数理解析研究所講究録
1051 巻 1998 年 20-28

渦度のClebsch変数表示による減衰乱流の理論
A Theory on the Decaying Turbulence using Clebsch Variables

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Abstract

Two-dimensional turbulent flows are known to have organized structures. There are some theories to explain the organized structures (Robert and Sommeria [4], Joyce and Montgemery [3]). They employ a theory of equilibrium statistical mechanics.

The subject of this paper is the application of these theories to three-dimensional decaying turbulent flows. From the structure of Clebsch variables, the suggestion is given that the 'entropy ' S is pointwise defined and that it is a function of the norm of the vorticity field $\omega(x)$ at each point.

1 Introduction

One of the most striking features of two-dimensional turbulence is that it has coherent structures. Direct numerical simulations of the two-dimensional Navier-Stokes Equation show that a disordered initial state relax to a long-time state with coherent structures [2]. This feature has been explained by using theories of equilibrium statistical mechanics.

Unlike two-dimensional case, there are no global structures in three-dimensional Navier-Stokes turbulent flows as far as numerical simulations show. But it is known that the flows have structures which are local with respect to space and time. Regions in a turbulent flow with strong vorticity often have tube-like structures and are called vortex tubes. We will formally apply the method of equilibrium statistical mechanics to three-dimensional turbulent flows as in two-dimensional case and try to explain their local structures.

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2 Basic Equations

In this paper, we will work with the Navier-Stokes equations for a incompressible fluid in a domain M with dimension 2 or 3:

$$\begin{aligned}
&\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu \nabla^2 u \\
&\nabla \cdot u = 0 \\
&u \in \mathfrak{X}(\Omega), \quad \dim \Omega = 2, 3,
\end{aligned}$$

where u is the velocity field of the fluid. The boundary condition is

$$u \cdot n \text{ on } \partial\Omega, \quad n \text{ normal to } \partial\Omega,$$

when $\nu = 0$ (the Euler equation) and otherwise

$$u = 0 \quad \text{on} \quad \partial\Omega.$$

The equivalent equation for the vorticity field is

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = \nu \nabla^2 \omega.$$

If Ω is two-dimensional, the vorticity field is a scalar field and the third term in the left hand side of the equation vanishes. This is the significant difference between the two-dimensional flow and the three-dimensional flow. This equation for the vorticity field can be reformulated in terms of differential geometry:

$$\begin{aligned}
&\frac{\partial \omega}{\partial t} + L_u \omega = -\nu \Delta \omega \\
&d^* u^\flat = 0, \omega = du \\
&u^\flat \in \Lambda^1(\Omega), \omega \in \Lambda^2(\Omega)
\end{aligned}$$

$u^\flat$ is the 1-form correspond to the velocity field u and the vorticity field ω is a 2-form. L_u denotes the Lie derivative with respect to u. d^* is the adjoint operator of the exterior derivative d and $\Delta = dd^* + d^*d$ is the Laplace-de Rham operator. In absence of the viscosity ν, the time derivative d/dt and the Lie derivative L_u cancel each other out for the 2-form ω, which means that the vorticity 2-form ω is advected by the velocity field u. The 2-form is a one-dimensional vector (scalar) field when the domain M is two-dimensional, as is three-dimensional when M is three-dimensional. We introduce the vector potential $\psi \in \Lambda^2(M)$ which is defined by

$$d^*\psi = u, \Delta\psi = \omega.$$

The vector potential ψ is not defined uniquely and we may add an arbitrary 2-form ζ satisfying $d^*\zeta = 0(\text{rot}\zeta = 0)$. In two-dimensional case, ψ is a scalar field and often called the flow function.

3 Two-dimensional turbulent flow

This section provides a brief review of the two theories on the coherent structure in two-dimensional turbulent flow. Both of them employ the method of equilibrium statistical mechanics. The vorticity field is regarded as a macroscopic state in some way and the entropy S is defined. The organized structure appears as a state of maximal entropy, with the constraints $E = \text{const.}$, $Q_i = \text{const.} (i = 1, 2, \dots)$, where E is the energy of the system and Q_i are the other constants of the motion. Then, with Lagrange multipliers β, α_i, the maximal entropy state satisfies the equation for the variations:

$$\delta S = \beta \delta E + \sum_i \alpha_i \delta Q_i.$$

Joyce and Montgomery [3] approximated the Euler system by N^+, N^--point vorticities system. N^+ point vorticities have positive charge a and N^- point vorticities have negative charge $-a$. Each point vorticity moves along the flow induced by the other vorticities. The two-dimensional domain Ω is divided into M cells with same areas. The macroscopic state is determined by a set of numbers $\{(N_i^+, N_i^-) | \; i = 1 \dots M\}$, where N_i^+, N_i^- are numbers of positive and negative vorticities in i-th cell. The probability W of the state is given by

$$W = \left\{ N! \prod_{i=1}^{M} \frac{1}{M^{N_i^+} N_i^+!} \right\} \left\{ N! \prod_{i=1}^{M} \frac{1}{M^{N_i^-} N_i^-!} \right\}.$$

