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THE ROTATION SETS VERSUS THE MARKOV PARTITIONS

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ABSTRACT. The rotation sets of homologically trivial homeomorphisms are studied. If a homologically trivial homeomorphism has Markov partitions, we show the relation between this homeomorphism and the Markov partitions, give the way to calculate the rotation set from the Markov partition, and show that if a homologically trivial homeomorphism has Markov partitions, the rotation set is a convex polygon.

§1 Introduction

In this paper we show the relation between the rotation set for homeomorphisms whose associated homomorphism f_* on $H_1(M;\mathbb{Z})$ is the identity map and the sofic system of the subshift of finite type associated with the Markov partition. We give the way to calculate the rotation set from the Markov partition and show that the rotation set is a convex polygon and explicit representation of every extremal point of this polygon.

We will define the rotation set for homeomorphisms whose associated homomorphism f_* on $H_1(M;\mathbb{Z})$ is the identity map in Section 2, overview the theory of the shift automorphisms in Section 3, define the Markov partitions and show some properties of the Markov partitions in Section 4, and show the relation between the rotation sets and the Markov partitions in Section 5.

Our main results are :

Theorem 5.2. *Let (M,f) be a homeomorphism whose associated homomorphism f_* on $H_1(M;\mathbb{Z})$ is the identity map, and suppose (M,f) has a Markov partition $\mathcal{R}=\{R_i\}$ of M. Suppose the itinerary $\mathcal{I}(x)$ of $x\in M$ is $\mathcal{I}(x)=\cdots i_{-2},i_{-1},i_0,i_1,i_2,\cdots$ and the image of the 2-block map $\mathcal{S}(\mathcal{I}(x))$ of $\mathcal{I}(x)$ is*

$$\mathcal{S}(\mathcal{I}(x))=\cdots[\alpha_{i_{-2},i_{-1}}][\alpha_{i_{-1},i_0}][\alpha_{i_0,i_1}][\alpha_{i_1,i_2}]\cdots$$

Then the homological rotation set $\rho(x,f)$ of f is given by

$$\rho(x,f)=\sum_{i,j}P(\alpha_{i,j})[\alpha_{i,j}]$$

where $P(\alpha_{i,j})$ is the appearance probability of the subsequence "R_iR_j" in the itinerary $\mathcal{I}(x)=\cdots i_{-2},i_{-1},i_0,i_1,i_2,\cdots$ of x if $P(\alpha_{i,j})$ exists.

and

Theorem 5.3. *Let (M, f) be a transitive homeomorphism whose associated homomorphism f_* on $H_1(M;\mathbb{Z})$ is the identity map and suppose (M, f) has a Markov partition $\mathcal{R} = \{R_i\}$ of M. Then the rotation set $\mathrm{Rot}(f)$ is a finite polygon and every extremal point is obtained by the pointwise rotation set of some periodic point.*

Bowen[1] constructed Markov partitions on the basic sets of the Axiom A diffeomorphisms, so we conclude that the rotation set of Axiom A diffeomorphisms with $f_* = \mathrm{id}$ is a finite polygon if f is restricted on one basic set. Thus we have

Corollary 5.4. *For an Axiom A diffeomorphism f with $f_* = id$, the homological rotation set is a finite union of finite polygons, and the mean rotation set is a finite polygon.*

Thurston[3] constructed Markov partitions of the pseudo Anosov diffeomorphisms. Thus we have

Corollary 5.5. *For a pseudo Anosov diffeomorphism f with $f_* = id$, the homological rotation set is a finite polygon.*

§2 The rotation set

Let M be a closed manifold and let f be a homeomorphism of M whose associated homomorphism f_* on $H_1(M;\mathbb{Z})$ is the identity map. Here we define the rotation set for f and show some properties.

