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静弾性方程式初期値問題の変分法的解法

A variational solution of the Cauchy problem in elastostatics

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An inverse problem in two-dimensional elasticity is considered. The purpose is to present a variational approach to identification of the boundary conditions for resolution of the Cauchy problem governed by the Navier equations in plane elastostatics. The Cauchy problem is featured by simultaneously prescribed displacement and traction on a part of the boundary of an elastic body. The boundary data may contain some noises. The problem is re-formulated as a minimization problem of a functional with constraints, then the minimization problem is recast into successive primary and dual boundary value problems with no constraints in the corresponding plane elasticity problem. Two variational formulations, *i.e.* displacement approach and traction approach, are described. It is suggested that our variational method is convergent and the proposed rocess is stable.

Key Words: Inverse Analysis, Cauchy Problem, Elastostatics, Variational method, Displacement Approach, Traction Approach, Optimization, Elastostatics.

1 INTRODUCTION

We consider a cross section of an isotropic, linearly elastic bounded body. The deformation of the body with small strains is assumed to be described on the cross section denoted by Ω. Using the rectangular coordinates $\boldsymbol{x}=(x_1,x_2)$ in Ω, we denote by u_i the i-th component of the displacement ($i=1,2$), and by ε_{ij} and σ_{ij} the ij-th component of strain and stress, respectively. The compatibility equations relating the displacements to the strains are described by

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \tag{1}$$

The constitutive equations representing Hooke's law are given by

$$\begin{aligned}\sigma_{ij} &= 2\mu\varepsilon_{ij}+\lambda\delta_{ij}\varepsilon_{kk} \qquad \text{for plane strain}, \\ \sigma_{ij} &= 2\mu\varepsilon_{ij}+\frac{2\lambda\mu}{\lambda+2\mu}\delta_{ij}\varepsilon_{kk} \qquad \text{for plane stress}\end{aligned} \tag{2}$$

with the Lamé constants μ and λ, Kronecker's symbol δ_{ij}, and the bulk strain ε_{kk}, in which Einstein's summation convention is used for repeated indices. The Lamé constants are related to Young's modulus E, the shear modulus G, and Poisson's ratio ν as

$$\begin{aligned}\lambda &= \frac{2\nu G}{1-2\nu}=\frac{\nu E}{(1+\nu)(1-2\nu)}, \\ \mu &= G=\frac{E}{2(1+\nu)}.\end{aligned}$$

The force equilibrium equations with no external body force are written by

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \ . \tag{3}$$

We let Ω be enclosed by a piecewise smooth boundary denoted by Γ with no singularities, which is composed of two connected non-zero measure parts Γ_d and $\Gamma_{id} = \Gamma \backslash \Gamma_d$, see Figure 1. On the boundary Γ_d, we prescribe both displacements as the Dirichlet data and tractions as the Neumann data:

$$u_i = \overline{u}_i \qquad \text{and} \qquad \sigma_{ij} n_j = \overline{S}_i \qquad \text{on } \Gamma_d \tag{4}$$

simultaneously, with the unit exterior normal $\boldsymbol{n} = (n_1, n_2)$ to the boundary Γ. The system of equations (1)–(3) with partially overprescribed boundary conditions as in (4) constitutes a Cauchy problem in elastostatics.

Suppose that the Lamé constants μ and λ are known *à priori*. We suppose also that the geometry of Ω and the locaton of Γ_d are known. We notice that, if the data $\bar{u}_i$ and $\bar{S}_i$ are exactly available, the displacement $u_i(\boldsymbol{x})$ satisfying the system of equations (1)–(3) as a solution of the Cauchy problem is uniquely determined [1]. We shall take the case into account when the data $\bar{u}_i$ and $\bar{S}_i$ involve some errors in the measurement. When the data are noisy, or when the boundary displacements and tractions in (4) are given arbitrarily in such a way that they are not consistent, there exist no solutions satisfying (1)–(4) at all.

Our problem, therefore, consists of identifying *proper* boundary displacements $u_i = \omega_i$ on Γ_{id}, so that the solution $u_i(\boldsymbol{x})$ of the system of equations (1)–(3) *reflects* the simultaneous boundary conditions (4) given on Γ_d.

