in $\mathbf{C}^2$ (see [L]).

In general, if X is compact in $\partial\Delta \times \mathbf{C}$ with $\pi_z(\hat{X}) = \overline{\Delta}$, then $\hat{X} \setminus X \subset \Delta \times \mathbf{C}$ is *pseudoconcave* in the sense of Oka; i.e., $(\Delta \times \mathbf{C}) \setminus (\hat{X} \setminus X)$ is pseudoconvex. In the terminology of set-valued functions, $\hat{X} \setminus X$ is the graph of an *analytic multifunction* over Δ (cf. [Sl]). Yamaguchi [Y] has shown in this setting that the function $z \to \log C(\hat{X}_z)$, where $\hat{X}_z := \{w : (z,w) \in \hat{X}\}$ is the fiber of $\hat{X}$ over z and $C(S)$ denotes the logarithmic capacity of the compact set S, is subharmonic on Δ. Thus, if there exists **one** z in Δ such that the fiber $\hat{X}_z$ is non-polar in $\mathbf{C}$, then $\hat{X} \setminus X$ is non-pluripolar as a subset of $\mathbf{C}^2$.

3. Final comments and open questions. Theorem 1 gives a concrete example of a compact set X with $\hat{X} \setminus X$ being non-pluripolar without containing any analytic structure. It is unknown if the Wermer example can be modified in this manner.

1. *Does there exist X compact in $\partial\Delta \times \mathbf{C}$ with $\pi_z(\hat{X}) = \overline{\Delta}$ such that $\hat{X} \setminus X$ contains no analytic structure but is non-pluripolar?*

From the discussion in section 3, once $\hat{X}_z$ is non-polar in $\mathbf{C}$ for **one** z in Δ, then $\hat{X} \setminus X$ is non-pluripolar in $\mathbf{C}^2$.

Suppose $S \subset \Delta \times \mathbf{C}$ is pseudoconcave. Sadullaev has shown [Sa] that S is pluripolar in $\mathbf{C}^2$ if and only if each fiber S_z is polar ("only if" follows from Yamaguchi's result).

2. *Let $S \subset \Delta \times \mathbf{C}$ be pseudoconcave with each fiber S_z being polar. Is it true that for each $r < 1$, $S^r := S \cap \{|z| < r\}$ is complete pluripolar; i.e., there exists u plurisubharmonic in $\{|z| < r\} \times \mathbf{C}$ such that*

$$S^r = \{(z,w) : u(z,w) = -\infty\}?$$

Is it true that $S \cap \{|z| \le r\}$ is polynomially convex for each $r < 1$?

Recall that for the Stolzenberg example, $P(\hat{X}) \neq C(\hat{X})$. Recently, Izzo [I] has constructed an example of a compact set X in the unit sphere ∂B in $\mathbf{C}^3$ which is polynomially convex ($\hat{X} = X$) but with $P(X) \neq C(X)$. Note that a subset of the unit sphere ∂B in $\mathbf{C}^N$ contains no analytic disk; thus there is no *analytic* obstruction to $P(X)$ being dense in $C(X)$. However, it is unkown if such an example can be constructed in $\mathbf{C}^2$.

3. *Suppose $X \subset \partial B \subset \mathbf{C}^2$ is compact and polynomially convex. Is $P(X) = C(X)$?*

We end this note by remarking that Alexander [A3] has recently constructed a compact set X in the unit torus $\partial\Delta \times \partial\Delta$ in $\mathbf{C}^2$ such that the origin $(0,0)$ lies in $\hat{X}$ but such that $\hat{X}$ contains no analytic structure.

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