Suppose O is a base point of f. Let $p : \hat{M} \to M$ be the maximal Abelian covering space and $F : \hat{M} \to \hat{M}$ be a lift of f. Let $\xi \in p^{-1}(x)$ on $\hat{M}$ be a lift of x on M. $\gamma(a, b)$ denotes a curve from a to b on $\hat{M}$. $\hat{\mathrm{O}}(a)$ denotes the lift of O which is the closest to a on $\hat{M}$. $\mathcal{H}_n(\xi, F, \mathrm{O})$ denotes the curve given by the concatenation of $\gamma(\hat{\mathrm{O}}(\xi), F^n(\xi))$ and $\gamma(F^n(\xi), \hat{\mathrm{O}}(F^n(\xi)))$. Since $(\hat{M}, p)$ is the maximal Abelian, every loop c on $\hat{M}$ is mapped to a null homologous loop $p(c)$ on M. Thus for every curve α on $\hat{M}$, the homology class of $p(\alpha)$ is uniquely determined by the start point and the end point of α. Let $[\alpha]$ be the homology class of α.

[Figure 2.1]

Proposition 2.1. *For $\xi, \eta \in p^{-1}(x)$, $p(\mathcal{H}_n(\xi, F, \mathrm{O})) = p(\mathcal{H}_n(\eta, F, \mathrm{O}))$*

Let $h_n(x, f, F, \mathrm{O})$ denote the loop on M which is the image of p of the curve $\mathcal{H}_n(\xi, F, \mathrm{O})$ on $\hat{M}$ and $[h_n(x, f, F, \mathrm{O})]$ denote the homology class of $h_n(x, f, F, \mathrm{O})$.

Proposition 2.2.

$$[h_n(x, f, F, \mathrm{O})] = \sum_{i=0}^{n-1} [h_1(f^i(x), f, F, \mathrm{O})]$$

[Figure 2.2]

Proposition 2.3. *Let F and G be lifts of f. Then the difference between the element $[h_n(x, f, F, \mathrm{O})]$ and $[h_n(x, f, G, \mathrm{O})]$ is $n\alpha(h)$ for some element $\alpha(h)$ of $H_1(M; \mathbb{Z})$, and $\alpha(h)$ does not depend on x.*

Proposition 2.4. *Let* A *and* B *are the different base points on M. Then $[h_n(x, f, F, \mathrm{A})]-[h_n(x, f, F, \mathrm{B})]$ is bounded element of $H_1(M; \mathbb{Z})$.*

Definition 2.5. *The (homological) pointwise rotation set $\rho(x, f, F)$* of x with respect to f and F is defined as

$$\rho(x, f, F) = \lim_{n\to\infty} \frac{[h_n(x, f, F, \mathrm{O})]}{n}$$

if limit exists.

Note that $\rho(x, f, F)$ is an element of $H_1(M; \mathbb{R})$.

Suppose F and G are lifts of f then there is an element h of the covering transformation $D(\hat{M}, p, M)$ which satisfies $G = h \circ F$. Let $\alpha(F, G)$ be an element of $H_1(M; \mathbb{Z})$ which is defined by h. Then the difference between $\rho(x, f, F)$ and $\rho(x, f, G)$ is equal to $\alpha(F, G)$ and $\alpha(F, G)$ is not depend on x.

The set $\mathrm{Rot}(f, F) = \{\rho(x, f, F) \mid x \in M \text{ and } \rho(x, f, F) \text{ exists}\}$ of the homological pointwise rotation set $\rho(x, f, F)$ is called *the rotation set of f*.

Remark. When f is homotopic to the identity map, this definition agrees with that of Franks [6].

Proposition 2.6. $\rho(x, f^N, F^N) = N \cdot \rho(x, f, F)$

Let us consider the mean rotation set with respect to the measure μ on M. Firstly let us recall the following theorem.

