数理解析研究所講究録
1046 巻 1998 年 45-52

SP-property for a pair of C*-algebras

琉球大理　大坂　博幸 (Hiroyuki Osaka)

Abstract

Recall that C*-algebra A has the SP-property if every non-zero hereditary C*-subalgebra of A has a non-zero projection. Let $1 \in A \subset B$ be a pair of C*-algebras.

In this paper we investigate a sufficient condition for B to have the SP-property under A holds. As an application, we will present the cancellation property for crossed products of simple C*-algebras by discrete groups.

This paper basically comes from joint works with Ja A Jeong ([7][8]).

1 The SP-Property

In this section we present a sufficient condition for B to have the SP-property under A holds.

The argument in [11, Lemma 10] gives the following general result.

Theorem 1.1 *Let $1 \in A \subset B$ be a pair of C*-algebras. Suppose that A has the SP-property and there is a conditional expectaion E from B to A. If for any non-zero positive element x in B and an arbitrary positive number $\varepsilon > 0$ there is an element y in B such that*

$$\|y^*(x - E(x))y\| < \varepsilon,$$
$$\|y^* E(x) y\| \geq \|E(x)\| - \varepsilon$$

then B has the SP-property. Moreover, every non-zero herediatery C-subalgebra of B has a projection which is a equivalent to some projecion in A in the sence of Murray-von Neumann*

Next we consider the following stronger assumption on a conditional expectation E from B to A.

Definition 1.2 *Let $1 \in A \subset B$ be a pair of C*-algebras. A conditional expectation E from B to A is called outer if for any element $x \in B$ with*

$E(x) = 0$ and any non-zero hereditary C^*-subalgebra C of A

$$\inf\{\|cxc\|; c \in C^+, \|c\| = 1\} = 0.$$

The following result comes from the same argument as in [10, Lemma 3.2] and Theorem 1.1.

Corollary 1.3 *Let $1 \in A \subset B$ be a pair of C^*-algebras. Suppose that A has the SP-property and there is a conditional expectaion E from B to A. If E is outer, then B has the SP-property.*

We present some examples of a pair of C*-algebras with an outer conditional expectations.

Example 1.4 *Let ρ be a corner endmorphism on a unital C^*-algebra A, and let E be a canonical conditional expectation from a crossed product $A \times_\rho \mathbf{N}$ by ρ to A. Suppose that*

$$\tilde{\mathbf{T}}(\rho) = \{\lambda \in \mathbf{T} | \hat{\rho}(I) = I \quad for \quad \forall I \in Prime(A \times_\rho \mathbf{N})\} = \mathbf{T}.$$

Then, E is outer.

Proof. See Jeong-Kodaka-Osaka [6]. □

Example 1.5 (Kishimoto[10]) *Let G be a discrete group and let α be a representation of G by automorphisms of a simple unital C^*-algebra A. Suppose α is outer. Then, a canonical conditional expectation from a crossed product $A \times_\alpha G$ to A is outer.*

In the case of a crossed product of a simple unital C*-algebra with the SP-property by a finite group G, we can deduce the SP-property for the crossed product algebra $A \times_\alpha G$ by any automorphism α on A.

Theorem 1.6 ([7]) *Let A be a simple unital C^*-algebra with the SP-property, and let α be an action by a finite group G. Then, a crossed product algebra $A \times_\alpha G$ has the SP-property.*

2 C*-Index Theory

In this section, we brief the C*-index theory by Watatani ([16]).

Let $1 \in A \subseteq B$ be a pair of C*-algebras. By a conditional expectation $E : B \to A$ we mean a positive faithful linear map of norm one satisfying

$$E(aba') = aE(b)a', \quad a, a' \in A, b \in B.$$

A finite family $\{(u_1, v_1), \cdots, (u_n, v_n)\}$ in $B \times B$ is called a quasi-basis for E if

$$\sum_{i=1}^{n} u_i E(v_i b) = \sum_{i=1}^{n} E(b u_i) v_i = b \text{ for } b \in B.$$

We say that a conditional expectation E is of index-finite type if there exists a quasi-basis for E. In this case the index of E is defined by

$$\mathrm{Index}E = \sum_{i=1}^{n} u_i v_i.$$

Note that Index$E$ does not depend on the choice of a quasi-basis and every conditional expectation E of index-finite type on a C*-algebra has a quasi-basis of the form $\{(u_1, u_1^*), \cdots, (u_n, u_n^*)\}$ ([16, Lemma 2.1.6]). Moreover, Index$E$ is always contained in the center of B, so that it is a scalar whenever B has the trivial center, in particular when B is simple.