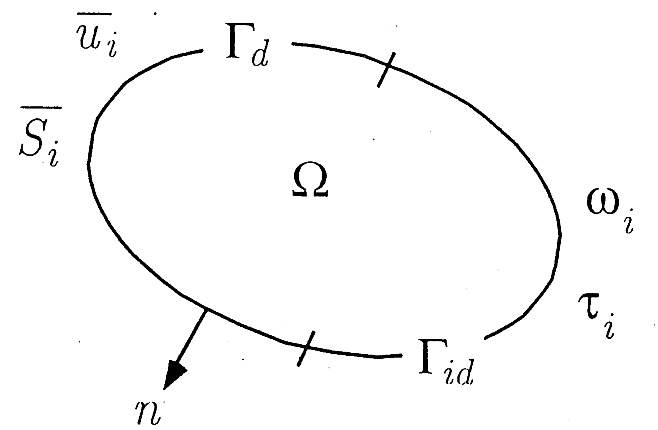

Figure 1. Cauchy problem in elastostatics.

In this paper the inverse problem under investigation is the conventional Cauchy problem. We present a variational approach, which is often employed in control theory [2], for the resolution of the inverse problem to identify boundary displacements. Our inverse problem is formulated as a minimization problem of a regularized least-squares functional with no constraints. By the use of the direct variational method combined with the gradient method, the minimization problem is recast into a series of well-posed primary and adjoint boundary value problems in elasticity.

2 VARIATIONAL FORMULATION

2.1 Displacement Approach

We will write $u_i(\boldsymbol{x}) = u_i(\boldsymbol{x};\boldsymbol{\omega})$ to show explicitly the dependence of the solution u_i on unknown boundary displacements $\boldsymbol{\omega} = (\omega_1, \omega_2)$ to be identified on Γ_{id}. Along the boundary, put $u_i = \bar{u}_i$ on Γ_d, $u_i = \omega_i$ on Γ_{id}, and assume that $u_i \in C(\Gamma)$.

Our strategy to find a proper ω_i is to consider the following object functional to be minimized:

$$J(\boldsymbol{\omega}) := \int_{\Gamma_d} [u_i(\boldsymbol{x};\boldsymbol{\omega}) - \bar{u}_i(\boldsymbol{x})]^2 \, d\Gamma + \eta \int_\Omega \sigma_{ij}\varepsilon_{ij} d\Omega \tag{5}$$

with a regularization parameter $\eta > 0$, among all admissible displacements $u_i(\boldsymbol{x};\boldsymbol{\omega})$ with the constraints $\sigma_{ij}n_j = \bar{S}_i$ on Γ_d. Here we regard $J : H^{1/2}(\Gamma_{id})^2 \ni \boldsymbol{\omega} \longmapsto \boldsymbol{R}_+ = [0, +\infty)$, and the sums are taken for repeated indices $i, j = 1, 2$.

The strain energy added to the integral of the square of the difference in (5) as a regularizer guarantees unique existence of the minimum of the functional $J(\boldsymbol{\omega})$ [3] even for noisy data. With a suitable choice of positive real numbers α_n for $n = 0, 1, 2, \cdots$, we will consider the minimizing process;

$$\boldsymbol{\omega}^{(n+1)} = \boldsymbol{\omega}^{(n)} - \alpha_n \boldsymbol{J}'(\boldsymbol{\omega}^{(n)}), \tag{6}$$

where the functional gradient $\boldsymbol{J}'(\boldsymbol{\omega})$ can be defined from the first variation;

$$J(\boldsymbol{\omega} + \delta\boldsymbol{\omega}) - J(\boldsymbol{\omega}) = \ < \boldsymbol{J}'(\boldsymbol{\omega}), \delta\boldsymbol{\omega} > + \, o(\| \, \delta\boldsymbol{\omega} \, \|) \tag{7}$$

with a real-valued functional $o(\| \, \delta\boldsymbol{\omega} \, \|)$ of higher order than $\| \, \delta\boldsymbol{\omega} \, \|$ as it tends to zero with the $(L^2)^2$–norm on Γ_{id}. Owing to (6), we require that $\boldsymbol{J}'(\boldsymbol{\omega}) \in H^{1/2}(\Gamma_{id})^2$ to keep $\boldsymbol{\omega}^{(n+1)}$ again in $H^{1/2}(\Gamma_{id})^2$.