Theorem (Birkhoff's ergodic theorem). *Let μ be a measure on M and suppose $f : (M, \mu) \to (M, \mu)$ is μ-preserving and $h \in L^1(\mu)$. Then*

$$\frac{1}{n} \sum_{i=0}^{n-1} h(f^i(x))$$

converges a.e. *to a function $h^* \in L^1(\mu)$.*

If f preserves the measure μ on M, from the Birkhoff's ergodic theorem, the limit

$$\lim_{n\to\infty} \frac{[h_n(x, f, F, \mathrm{O})]}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} [h_1(f^i(x), f, F, \mathrm{O})]$$

exists for almost every x with respect to μ.

In the following arguments, μ denotes an f-invariant measure on M. Let $\rho_\mu(f,F)$ be the *mean rotation set* defined as

$$\rho_\mu(f,F) = \int_M \rho(x,f,F)d\mu$$

Proposition 2.7 (The ergodic theorem). *The mean rotation set satisfies the following equality.*

$$\rho_\mu(f,F) = \int_M [h_1(x,f,F,\mathrm{O})]d\mu$$

Lemma 2.8. *Let M be a compact manifold and let f and g be homeomorphisms of M whose associated homomorphism f_* on $H_1(M;\mathbb{Z})$ is the identity map. We also suppose f and g have the common invariant measure μ on M. Let $\widehat{g\circ f}$ be a lift of the composition $g\circ f$ to $\hat{M}$. Then the rotation set $\rho_\mu(g\circ f, \widehat{g\circ f})$ of the composition $g\circ f$ of f and g is equal to the sum $\rho_\mu(f,F)+\rho_\mu(g,G)$ of the rotation set $\rho_\mu(f,F)$ of f and that $\rho_\mu(g,G)$ of g with $H_1(M;\mathbb{Z})$ translation ambiguity.*

§3 Shift Automorphisms

Let k be a positive integer and $[k]$ be the set of numbers $\{1,2,\cdots,k\}$ with the discrete topology. $[k]$ is called *the alphabet set.* Let $\Sigma(k)$ be the product space $[k]^{\mathbb{Z}}$. Then an element of $\Sigma(k)$ is an infinite sequence $a = \cdots a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots$, where every a_n is contained in $[k] = \{1,2,\cdots,k\}$. The product topology on $\Sigma(k)$ induces the metric $d(a,b) = \sum_{n=-\infty}^{\infty} 2^{-2|n|-1}\delta_n(a,b)$, where $\delta_n(a,b) = 0$ if $a_n = b_n$ otherwise $\delta_n(a,b) = 1$. Then $\Sigma(k)$ is compact, totally disconnected and has no isolated points, thus $\Sigma(k)$ is homeomorphic to the Cantor set.

Let the shift σ be the homeomorphism on $\Sigma(k)$ defined as $(\sigma(a))_n = a_{n+1}$ where $(\sigma(a))_n$ denotes the n-th digit of the infinite sequence $\sigma(a)$. The shift moves the sequence one place to the left.

Next we define the subshift of finite type.

Definition 3.1. Let $A = (A_{ij})$ be a $k\times k$ matrix of 0-1 entries. We define the subspace Σ_A of $\Sigma(k)$ as

$$\Sigma_A = \{a\in\Sigma(k) \mid A_{a_i a_{i+1}} = 1 \text{ for every } i\}$$

Then Σ_A is a closed σ-invariant subspace of $\Sigma(k)$. The restriction of σ on Σ_A is also written as σ. We call the pair (Σ_A,σ) a *subshift of finite type* and the matrix A is called the *transition matrix.* Every member of Σ_A is called an *admissible sequence.* If for every i and j, there is a positive integer $n(ij)$ such that $A_{ij}^{n(ij)} \neq 0$ then the matrix A is called *transitive.*

For $\{c_i, c_{i+1}, \cdots, c_{i+j}\} \subset [k]$, the set

$$C(c_i, c_{i+1}, \cdots, c_{i+j}) = \{a \in \Sigma_A \mid a_k = c_k,\ i \leq k \leq i+j\}$$

is called *a cylinder.*

The subshift of finite type defines the oriented graph Γ whose vertices are $[k]$ and whose oriented edges are given by the matrix A as follows:

For a cylinder $C(c_0, c_1, \cdots, c_j)$, we attach the oriented path $\gamma(C(c_0, c_1, \cdots, c_j)) = c_0 \to c_1 \to c_2 \to \cdots \to c_j$ of Γ to this sequence. Inversely, we can determine the cylinder C from the finite path on Γ.