The entropy S is defined by

$$S = \log W.$$

The energy of the state is given by

$$E = \frac{1}{2} \sum_{i \neq j}^{M} a^2 (N_i^+ - N_i^-)(N_j^+ - N_j^-) G(r_i, r_j).$$

r_i denotes the position of the center of cell and $G(r_i, r_j) = -\{(\nabla^2)^{-1} \delta(r_i)\}(r_j)$ is the Green function. Condition of the extremum entropy state is

$$\delta S = \alpha^+ \delta N^+ + \alpha_i^- \delta N^- + \beta \delta E.$$

After taking a continuum limit ($M \to \infty$), one obtain the sinh-Poisson equation:

$$\omega = -\nabla^2 \psi = C a \sinh(a \beta \psi), \qquad C \quad \text{is a constant.} \tag{1}$$

In Robert and Sommeria [4], the initial vorticity field is approximated by the field made of n patches Ω_i of value a_i. Each patches Ω_i preserves its area $|\Omega_i|$and its vorticity a_i during the fluid motion but they in general become more and more intricate. Let $e_i(x)$ denote the probability of finding the value a_i at the point x, then the macroscopic state is defined by $e(x) = (e_1(x), \dots, e_n(x))$. The entropy S is defined by the Shannon classical entropy integrated over the domain Ω:

$$S = -\int_\Omega \sum_i e_i(x) \log e_i(x) dx.$$

The Energy E is determined from the macroscopic vorticity field

$$\omega(x) = \sum_i a_i e_i(x),$$

namely

$$E = \int_\Omega \psi \omega dx, \qquad where \quad \psi = -(\nabla^2)^{-1}\omega.$$

The other constraints are that the areas:

$$F_i = \int_\Omega e_i(x) dx = |\Omega_i|$$

are constant. Then the entropy extremum condition is the equation for the variations:

$$\delta S = \sum_{i=1}^{n-1} \alpha_i \delta F_i + \beta \delta dE.$$

The solution must satisfy the flowing equation:

$$\omega = -\nabla^2 \psi = -\frac{1}{\beta}\frac{d}{d\psi} \log Z(\psi), \tag{2}$$

$$\text{where} \qquad Z(\psi) = \sum_{i=1}^{n} \exp(-\alpha - \beta a_i \psi). \tag{3}$$

The general case of an initial vorticity field ω belonging to the space $L^\infty(\Omega)$ is also studied in [4].

The maximum entropy state is a solution of (1) or (2). The equations (1) and (2) are known to have solutions with global coherent structures [5]. Direct numerical simulations of the two-dimensional Navier-Stokes turbulences [2] have good agreements with the equations (1).

4 Clebsch variables

When the domain Ω is three dimensional, we cannot adopt the methods in section 3 straightforwardly. Since the vorticity 2-form ω is not a scalar field but a vector field, point vortices or vortex patches do not make sense any more. Advected by the velocity field u, the vorticity 2-form ω is 'stretched' and the value is changed. In this section, we introduce Clebsch variables as the substitute for the vorticity field. Clebsch variables are 2-scalar field which is advected by the velocity field u. Then we will try to apply the methods in section 3 using the Clebsch variables in section 5.

The classical Clebsch variables (λ, μ) are the 2-scalar field which describe the vorticity field:

$$\begin{aligned}
&(\lambda, \mu) : \Omega \to R^2, \\
&\omega = d\lambda \wedge d\mu, \\
&= \sum_{i,j} \frac{d\lambda}{dx_i} dx_i \wedge \frac{d\mu}{dx_j} dx_j \\
&= \sum_{i,j} \left(\frac{d\lambda}{dx_i}\frac{d\mu}{dx_j} - \frac{d\lambda}{dx_j}\frac{d\mu}{dx_i} \right) dx_i \wedge dx_j.
\end{aligned}$$