The key for the success of the minimizing process in (6) is to seek a concrete expression of $\boldsymbol{J}'(\boldsymbol{\omega})$. We notice that

$$\begin{aligned}
& J(\boldsymbol{\omega} + \delta\boldsymbol{\omega}) - J(\boldsymbol{\omega}) \\
= & \int_{\Gamma_d} \left\{ [u_i(\boldsymbol{x};\boldsymbol{\omega}+\delta\boldsymbol{\omega}) - \bar{u}_i(\boldsymbol{x})]^2 - [u_i(\boldsymbol{x};\boldsymbol{\omega}) - \bar{u}_i(\boldsymbol{x})]^2 \right\} d\Gamma \\
& + \eta \int_\Omega \{ \sigma_{ij}(\boldsymbol{x};\boldsymbol{\omega}+\delta\boldsymbol{\omega})\varepsilon_{ij}(\boldsymbol{x};\boldsymbol{\omega}+\delta\boldsymbol{\omega}) \\
& \qquad - \sigma_{ij}(\boldsymbol{x};\boldsymbol{\omega})\varepsilon_{ij}(\boldsymbol{x};\boldsymbol{\omega}) \} \, d\Omega \\
= & \int_{\Gamma_d} [u_i(\boldsymbol{x};\boldsymbol{\omega}+\delta\boldsymbol{\omega}) + u_i(\boldsymbol{x};\boldsymbol{\omega}) - 2\bar{u}_i(\boldsymbol{x})] \\
& \qquad [u_i(\boldsymbol{x};\boldsymbol{\omega}+\delta\boldsymbol{\omega}) - u_i(\boldsymbol{x};\boldsymbol{\omega})] \, d\Gamma \\
& + \eta \int_\Omega \{ \sigma_{ij}(\boldsymbol{x};\boldsymbol{\omega}+\delta\boldsymbol{\omega})\varepsilon_{ij}(\boldsymbol{x};\boldsymbol{\omega}+\delta\boldsymbol{\omega}) \\
& \qquad - \sigma_{ij}(\boldsymbol{x};\boldsymbol{\omega}+\delta\boldsymbol{\omega})\varepsilon_{ij}(\boldsymbol{x};\boldsymbol{\omega}) \\
& \qquad + \sigma_{ij}(\boldsymbol{x};\boldsymbol{\omega}+\delta\boldsymbol{\omega})\varepsilon_{ij}(\boldsymbol{x};\boldsymbol{\omega}) - \sigma_{ij}(\boldsymbol{x};\boldsymbol{\omega})\varepsilon_{ij}(\boldsymbol{x};\boldsymbol{\omega}) \} \, d\Omega \\
= & \int_{\Gamma_d} [\delta u_i(\boldsymbol{x};\boldsymbol{\omega}) + 2u_i(\boldsymbol{x};\boldsymbol{\omega}) - 2\bar{u}_i(\boldsymbol{x})] \, \delta u_i(\boldsymbol{x};\boldsymbol{\omega}) d\Gamma \\
& + \eta \int_\Omega \{ \sigma_{ij}(\boldsymbol{x};\boldsymbol{\omega}+\delta\boldsymbol{\omega})\delta\varepsilon_{ij}(\boldsymbol{x};\boldsymbol{\omega}) \\
& \qquad + \delta\sigma_{ij}(\boldsymbol{x};\boldsymbol{\omega})\varepsilon_{ij}(\boldsymbol{x};\boldsymbol{\omega}) \} \, d\Omega \\
= & \int_{\Gamma_d} 2 \, [u_i(\boldsymbol{x};\boldsymbol{\omega}) - \bar{u}_i(\boldsymbol{x})] \, \delta u_i(\boldsymbol{x};\boldsymbol{\omega}) d\Gamma
\end{aligned}$$