An oriented path $a_0 \to a_1 \to a_2 \to \cdots \to a_m$ on Γ is called *a simple loop* when the path satisfies $a_0 = a_m$ and $a_i \neq a_j$ for every $0 \leq i < j \leq m-1$.

Let V be the vector space hulled by $[k]$ and for a in (Σ_A, σ), we define elements $v_n(a)$ and $v(a)$ of V as follows:

$$v_n(a) = \frac{a_0 + \cdots + a_n}{n+1}$$
$$v(a) = \lim_{n\to\infty} v_n(a) \quad \text{if limit exists}$$

Suppose $a = \cdots a_0, a_1, \cdots, a_{p-1}, a_0, a_1, \cdots, a_{p-1}, \cdots \in \Sigma_A$ is periodic with period p. Let l_i be a finite loop on Γ given by $l_i = (a_0 \to a_1 \to \cdots \to a_{p-1} \to a_0)$. Then we define the barycenter $g_i = v(a) = \dfrac{a_0 + a_1 + \cdots + a_{p-1}}{p} \in V$ corresponds to this periodic point, and $G \subset V$ denotes the set of these barycenters. Note that every periodic point of (Σ_A, σ) corresponds to some loop of Γ.

Lemma 3.2. *Let Σ_A be a subshift of finite type of the transitive transition matrix A with alphabet set $[k] = \{1, 2, \cdots, k\}$ and Γ be the graph given by the transition matrix A. Let V be the vector space hulled by $\{1, 2, \cdots, k\}$. g_i denotes the barycenter in V corresponding to the simple loop l_i of Γ.*

For every admissible sequence $a = \cdots a_{-1}a_0a_1 \cdots$ of Σ_A, we define the vector v_n^+, v_n, v^+ and v in V as

$$v_n^+(a) = \frac{1}{n+1}(a_0 + a_1 + \cdots + a_{n-1} + a_n)$$
$$v^+(a) = \lim_{n\to\infty} v_n^+(a) \qquad \textit{if limit exists}$$
$$v_n(a) = \frac{1}{2n+1}(a_{-n} + a_{-n+1} + \cdots + a_{-1} + a_0 + a_1 + \cdots + a_{n-1} + a_n)$$
$$v(a) = \lim_{n\to\infty} v_n(a) \qquad \textit{if limit exists}$$

Then the set $P^+ = \{v^+(a) | a \in \Sigma_A\}$ agrees with $P = \{v(a) | a \in \Sigma_A\}$, and each of them is equal to the closure of the convex hull $\mathrm{Conv}(G)$ *of G. Moreover,* $\mathrm{Conv}(G)$ *is a convex finite polygon.*

§4 Markov partitions

In this section, we define a Markov partition and show some properties.
Let M be a compact manifold and $f : M \to M$ be a homeomorphism of M.

Definition 4.1. Let M_0 be a subset of M. An (at most countable) partition α of M is called *a topological generator for* M_0 if the following conditions are satisfied:

(1) The union $\bigcup_{A\in\alpha} \mathrm{Int}A$ of the interior $\mathrm{Int}A$ is dense in M and $\mathrm{Int}A \neq \emptyset$ for every $A \in \alpha$.