Let $E : B \to A$ be a conditional expectation. Then $B_A (= B)$ is a pre-Hilbert module over A with an A-valued inner product

$$\langle x, y \rangle = E(x^* y), \quad x, y \in B_A.$$

Let $\mathcal{E}$ be the completion of B_A with respect to the norm on B_A defined by

$$\|x\|_{B_A} = \|E(x^* x)\|_A^{1/2}, \quad x \in B_A.$$

Then $\mathcal{E}$ is a Hilbert C^*-module over A. Since E is faithful, the canonical map $B \to \mathcal{E}$ is injective. Let $L_A(\mathcal{E})$ be the set of all (right) A-module homomorphisms $T : \mathcal{E} \to \mathcal{E}$ with an adjoint A-module homomorphism $T^* : \mathcal{E} \to \mathcal{E}$ such that

$$\langle T\xi, \zeta \rangle = \langle \xi, T^* \zeta \rangle \quad \xi, \zeta \in \mathcal{E}.$$

Then $L_A(\mathcal{E})$ is a C^*-algebra with the operator norm $\|T\| = \sup\{\|T\xi\| : \|\xi\| = 1\}$. There is an injective $*$-homomorphism $\lambda : B \to L_A(\mathcal{E})$ defined by

$$\lambda(b)x = bx$$

for $x \in B_A$, $b \in B$, so that B can be viewed as a C^*-subalgebra of $L_A(\mathcal{E})$. Note that the map $e_A : B_A \to B_A$ defined by

$$e_A x = E(x), \quad x \in B_A$$

is bounded and thus it can be extended to a bounded linear operator, denoted by e_A again, on $\mathcal{E}$. Then $e_A \in L_A(\mathcal{E})$ and $e_A = e_A^2 = e_A^*$, that is, e_A is a projection in $L_A(\mathcal{E})$.

The *(reduced) C^*-basic construction* is a C^*-subalgebra of $L_A(\mathcal{E})$ defined to be

$$C^*(B, e_A) = \overline{span\{\lambda(x)e_A\lambda(y) \in L_A(\mathcal{E}) : x,\ y \in B\ \}}^{\|\cdot\|}$$

see [16, Definition 2.1.2].

Then,

Lemma 2.1 ([16, Lemma 2.1.4]) *(1)* $e_A C^*(B, e_A)e_A = \lambda(A)e_A$.

(2) $\psi : A \to e_A C^*(B, e_A)e_A$, $\psi(a) = \lambda(a)e_A$, *is a $*$-isomorphism (onto).*

Lemma 2.2 ([16, Lemma 2.1.5]) *The following are equivalent:*

(1) $E : B \to A$ *is of index-finite type*

(2) $C^*(B, e_A)$ *has an identity and there exists a number c with* $0 < c < 1$ *such that*

$$E(x^*x) \geq c(x^*x) \quad x \in B.$$

The above inequality was shown first in [13] by Pimsner and Popa for the conditional expectation $E_N : M \to N$ from a type II_1 factor M onto its subfactor N (c can be taken as the inverse of the Jones index $[M : N]$).

The conditional expectation $E_B : C^*(B, e_A) \to B$ defined by

$$E_B(\lambda(x)e_A\lambda(y)) = (\mathrm{Index}E)^{-1}xy, x, y \in B$$

is called *the dual conditional expectation* of $E : B \to A$. If E is of index-finite tyle, so is E_B with a quasi-basis $\{(w_i, w_i^*)\}$, where $w_i = \sqrt{\mathrm{Index}E}u_i e_A$, and $\{(u_i, u_i^*)\}$ are quasis-basis for E ([16, Proposition 2.3.4]).

3 The Stable Rank for C*-Crossed Products

Let α be an action of a finite group G on a unital C^*-algebra A by automorphisms, and let $A \times_\alpha G$ be its crossed product, that is, it is the universal C^*-algebra generated by a copy of A and implementing unitaries $\{u_g | g \in G\}$ with $\alpha_g(a) = u_g a u_g^*$ for every $g \in G$ and $a \in A$. Then there exists a canonical conditional expectation $E : A \times_\alpha G \to A$ defined by

$$E(\sum_g a_g u_g) = a_e,$$

for $a_g \in A$ and $g \in G$, where e denotes the identity of the group G.

Lemma 3.1 *Under this situation, the canonical conditional expectation E is of index-finite type with a quasi-basis $\{(u_g, u_g^*) : g \in G\}$ and* Index*(E)*$=\sum_{g\in G} u_g u_g^* = |G|$, *the order of* G.