$$
\begin{aligned}
&+\eta \int_{\Omega} \{\sigma_{ij}(\boldsymbol{x};\boldsymbol{\omega})\delta\varepsilon_{ij}(\boldsymbol{x};\boldsymbol{\omega}) + \delta\sigma_{ij}(\boldsymbol{x};\boldsymbol{\omega})\varepsilon_{ij}(\boldsymbol{x};\boldsymbol{\omega})\}\, d\Omega \\
&\qquad + o(\| \delta\boldsymbol{\omega} \|) \\
= &\int_{\Gamma_d} 2\left[u_i(\boldsymbol{x};\boldsymbol{\omega}) - \bar{u}_i(\boldsymbol{x})\right] \delta u_i(\boldsymbol{x};\boldsymbol{\omega}) d\Gamma \\
&\qquad + \eta \int_{\Omega} 2\sigma_{ij}(\boldsymbol{x};\boldsymbol{\omega})\delta\varepsilon_{ij}(\boldsymbol{x};\boldsymbol{\omega}) d\Omega + o(\| \delta\boldsymbol{\omega} \|) \\
= &\int_{\Gamma_d} 2\left[u_i(\boldsymbol{x};\boldsymbol{\omega}) - \bar{u}_i(\boldsymbol{x})\right] \delta u_i(\boldsymbol{x};\boldsymbol{\omega}) d\Gamma \\
&\qquad + \eta \int_{\Omega} 2\sigma_{ij}(\boldsymbol{x};\boldsymbol{\omega}) \frac{\partial \delta u_i}{\partial x_j}(\boldsymbol{x};\boldsymbol{\omega}) d\Omega + o(\| \delta\boldsymbol{\omega} \|) \\
= &\int_{\Gamma_d} 2\left[u_i(\boldsymbol{x};\boldsymbol{\omega}) - \bar{u}_i(\boldsymbol{x})\right] \delta u_i(\boldsymbol{x};\boldsymbol{\omega}) d\Gamma \\
&\qquad + \eta \int_{\Gamma} 2\sigma_{ij}(\boldsymbol{x};\boldsymbol{\omega}) n_j \delta u_i(\boldsymbol{x};\boldsymbol{\omega}) d\Gamma + o(\| \delta\boldsymbol{\omega} \|).
\end{aligned}
$$

Here we have put

$$
\delta u_i(\boldsymbol{x};\boldsymbol{\omega}) = u_i(\boldsymbol{x};\boldsymbol{\omega} + \delta\boldsymbol{\omega}) - u_i(\boldsymbol{x};\boldsymbol{\omega}), \tag{8}
$$

and correspondingly for $\delta\varepsilon_{ij}$ and $\delta\sigma_{ij}$. Moreover, we used the relations;

$$
\begin{aligned}
\sigma_{ij}\delta\varepsilon_{ij} &= (2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk})\delta\varepsilon_{ij} \\
&= 2\mu\varepsilon_{ij}\delta\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta\varepsilon_{ll} \\
&= (2\mu\delta\varepsilon_{ij} + \lambda\delta_{ij}\delta\varepsilon_{ll})\varepsilon_{ij} = \delta\sigma_{ij}\varepsilon_{ij},
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_{ij}\delta\varepsilon_{ij} &= \sigma_{ij}\frac{1}{2}\left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i}\right) \\
&= \frac{1}{2}\left(\sigma_{ij}\frac{\partial \delta u_i}{\partial x_j} + \sigma_{ji}\frac{\partial \delta u_j}{\partial x_i}\right) \\
&= \frac{1}{2}\left(\sigma_{ij}\frac{\partial \delta u_i}{\partial x_j} + \sigma_{ij}\frac{\partial \delta u_i}{\partial x_j}\right) \\
&= \sigma_{ij}\frac{\partial \delta u_i}{\partial x_j}
\end{aligned}
$$

from the symmetry $\sigma_{ij} = \sigma_{ji}$. In the last equality we used the Gauss divergence theorem and (3).

We notice that the stresses $\delta\sigma_{ij}$ induced by the displacements δu_i satisfy

$$
\frac{\partial \delta\sigma_{ij}}{\partial x_j} = 0 \qquad \text{in } \Omega, \tag{9}
$$

$$
\delta\sigma_{ij} n_j = 0 \qquad \text{on } \Gamma_d, \tag{10}
$$

$$
\delta u_i = \delta\omega_i \qquad \text{on } \Gamma_{id}. \tag{11}
$$

Equation (10) follows the constraints $\sigma_{ij} n_j = \bar{S}_i$ on Γ_d imposed in the admissible space.