(2) if $x \in M_0$ and every sequence $A_{i_k} \in \alpha$ satisfies that

$$x \in \bigcap_{n\in\mathbb{Z}} \overline{\bigcap_{k=-n}^{n} f^k(\mathrm{Int}A_{i_k})}$$

then

$$\{x\} = \bigcap_{n\in\mathbb{Z}} \overline{\bigcap_{k=-n}^{n} f^k(\mathrm{Int}A_{i_k})}$$

where $\overline{A}$ is the closure of A.

Definition 4.2. A finite topological generator $\alpha = (A_1, A_2, \cdots, A_N)$ for M satisfying $A_i \subset \overline{\mathrm{Int}A_i}$ for every i is called *a Markov partition for* M if the following conditions are satisfied:

(1) (Local product structure) For every A_i, there exist compact spaces E_i and F_i, and a topological isomorphism

$$\varphi_i : \overline{A_i} \to E_i \times F_i$$

such that for every $x \in \mathrm{Int}A_i$ with $f(x) \in \mathrm{Int}A_j$ for some $1 \leq j \leq N$,

$$f(E_i(x) \cap \mathrm{Int}A_i) \supset E_j(f(x)) \cap \mathrm{Int}A_j$$
$$f^{-1}(F_j(f(x)) \cap \mathrm{Int}A_j) \supset F_i(x) \cap \mathrm{Int}A_i$$

where $E_i(x) = \{y \in E_i \mid \varphi_i(y) \in E_i \times \{x_2\}\}$ and $F_i(x) = \{y \in F_i \mid \varphi_i(y) \in \{x_1\} \times F_i\}$.

(2) (Boundary condition) There exists a decomposition

$$M \setminus \bigcup_{i=1}^{N} \mathrm{Int}A_i = B^+ \cup B^-$$

(not necessarily disjoint union) such that

$$f(B^+) \subset B^+, \quad f^{-1}(B^-) \subset B^-$$

When (M, f) has a Markov partition, this defines the transition matrix $T = (t_{ij})$ of $l \times l$ matrix which is defined as

$$t_{ij} = \begin{cases} 1 & \text{if } f(\mathrm{Int}R_i) \cap R_j \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

This transition matrix T defines the subshift of finite type (Σ_T, σ).

The *itinerary* $\mathcal{I}(x) = \cdots i_{-1} i_0 i_1 i_2 \cdots$ of x is an element of Σ_A defined as $i_k = l$ if $f^k(x) \in R_l$ for every x in M. When $f^k(x)$ is on ∂R_l and ∂R_m, define $i_k = l$ or $i_k = m$ properly.

Let M be a manifold and f be a homeomorphism on M. Suppose f on M has Markov partitions and let $\mathcal{R} = \{R_1, \cdots, R_l\}$ be a Markov partition of M. Then we can define the semiconjugacy of σ on the subshift of finite type (Σ_T, σ) and f on M as follows;

For every a in Σ_T, we can find x in M whose itinerary $\mathcal{I}(x)$ is $a = \cdots i_{-1} i_0 i_1 i_2 \cdots$, where $i_k = j$ if $f^k(x) \in R_j$. θ denotes this map. Since every x in M has its itinerary, θ is clearly surjective. We have the two representation of a rational decimal which makes θ finite to 1. The continuity of θ is given from the direct calculations [16].

Thus we have the following semiconjugacy

$$\begin{array}{ccc} \Sigma_T & \xrightarrow{\sigma} & \Sigma_T \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ M & \xrightarrow[f]{} & M \end{array}$$

where θ is at most k^2 to 1.

§5 The rotation set v.s. the Markov partition

Here we show the relation between the rotation set $\rho(x, f)$ and the Markov partitions.

In this section we suppose (M, f) has a Markov partition $\mathcal{R} = \{R_1, R_2, \cdots, R_l\}$ of M. $(M, f, \mathcal{R})$ induces a subshift of finite type and (Σ_A, σ) denotes this subshift of finite type.