Let $B = A \times_\alpha G$ and $n = |G|$. Then, a dual conditional expectation E_B is of index-finite type with a quasi-basis $\{(w_g, w_g^*) : g \in G\}$, where $w_g = \sqrt{n} u_g e_A$ (see section 2).

The following fact comes from a simple computation.

Lemma 3.2 ([8]) *The expression $x = \sum_{g\in G} w_g b_g$ ($b_g \in B$) is unique for each $x \in C^*(B, e_A)$.*

Let A be a unital C^*-algebra and $Lg_n(A)$ denote the n-tuples $(x_1, \ldots, x_n)$ in A^n which generate A as a left ideal. The *topological stable rank* of A ($sr(A)$) is defined to be the least integer for which $Lg_n(A)$ is dense in A^n. If there does not exist such an integer then $sr(A)$ is defined to be ∞. For a non unital C^*-algebra A we define $sr(A) = sr(\tilde{A})$ where $\tilde{A}$ is the unitization of A. See [15] for details about stable rank. It is not hard to see that for a unital C^*-algebra A $sr(A) = 1$ if and only if the set of invertible elements is dense in A.

Theorem 3.3 ([8]) *Let G be a finite group, and α be an action of G on a unital C^*-algebra A with $sr(A) = 1$. Then $sr(A \times_\alpha G) \leq |G|$.*

Proof. Let $n = |G|$, and $(b_{g_1}, \ldots, b_{g_n}) \in B^n$, where $B = A \times_\alpha G$. Put $y = \sum_{g\in G} w_g b_g \in C^*(B, e_A)$. Since $C^*(B, e_A)$ is strong Morita equivalent to A and $sr(A) = 1$, we have $sr(C^*(B, e_A)) = 1$ ([16, Proposition 1.3.4.]). Approximate y by invertible elements x in $C^*(B, e_A)$, and write $x = \sum_{g\in G} w_g c_g$, $c_g \in B$. Then by Lemma 3.2, $(c_{g_1}, \ldots, c_{g_n})$ is close to $(b_{g_1}, \ldots, b_{g_n})$. Note that

$$x^* x = n \sum_g c_g^* e_A c_g.$$

By Lemma 2.2

$$E_B(x^* x) \geq \frac{1}{n} x^* x, \quad x \in C^*(B, e_A).$$

Since $E_B(x^* x) = \sum_g c_g^* c_g$, it follows that

$$\sum_g c_g^* c_g \geq \frac{1}{n} x^* x$$

which is invertible in $C^*(B, e_A)$. Therefore $\sum_g c_g^* c_g$ is invertible in B, that is, $(c_{g_1}, \ldots, c_{g_n}) \in Lg_n(B)$. □

Remark 3.4 *If $sr(A) = m$ then it can be shown that $sr(A \times_\alpha G) \leq |G|m$ whenever A is a simple unital C^*-algebra. Indeed, it can come from the following two facts;* (i) *$C^*(B, e_A)$ is isomorphic to the matrix algebra $M_n(A)$ ([16]),* (ii) *$sr(M_n(A)) = \{\frac{sr(A)-1}{n}\} + 1$, where $\{t\}$ denotes the least integer which is greater than or equal to t ([15]).*

4 The Cancellation Property

A C^*-algebra A is said to have *cancellation of projections* if for any projections p, q, r in A with $p \perp r$, $q \perp r$, $p + r \sim q + r$, we have $p \sim q$. If $M_n(A)$ has cancellation of projections for each $n = 1, 2, \ldots$, then we simply say that A has *cancellation*. Note that every C^*-algebra with cancellation is stably finite, that is, every matrix algebra $M_n(A)$ with entries from A contains no infinite projections for $n = 1, 2, \ldots$. It can be shown that if A is a C^*-algebra with $sr(A) = 1$ then it has cancellation. In the previous section we proved that the stable rank of the C^*-crossed product $A \times_\alpha G$ is bounded by the order of the group G if sr(A)=1, and actually it seems that the crossed product has stable rank 1, and therefore it would be natural to ask if it has cancellation.

Theorem 4.1 ([2, Theorem 4.2.2]) *Let A be a simple unital C^*-algebra. Suppose A contains a sequence (p_k) of projections such that*

1. for each k there is a projection r_k such that $2p_{k+1} \oplus r_k$ is equivalent to a subprojection of $p_k \oplus r_k$,

2. there is a constant K such that $sr(p_k A p_k) \leq K$ for all k.

Then A has cancellation.