We now introduce the adjoint displacement $(\hat{u}_1(\boldsymbol{x}), \hat{u}_2(\boldsymbol{x})) \in H^1(\Omega)^2$ and the corresponding adjoint stresses $\hat{\sigma}_{ij}$, as being the solution of the system of equations;

$$
\frac{\partial \hat{\sigma}_{ij}}{\partial x_j} = 0 \qquad \text{in } \Omega, \tag{12}
$$

subject to the boundary conditions;

$$
\hat{\sigma}_{ij} n_j = 2\left[u_i(\boldsymbol{x};\boldsymbol{\omega}) - \bar{u}_i(\boldsymbol{x})\right] + 2\eta\bar{S}_i \quad \text{on } \Gamma_d, \tag{13}
$$

$$
\hat{u}_i = 0 \qquad \text{on } \Gamma_{id}. \tag{14}
$$

Using the Gauss divergence theorem, we know that

$$\int_\Omega \frac{\partial \hat{\sigma}_{ij}}{\partial x_j}\delta u_i d\Omega = \int_\Gamma \hat{\sigma}_{ij} n_j \delta u_i d\Gamma - \int_\Omega \hat{\sigma}_{ij}\frac{\partial \delta u_i}{\partial x_j} d\Omega .$$

From the relations;

$$\begin{aligned}
\hat{\sigma}_{ij}\frac{\partial \delta u_i}{\partial x_j} &= \hat{\sigma}_{ij}\delta\varepsilon_{ij} \;=\; (2\mu\hat{\varepsilon}_{ij} + \lambda\delta_{ij}\hat{\varepsilon}_{kk})\delta\varepsilon_{ij} \\
&= 2\hat{\varepsilon}_{ij}\mu\delta\varepsilon_{ij} + \hat{\varepsilon}_{kk}\lambda\delta\varepsilon_{\ell\ell} \;=\; \hat{\varepsilon}_{ij}(2\mu\delta\varepsilon_{ij} + \lambda\delta_{ij}\delta\varepsilon_{\ell\ell}) \\
&= \hat{\varepsilon}_{ij}\delta\sigma_{ij} \\
&= \frac{\partial \hat{u}_i}{\partial x_j}\delta\sigma_{ij} ,
\end{aligned}$$

we get

$$\begin{aligned}
\int_\Omega \frac{\partial \hat{\sigma}_{ij}}{\partial x_j}\delta u_i d\Omega &= \int_\Gamma \hat{\sigma}_{ij} n_j \delta u_i d\Gamma - \int_\Omega \frac{\partial \hat{u}_i}{\partial x_j}\delta\sigma_{ij} d\Omega \\
&= \int_\Gamma \hat{\sigma}_{ij} n_j \delta u_i d\Gamma - \int_\Gamma \hat{u}_i \delta\sigma_{ij} n_j d\Gamma + \int_\Omega \hat{u}_i \frac{\partial \delta\sigma_{ij}}{\partial x_j} d\Omega .
\end{aligned}$$

Therefore, from (12), (10), (14), and (9) we obtain

$$0 = \int_{\Gamma_d} \hat{\sigma}_{ij} n_j \delta u_i d\Gamma + \int_{\Gamma_{id}} \hat{\sigma}_{ij} n_j \delta u_i d\Gamma . \tag{15}$$

Consequently, from (13), (15), (11), and using the traction condition in (4), we know that

$$\begin{aligned}
J(\omega + \delta\omega) - J(\omega) &= \int_{\Gamma_d} \hat{\sigma}_{ij} n_j \delta u_i d\Gamma - 2\eta \int_{\Gamma_d} \bar{S}_i \delta u_i d\Gamma \\
&\qquad + \eta \int_\Gamma 2\sigma_{ij} n_j \delta u_i d\Gamma + o(\| \delta\omega \|) \\
&= -\int_{\Gamma_{id}} \hat{\sigma}_{ij} n_j \delta\omega_i d\Gamma + \eta \int_{\Gamma_{id}} 2\sigma_{ij} n_j \delta\omega_i d\Gamma + o(\| \delta\omega \|) \\
&= \int_{\Gamma_{id}} (-\hat{\sigma}_{ij} n_j + 2\eta S_i)\delta\omega_i d\Gamma + o(\| \delta\omega \|) .
\end{aligned}$$

Now we know the explicit form

$$J'_i(\omega) = -\hat{\sigma}_{ij} n_j + 2\eta S_i \qquad \text{on} \quad \Gamma_{id} . \tag{16}$$

Using this result, we can summarize an algorithm for the minimization in the displacement approach as follows:

[1] *Given* $\omega^{(0)}$.