Let O be a base point on M and $(\hat{M}, p, M)$ be the maximal Abelian covering space. We attach the itinerary $\mathcal{I}(x) \in \Sigma_A$ to every x in M. For the Markov partition, take a point x_i in the interior $\mathrm{Int}R_i$ of R_i and let this point denote the representative point of the rectangle R_i. Let $\xi_i \in p^{-1}(x_i)$ be a lifted point of $x_i \in \mathrm{Int}R_i$ and let $\hat{R}_i$ be the lifted rectangle of R_i which contains ξ_i. Suppose $\mathrm{Int}f(R_i) \cap \mathrm{Int}R_j \neq \emptyset$, there is a lift $\hat{R}_j$ of R_j such that $\mathrm{Int}F(\hat{R}_i) \cap \mathrm{Int}\hat{R}_j \neq \emptyset$. Let ξ_{ij} be a point in $\mathrm{Int}F(\hat{R}_i) \cap \mathrm{Int}\hat{R}_j$ and $\xi_j \in p^{-1}(x_j)$ be the representative point of $\hat{R}_j$. We connect $F(\xi_i)$ and ξ_j by a simple curve and g_{ij} denotes this simple curve. Then we have the curve A_{ij} on $\hat{M}$ by the concatenation of $\gamma(\hat{\mathrm{O}}(\xi_i), F(\xi_i))$, g_{ij} and $\gamma(\xi_j, \hat{\mathrm{O}}(\xi_j))$ which defines the loop α_{ij} on M. Let $[\alpha_{ij}] \in H_1(M; \mathbb{Z})$ denote the homology class of the loop α_{ij}.

[figure 5.1]

Definition 5.1. 2-block map $\mathcal{S} : \Sigma_A \to (H_1(M;\mathbb{Z}))^{\mathbb{Z}}$ of the subshift of finite type Σ_A to $(H_1(M;\mathbb{Z}))^{\mathbb{Z}}$ is a homomorphism defined as follows.

Suppose $a = \cdots a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots$ is an element of Σ_A. For every pair (a_i, a_{i+1}), $i \in \mathbb{Z}$ of $a \in \Sigma_A$, there is an element $[\alpha_{a_i,a_{i+1}}]$ of $H_1(M;\mathbb{Z})$. Then $\mathcal{S}(a)$ is defined by

$$\mathcal{S}(a) = \cdots [\alpha_{a_{-2},a_{-1}}][\alpha_{a_{-1},a_0}][\alpha_{a_0,a_1}][\alpha_{a_1,a_2}] \cdots$$

Note that $(\mathcal{S}(\Sigma_A), \sigma)$ is a typical example of the sofic system.[21]

Theorem 5.2. *Let (M, f) be a homeomorphism whose associated homomorphism f_* on $H_1(M;\mathbb{Z})$ is the identity map, and suppose (M, f) has a Markov partition $\mathcal{R} = \{R_i\}$ of M. Suppose the itinerary $\mathcal{I}(x)$ of $x \in M$ is $\mathcal{I}(x) = \cdots i_{-2}, i_{-1}, i_0, i_1, i_2, \cdots$ and the image of the 2-block map $\mathcal{S}(\mathcal{I}(x))$ of $\mathcal{I}(x)$ is*

$$\mathcal{S}(\mathcal{I}(x)) = \cdots [\alpha_{i_{-2},i_{-1}}][\alpha_{i_{-1},i_0}][\alpha_{i_0,i_1}][\alpha_{i_1,i_2}] \cdots$$

Then the homological rotation set $\rho(x, f)$ of f is given by

$$\rho(x, f) = \sum_{i,j} P(\alpha_{i,j})[\alpha_{i,j}]$$

where $P(\alpha_{i,j})$ is the appearance probability of the subsequence "R_iR_j" in the itinerary $\mathcal{I}(x) = \cdots i_{-2}, i_{-1}, i_0, i_1, i_2, \cdots$ of x if $P(\alpha_{i,j})$ exists.