Theorem 4.2 ([8]) *Let A be a simple unital C^*-algebra with $sr(A) = 1$ and SP-property. If G is a finite group and α is an action of G on A then the crossed product $A \times_\alpha G$ has cancellation.*

Sketch of a proof.

We give a proof in the case that $A \times_\alpha G$ is simple.

Since the fixed point algebra A^α can be identified with a hereditary C^*-subalgebra of the crossed product it has the SP-property by Theorem 1.6. Thus there is a sequence of projections $\{p_k\} \in A^\alpha$ such that $2[p_{k+1}] \leq [p_k]$ by [9, Lemma 2.2], where $[p]$ denotes the equivalence class of p. Since

$p_k \in A^\alpha$, $p_k(A \times_\alpha G)p_k$ is isomorphic to $(p_kAp_k) \times_\alpha G$ for each $k \in N$. Note that each p_kAp_k has stable rank one. By Theorem 3.3 $sr(p_kAp_k \times_\alpha G) \leq |G|$. Therefore, the assertion follows from Theorem 4.1 ($K = |G|, r_k = 0$). □

Recall that a unital C^*-algebra A has *real rank zero*, $RR(A) = 0$, if the set of invertible self-adjoint elements is dense in A_{sa}. It is well known that $RR(A) = 0$ is equivalent to say that every non-zero hereditary C^*-subalgebra contains an approximate identity consisting of projections (HP) ([3]). From [2, Section 4] where the HP-property is studied for simple C^*-algebras we can deduce the following.

Corollary 4.3 ([8]) *Under the assumptions of the above theorem, if $RR(A \times_\alpha G) = 0$ then its stable rank is one.*

For crossed products by the integer group Z we have the following cancellation theorem:

Theorem 4.4 ([8]) *Let A be a simple unital C^*-algebra with $sr(A) = 1$ and SP-property. If α is an outer action of the integer group Z on A such that $\alpha_* = id$ on the K_0 group $K_0(A)$ of A then the crossed product $A \times_\alpha Z$ has cancellation.*

Example 4.5 *If A is a UHF algebra or an irrational rotation algebra then the identity map is the only possible homomorphism on its K_0 group. Therefore the theorem says that any crossed product $A \times_\alpha Z$ has cancellation.*

Corollary 4.6 ([8]) *Under the same assumption of Theorem 3.5 if $RR(A \times_\alpha Z) = 0$, then its stable rank is one.*

参考文献

[1] B. Blackadar, *A simple unital projectionless C^*-algebras*, J. Operator Theory 5(1981), 63 - 71.

[2] B. Blackadar, *Comparison Theory for simple C^*-algebras*, Operator algebras and Applications, LMS Lecture Notes, no. 135, Cambridge University Press, 1988.

[3] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. 99(1991), 131 - 149.

[4] L. G. Brown and G. K. Pedersen, *On the geometry of the unit ball of a C^*-algebra*, J. reine angew. Math. 469(1995), 113 - 147.

[5] J. Cuntz, *K-theory for certain C^*-algebras*, Annals of Math. 113(1981), 181 - 197.

[6] J. A Jeong, K. Kodaka and H. Osaka, *Purely infinite simple C^*-crossed products II*, Canad. Math. Bull. 39(1986), 203 - 210.

[7] J. A Jeong and H. Osaka, *Extremally rich C^*-crossed products and cancellation property*, to apper in J. Australian Math. Soc.

[8] J. A Jeong and H. Osaka, *Stable rank of crossed products by finite groups*, preprint.

[9] H. Lin and S. Zhang, *Certain simple C^*-algebras with non-zero real rank whose corona algebras have real rank zero*, Houston J. Math. 18(1992), 57 - 71.

[10] A. Kishimoto, *Outer automorphisms and reduced crossed products of simple C^*-algebras*, Commun. Math. Phys. 81(1981), 429 - 435.

[11] A. Kishimoto and A. Kumjian, *Crossed products of Cunts algebras by quasi-free automorphihsms*, Operator Algebras and their Applications, The Fields Institute Communications 13(1997), 173 - 192.

[12] M. Izumi, *Index theory of simple C^*-algebras*, Workshop "Subfactors and their applications", The Fields Institute , March 1995.

[13] M. Pimsner and S. Popa, *Entropy and index for subfactors*, Ann. Sci. Ecole Norm. Sup. (4) 19(1986), 57 - 106.

[14] M. A. Rieffel, *Actions of finite groups on C^*-algebras*, Math. Scand. 47(1980), 157 - 176.

[15] M. A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. 46(1983), 301 - 333.

[16] Y. Watatani, *Index for C^*-algebras*, Memories of the Amer. Math. Soc. 424(1990).