[2] *For* $n = 0, 1, 2, \cdots,$ *do:*

[2.1] *Solve* $\dfrac{\partial \sigma_{ij}^{(n)}}{\partial x_j} = 0$ *with* $\sigma_{ij}^{(n)} n_j|_{\Gamma_d} = \bar{S}_i$, $u_i^{(n)}|_{\Gamma_{id}} = \omega_i^{(n)}$ *to find* $u_i^{(n)}(x)$ *on* Γ_d *and* $S_i^{(n)}(x)$ *on* Γ_{id}.

[2.2] *Solve* $\dfrac{\partial \hat{\sigma}_{ij}^{(n)}}{\partial x_j} = 0$
with $\hat{\sigma}_{ij}^{(n)} n_j|_{\Gamma_d} = 2[u_i^{(n)}(x) - \bar{u}_i(x)] + 2\eta\bar{S}_i$,
$\hat{u}_i^{(n)}|_{\Gamma_{id}} = 0$

to find $\boldsymbol{J}'(\boldsymbol{\omega}^{(n)})$ *with the components*
$J_i'(\omega^{(n)}) = -\hat{S}_i^{(n)} + 2\eta S_i^{(n)}$ *on* Γ_{id} .

[2.3] *Update* $\boldsymbol{\omega}^{(n+1)} = \boldsymbol{\omega}^{(n)} - \alpha_n \boldsymbol{J}'(\boldsymbol{\omega}^{(n)})$.

2.2 Traction Approach

In the previous subsection, we considered the identification of the boundary displacements $\boldsymbol{\omega} = (\omega_1, \omega_2)$ on Γ_{id}. We will consider in this subsection the identification of boundary traction $\boldsymbol{\tau} = (\tau_1, \tau_2)$ on Γ_{id}. Here we express $u_i(\boldsymbol{x}) = u_i(\boldsymbol{x}; \boldsymbol{\tau})$ to stress the dependence of the solution u_i on unknown traction $\boldsymbol{\tau}$ to be identified.

Our objective is to find a proper τ_i, which minimizes the following functional

$$K(\boldsymbol{\tau}) := \int_{\Gamma_d} [S_i(\boldsymbol{x}; \boldsymbol{\tau}) - \bar{S}_i(\boldsymbol{x})]^2 d\Gamma + \eta \int_\Omega \sigma_{ij}\varepsilon_{ij} d\Omega \tag{17}$$

among all admissible tractions $S_i(\boldsymbol{x}; \boldsymbol{\tau})$ with the constraints $u_i = \bar{u}_i$ on Γ_d. Here we regard $K : H^{1/2}(\Gamma_{id})^2 \ni \boldsymbol{\tau} \mapsto \boldsymbol{R}_+$.

Along the same line of argument as in the preceding displacement approach, with the suitable choice of positive real numbers α_n for $n = 1, 2, \cdots$, we will consider the minimizing process;

$$\boldsymbol{\tau}^{(n+1)} = \boldsymbol{\tau}^{(n)} - \alpha_n \boldsymbol{K}'(\boldsymbol{\tau}^{(n)}), \tag{18}$$

where $\boldsymbol{K}'(\boldsymbol{\tau}) \in H^{1/2}(\Gamma_{id})^2$ can be defined from the first variation

$$K(\boldsymbol{\tau} + \delta\boldsymbol{\tau}) - K(\boldsymbol{\tau}) \;=\; < \boldsymbol{K}'(\boldsymbol{\tau}), \delta\boldsymbol{\tau} > + o\,(\|\, \delta\boldsymbol{\tau} \,\|). \tag{19}$$