Noting that a homomorphic image of the convex finite polygon is also the convex finite polygon, from the Lemma 3.2, we have

Theorem 5.3. *Let (M, f) be a transitive homeomorphism whose associated homomorphism f_* on $H_1(M;\mathbb{Z})$ is the identity map and suppose (M, f) has a Markov partition $\mathcal{R} = \{R_i\}$ of M. Then the rotation set* $\mathrm{Rot}(f)$ *is a finite polygon and every extremal point is obtained by the pointwise rotation set of some periodic point.*

Bowen[1] constructed Markov partitions on the basic sets of the Axiom A diffeomorphisms, so we conclude that the rotation set of Axiom A diffeomorphisms with $f_* = \mathrm{id}$ is a finite polygon if f is restricted on one basic set. Thus we have

Corollary 5.4. *For an Axiom A diffeomorphism f with $f_* = id$, the homological rotation set is a finite union of finite polygons, and the mean rotation set is a finite polygon.*

Thurston[3] constructed Markov partitions of the pseudo Anosov diffeomorphisms. Thus we have

Corollary 5.5. *For a pseudo Anosov diffeomorphism f with $f_* = id$, the homological rotation set is a finite polygon.*

Acknowledgements

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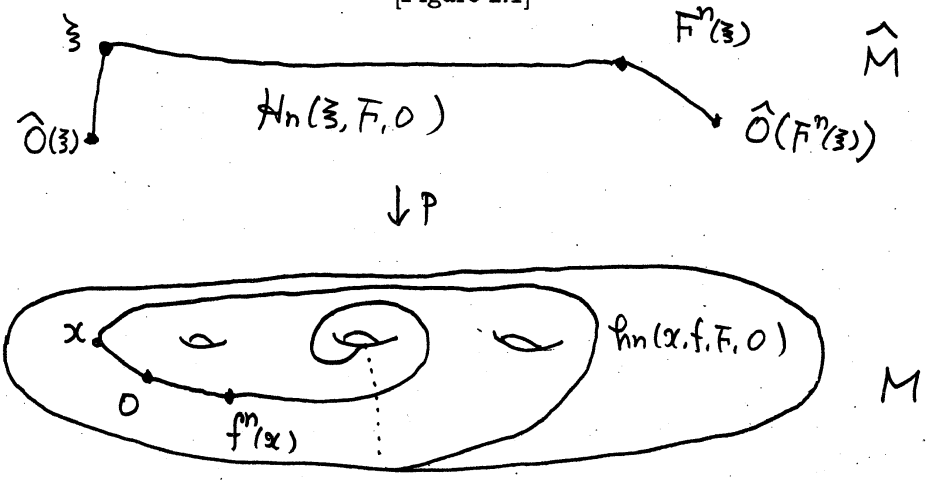
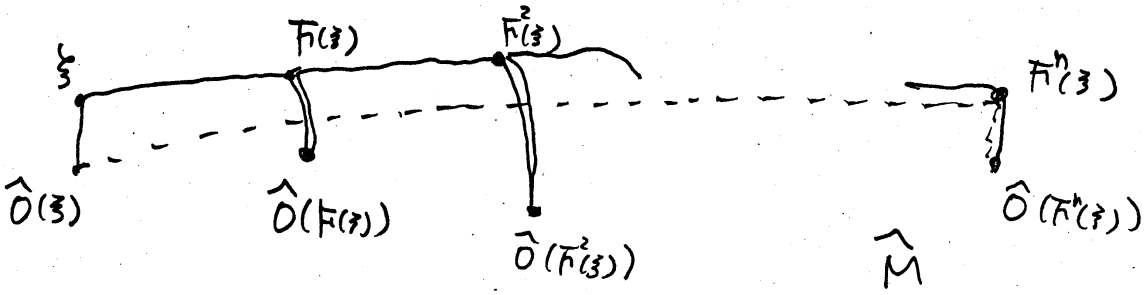
[Figure 2.1]



[Figure 2.2]



[Figure 5.1]