Clebsch variables are not determined uniquely from the vorticity 2-form ω and can be chosen to satisfy the following time evolution equation :

$$\frac{\partial}{\partial t}(\lambda, \mu) = -(u \cdot \nabla)(\lambda, \mu).$$

The equation means that each classical Clebsch variable is advected by the velocity field u. The vector potential ψ is written in terms of the classical Clebsch variables by

$$\psi = \Delta^{-1}(d\lambda \wedge d\mu).$$

Although the classical Clebsch variables satisfy the property above, there arise some problems to construct the theory. The first is that the classical Clebsch variables are not uniquely defined as mentioned above. Since the derivatives of classical Clebsch variables determine the vorticity 2-form ω, we can always add arbitrary constants to them. In fact, there is more arbitrariness. The vorticity field is invariant under the action on $R^2 = (\lambda, \mu)$ preserving the symplectic 2-form $\lambda \wedge \mu$. The second is that the classical Clebsch variables can only describe the vorticity fields which are helicity free:

$$H = \int_\Omega u \wedge \omega = 0.$$

Turbulent flows are generally has non-zero helicity and the classical Clebsch variables can not describe those flow.

In order to resolve second problem, we generalized the Clebsch variables to $2n$-scalar field:

$$\begin{aligned}
&(\lambda_i, \mu_i) : M \to R^{2n} \\
&\omega = \sum_i^n d\lambda_i \wedge d\mu_i.
\end{aligned}$$

They are the former classical Clebsch variables when $n = 1$. The flows described by the generalized Clebsch variables can have non-zero helicity, but it is not certain if there is a large number n enough to describe turbulent flows. This is an open question.

The first problem is more essential, which tells that the actual value of the Clebsch variables are of little importance and that only their derivative have meanings. We keep this fact in mind in constructing the theory for three-dimensional turbulence in the following section.

5 Three-dimensional Turbulent flow

In this section, an attempt is made to construct the theory of three-dimensional turbulent flows using the method in equilibrium statistical mechanics. We consider the force free case and suppose that the turbulent flow decays into a certain state. The state may well not have global coherent structures like in the two-dimensional flows.

For simplicity, let the domain Ω has no boundary and be T^3, that is a cube with the periodic boundary condition. First we assume that the Clebsch variables are classical:

$$(\lambda, \mu) : T^3 \to R^2,$$

then the inverse image of a point $(\lambda, \mu) \in R^2$ is a vortex line: a integral curve with orientation along the vorticity vector field. So classical Clebsch variables can be interpreted as the indices of vortex lines. The inverse image $B \subset T^3$ of a connected area $A \subset R^2$ is a collection of vortex lines. Let B' be a connected part of intersection of B with a plane, then

$$\int_{B'} |\omega_n| dx = |A|$$

where ω_n is the normal component of the vorticity vector ω and $|A|$ is the area of A. Let C be a small ball whose center is a point $x \in T^3$ and $D \in R^2$ be the image of C through Clebsch variables. We assume that all vortex lines intersect with the ball C at most once. Then the following equation is satisfied:

$$\frac{1}{2} \int_{\partial C} |\omega_n| = |D|,$$

since each vortex line intersects with ∂C twice.

Let J be a set of grid points in R^2:

$$J = \{(i\eta, j\eta) | i, j \in Z\},$$

where η is a grid size. Each point corresponds to a vortex line. The microscopic state is defined by the map $P_x : J \to \{-1, 0, 1\}$ and the direction $d_x \in S^2$ at each point x. We use the approximation that vortex lines penetrate the ball C in the direction d_x or $-d_x$. $P_x(\lambda, \mu) = 1, -1$ if the vortex line correspond to (λ, μ) penetrate x in the direction $d_x, -d_x$ respectively. $P_x(\lambda, \mu) = 0$ if the corresponding vortex line does not intersect with the ball C. The macroscopic vorticity vector ω at x is defined by

$$\omega(x) = \sum_{(\lambda,\mu)\in J} P_x(\lambda, \mu)\eta^2 d_x. \tag{4}$$

We will define the 'entropy' pointwise. The 'entropy' of a macroscopic state should be proportional to the log of the measure of corresponding microscopic states. It should not depend on the direction d_x, so that the 'entropy' should be a function of the norm of the vorticity vector in the approximation here. Let the function be f:

$$S(\omega = a) = f(|a|),$$
$$S = \int_{T^3} f(|\omega|)dx,$$

The energy of the state is determined by the macroscopic vorticity field ω:

$$E = \frac{1}{2}\int_{T^3} \omega \cdot \psi dx.$$

The above results may be extended to the generalized Clebsch variables, because the corresponding vorticity field is a n-fold superposition of the vorticity fields described by classical Clebsch variables. So we may remove the helicity free condition. Since vortices 'stretch', there is no constraint that corresponds to the invariance of vorticities a_i of vortex patches Ω_i. Then the equation of the variations for the state of entropy extremum becomes

$$\delta S = \beta\delta E, \tag{5}$$
$$\int_\Omega \frac{df}{d|\omega|}\frac{\omega}{|\omega|} \cdot \delta\omega dx = \beta \int_\Omega \psi \cdot \delta\omega dx,$$
$$\frac{df}{d|\omega|}\frac{\omega}{|\omega|} = \beta\psi + \zeta, \qquad d^*\zeta = 0(\mathrm{rot}\zeta = 0). \tag{6}$$