To seek a concrete expression of $\boldsymbol{K}'(\boldsymbol{\tau})$ in a similar way as regard to $\boldsymbol{J}'(\boldsymbol{\omega})$, we notice that

$$\begin{aligned} K(\boldsymbol{\tau} + \delta\boldsymbol{\tau}) - K(\boldsymbol{\tau}) &= \int_{\Gamma_d} 2[S_i(\boldsymbol{x}; \boldsymbol{\tau}) - \bar{S}_i(\boldsymbol{x})] \delta S_i(\boldsymbol{x}; \boldsymbol{\tau}) d\Gamma \\ &\quad + \eta \int_\Omega 2\sigma_{ij}(\boldsymbol{x}; \boldsymbol{\tau}) \frac{\partial \delta u_i}{\partial x_j}(\boldsymbol{x}; \boldsymbol{\tau}) d\Omega + o\,(\|\, \delta\boldsymbol{\tau} \,\|), \end{aligned}$$

where we put variations in the boundary traction by

$$\delta S_i(\boldsymbol{x}; \boldsymbol{\tau}) = S_i(\boldsymbol{x}; \boldsymbol{\tau} + \delta\boldsymbol{\tau}) - S_i(\boldsymbol{x}; \boldsymbol{\tau}),$$

and $\delta u_i(\boldsymbol{x}; \boldsymbol{\tau})$ are corresponding variations in the displacement.

Using the relation

$$\sigma_{ij} \frac{\partial \delta u_i}{\partial x_j} = \frac{\partial u_i}{\partial x_j} \delta\sigma_{ij},$$

and by the Gauss divergence theorem, it becomes

$$\begin{aligned} K(\boldsymbol{\tau} + \delta\boldsymbol{\tau}) - K(\boldsymbol{\tau}) &= \int_{\Gamma_d} 2[S_i(\boldsymbol{x}; \boldsymbol{\tau}) - \bar{S}_i(\boldsymbol{x})] \delta S_i(\boldsymbol{x}; \boldsymbol{\tau}) d\Gamma \\ &\quad + 2\eta \int_\Gamma u_i \delta\sigma_{ij} n_j d\Gamma - 2\eta \int_\Omega u_i \frac{\partial \delta\sigma_{ij}}{\partial x_j} d\Omega + o\,(\|\, \delta\boldsymbol{\tau} \,\|). \end{aligned}$$

The stresses $\delta\sigma_{ij}$ induced by the displacements δu_i satisfy

$$\frac{\partial \delta\sigma_{ij}}{\partial x_j} = 0 \quad \text{in} \quad \Omega, \tag{20}$$

$$\delta u_i = 0 \quad \text{on} \quad \Gamma_d, \tag{21}$$

$$\delta S_i = \delta\tau_i \quad \text{on} \quad \Gamma_{id}. \tag{22}$$

We now introduce the adjoint system

$$\frac{\partial \hat{\sigma}_{ij}}{\partial x_j} = 0 \quad \text{in} \quad \Omega, \tag{23}$$

subject to the boundary conditions;

$$\hat{u}_i = 2[S_i(\boldsymbol{x};\boldsymbol{\tau}) - \bar{S}_i(\boldsymbol{x})] + 2\eta\bar{u}_i \quad \text{on} \quad \Gamma_d, \tag{24}$$

$$\hat{S}_i = 0 \quad \text{on} \quad \Gamma_{id}. \tag{25}$$

From (23), (20), (21), (25), and (22) we can see that

$$\begin{aligned} 0 &= \int_\Omega \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} \delta u_i d\Omega \\ &= \int_\Gamma \hat{\sigma}_{ij} n_j \delta u_i d\Gamma - \int_\Omega \hat{\sigma}_{ij} \frac{\partial \delta u_i}{\partial x_j} d\Omega \\ &= \int_\Gamma \hat{S}_i \delta u_i d\Gamma - \int_\Omega \frac{\partial \hat{u}_i}{\partial x_j} \delta \sigma_{ij} d\Omega \\ &= \int_\Gamma \hat{S}_i \delta u_i d\Gamma - \int_\Gamma \hat{u}_i \delta \sigma_{ij} n_j d\Gamma + \int_\Omega \hat{u}_i \frac{\partial \delta \sigma_{ij}}{\partial x_j} d\Omega \\ &= -\int_{\Gamma_d} \hat{u}_i \delta S_i d\Gamma - \int_{\Gamma_{id}} \hat{u}_i \delta \tau_i d\Gamma, \end{aligned}$$

which yields the relation;

$$\int_{\Gamma_d} \hat{u}_i \delta S_i d\Gamma = -\int_{\Gamma_{id}} \hat{u}_i \delta \tau_i d\Gamma. \tag{26}$$

Consequently, from (24), (20), (22), and (26) we know that

$$\begin{aligned} K(\boldsymbol{\tau} + \delta\boldsymbol{\tau}) - K(\boldsymbol{\tau}) &= \int_{\Gamma_d} \hat{u}_i \delta S_i d\Gamma - 2\eta \int_{\Gamma_d} \bar{u}_i \delta S_i d\Gamma + 2\eta \int_\Gamma u_i \delta S_i d\Gamma + o(\| \delta\boldsymbol{\tau} \|) \\ &= \int_{\Gamma_d} \hat{u}_i \delta S_i d\Gamma + 2\eta \int_{\Gamma_{id}} u_i \delta \tau_i d\Gamma + o\,(\| \delta\boldsymbol{\tau} \|) \\ &= \int_{\Gamma_{id}} (-\hat{u}_i + 2\eta u_i) \delta \tau_i d\Gamma + o\,(\| \delta\boldsymbol{\tau} \|). \end{aligned}$$

Therefore we obtain $\boldsymbol{K}'(\boldsymbol{\tau})$ in the explicit form

$$K_i'(\boldsymbol{\tau}) = -\hat{u}_i + 2\eta u_i. \tag{27}$$

Using this result, we can summarize an algorithm for the minimization in the traction approach as follows:

[1] *Given* $\boldsymbol{\tau}^{(0)}$.

[2] *For* $n = 0, 1, 2, \cdots$, *do:*

[2.1] *Solve* $\dfrac{\partial \sigma_{ij}^{(n)}}{\partial x_j} = 0$ *with* $u_i^{(n)}|_{\Gamma_d} = \bar{u}_i$, $\sigma_{ij}^{(n)} n_j|_{\Gamma_{id}} = \tau_i^{(n)}$
to find $S_i^{(n)}(\boldsymbol{x})$ *on* Γ_d *and* $u_i^{(n)}(\boldsymbol{x})$ *on* Γ_{id}.

[2.2] *Solve* $\dfrac{\partial \hat{\sigma}_{ij}^{(n)}}{\partial x_j} = 0$
with $\hat{u}_i^{(n)}|_{\Gamma_d} = 2[S_i^{(n)}(\boldsymbol{x};\boldsymbol{\tau}) - \bar{S}_i(\boldsymbol{x})] + 2\eta\bar{u}_i$,
$\hat{\sigma}_{ij}^{(n)} n_j|_{\Gamma_{id}} = 0$
to find $K_i'(\boldsymbol{\tau}^{(n)}) = -\hat{u}_i^{(n)} + 2\eta u_i^{(n)}$ *on* Γ_{id}.

[2.3] *Update* $\boldsymbol{\tau}^{(n+1)} = \boldsymbol{\tau}^{(n)} - \alpha_n \boldsymbol{K}'(\boldsymbol{\omega}^{(n)})$.

3 CONCLUSIONS

We have considered the Cauchy problem of the Navier equations in elastostatics, regarded as a boundary inverse problem. The problem consists of identifying either unknown displacements or unknown tractions on a part of the boundary of the elastic material, when displacements and tractions are simultaneously prescribed as the Cauchy data on the rest of the boundary. Theoretically, when the data are exactly available, the unknown displacement or traction is uniquely determined. We included the case when noises are likely to be contained in the data. In order to make the unknown displacement or traction uniquely determined even for the noisy data, we considered regularization of the Tikhonov-type in the objective functional to be minimized.

Our inverse problem is recast by the use of the variational method into an infinite number of iterative processes consisting of direct primary and adjoint mixed boundary value problems in elastostatics. The process yields either a boundary displacement or traction, at which the objective functional attains its minimum.

Simple numerical examples suggested that our variational method of solution to the inverse problem is convergent to the minimum of the objective functional, and our numerical process is stable irrespective of measurement errors in the data.

参考文献

[1] Ang, D. D., Trong, D. D., and Yamamoto, M.: Unique continuation and identification of boundary of an elastic body. *Journal of Inverse and Ill-posed Problems*, Vol.3, pp.417-428 (1995).

[2] Lions, J. L.: *Contrôle Optimal de Systèmes Gouvernés par des Équations aux Derivées Partielles.* Dunod, Paris (1986).

[3] Ekeland, I. and Temam, R.: *Convex Analysis and Variational Problems.* North-Holland Publishing Company, Amsterdam (1976).