An arbitrary rotation free 2-form ζ appears in the equation since the variation $\delta\omega$ is divergence-free ($d\omega = 0$). Operating d^* (or rot) on (6), we obtain

$$d^*\left(\frac{df}{d|\omega|}\frac{\omega}{|\omega|}\right) = \beta d^*\psi \quad \text{or} \quad \mathrm{rot}\left(\frac{df}{d|\omega|}\frac{\omega}{|\omega|}\right) = \beta u. \tag{7}$$

6 Discussions

The state of 'entropy' extremum satisfy the relation (6) between the vector potential ψ and the vorticity field ω. The vector potential ψ is parallel to the vorticity field

ωat each point if we leave out the rotation free component ζ. If the domain Ω is two-dimensional, the vector potential ψ and the vorticity field ω are both scalar and parallel to each other. So (6) may suggest the two-dimensional structures in the three-dimensional turbulent flows.

A bottleneck in further study is the arbitrariness of the function $f(|\omega|)$. If we can determine the function f, then the equation (7) can be checked against the data of direct numerical simulations of Navier-Stokes turbulence. Recall that 'entropy' S is approximately proportional to the log of the probability density $P(|\omega|)$:

$$S(|\omega|) \sim \log P(|\omega|).$$

In (4), the macroscopic vorticity ω is defined as a summation. If $P_x(\lambda, \mu)$ are mutually independent as probability variables and the summation is over a large number, then the probability distribution is approximated by the normal distribution from the central limit theorem:

$$P(|\omega|) \propto \exp\left(-\frac{|\omega|^2}{2\sigma}\right)$$
$$f(|\omega|) = -\frac{|\omega|^2}{2\sigma}$$

In this case, the relation between the vector potential ψ and the vorticity ω become

$$\Delta\psi = \omega = -\beta\sigma\psi.$$

This means that the vector potential ψ is an eigenfunction of the Laplace-de Rham operator Δ, which is unrealistic. So the assumption of the mutually independence or the large number summation in $P_x(\lambda, \mu)$ is not appropriate. Indeed, the probability distribution of the vorticity in direct numerical simulations is not normal but rather has exponential tails [1]. The function f for the 'entropy ' corresponds to this exponential distribution may be of the form

$$f(|\omega|) = -\alpha\sqrt{|\omega|^2 + 1}.$$

Another possibility is to reproduce the two-dimensional result $|\omega| = \sinh(\beta|\psi|)$ by letting the function f be of the form

$$f(|\omega|) = -\omega \log\left(\sqrt{|\omega|^2 + 1} + \omega\right) + \sqrt{|\omega|^2 + 1}.$$

This ambiguity is the fault of the theory at present.

Another problem is that a solution of (7) is generally not a stationary solution of the Euler equation as in the two-dimensional case. This implies, in a sense, that the statistical condition and the dynamical condition are not compatible in the three-dimensional turbulence. Although (7) can not be satisfied at all points $x \in \Omega$ and all time t, the possibility remains that the considerably many points satisfy the relation (7).

We conclude this paper with giving the remark on the additional constraint. The helicity is a constant of the motion in the three-dimensional Euler equation. Perhaps the constraint that the helicity H is constant should be added in the equation (5), which gives

$$\delta S = \alpha\delta H + \beta\delta E,$$
$$\mathrm{rot}\left(\frac{df}{d|\omega|}\frac{\omega}{|\omega|}\right) = 2\alpha\omega + \beta u.$$

References

[1] Cao,N. ,Chen,S. ,Sreenivasaan,K.R. ,Properties of velocity circulation in three-dimensional turbulence, *Phys. Rev. Lett.* ,76 (1996),616-619.

[2] Montgomery,D. et al. , Relaxation in two dimensions and the "sinh-Poisson" equation, *Phys. Fluids A*, 4 No.1 (1992) 3-6.

[3] Joyce,G. and Montgomery,D. , Statistical mechanics of negative temperature states, *Phys. Fluids* , 17(1974), 1139.

[4] Robert,R. and Somméria,J. , Statistical equilibrium states for two-dimensional flows, *J. Fluid. Mech.* , 229, 291-310.

[5] Ting,A.C. ,Chen, H.H. and Lee, Y.C. , Exact solutions of a nonlinear boundary value problem: the vortices of the two-dimensional sinh-poisson equation, *Physica* 26D (1987), 37-66.