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Smoothing effect in Gevrey classes for Schrödinger equations

Kunihiko Kajitani
Institute of Mathematics
University of Tsukuba
305 Tsukuba Ibaraki Japan

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Introduction

We shall investigate Gevrey smoothing effects of the solutions to the Cauchy problem for Schrödinger type equations. Roughly speaking,we shall prove that if the initial data decay as $e^{-c<x>^{\kappa}}(0<\kappa\leq 1, c>0)$, then the solutions belong to Gevrey class $\gamma^{1/\kappa}$ with respect to the space variables. Let $T>0$. We consider the following Cauchy problem,

(1) $$\frac{\partial}{\partial t}u(t,x)-i\Delta u(t,x)-b(t,x,D)u(t,x)=0, t\in[-T,T], x\in R^{n},$$

(2) $$u(0,x)=u_{0}(x), x\in R^{n},$$

where

(3) $$b(t,x,D)u=\sum_{j=1}^{n}b_{j}(t,x)D_{j}u+b_{0}(t,x)u,,$$

and $D_{j}=-i\frac{\partial}{\partial x_{j}}$. We assume that the coefficients $b_{j}(t,x)$ satisfy

(4) $$|D_{x}^{\alpha}b_{j}(t,x)|\leq C_{b}(\rho_{b}<x>)^{-|\alpha|}|\alpha|!^{s},$$

for $(t,x)\in[-T,T]\times R^{n}, \alpha\in N^{n}$, where $<x>=(1+|x|^{2})^{1/2}$. Moreover we assume that there is $\kappa\in(0,1]$ such that

(5) $$\lim_{|x|\to\infty}Reb_{j}(t,x)<x>^{1-\kappa}=0, \text{uniformly in } t\in[-T,T].$$

For $\rho\geq 0$ let define a exponential operator $e^{\rho<D>^{\kappa}}$ as follows,

$$e^{\rho<D>^{\kappa}}u(x)=\int_{R^{n}}e^{ix\xi+\rho<\xi>^{\kappa}}\hat{u}(\xi)\bar{d}\xi$$

where $\hat{u}(\xi)$ stands for a Fourier transform of u and $\bar{d}\xi=(2\pi)^{-n}d\xi$. For $\varepsilon\in R$ denote $\phi_{\varepsilon}=x\xi-i\varepsilon x\xi<x>^{\sigma-1}<\xi>^{\delta-1}$, where $\sigma+\delta=\kappa$ and we define

$$I_{\phi_{\epsilon}}(x,D)u(x)=\int_{R^{n}}e^{i\phi_{\epsilon}(x,\xi)}\hat{u}(\xi)d\xi.$$

Then our main theorem follows.

Theorem. *Assume (4)-(5) are valid and there is $\varepsilon > 0$ such that $I_{\phi_\varepsilon} u_0 \in L^2(R^n)$. Then if $d\kappa \le 1$, there exists a solution of (1)-(2) satisfying that there are $C > 0, \rho > 0$ and $\delta > 0$ such that*

(6) $$|\partial_x^\alpha u(t,x)| \le C(\rho|t|)^{-|\alpha|}|\alpha|!^s e^{\delta<x>^\kappa},$$

for $(t,x) \in [-T,T]\backslash 0 \times R^n, \alpha \in N^n$.

Remark. (i) Kato T. and Yajima in [12] considered the smoothing effect phenomena . A. Jensen in [6] and Hayashi,Nakamitsu & Tsutsumi in [5] showed that if $<x>^k u_0(x) \in L^2(R^n)$, the solution u of (1)-(2) belongs to H^k_{loc} for $t \ne 0$, Hayashi & Saitoh in [4] proved that if $e^{\delta<x>^2}u_0$ $(\delta > 0)$ is in $L^2(R^n)$, the solution u is analytic in x for $t \ne 0$ and De Bouard, Hayashi & Kato in [1], Kato & Taniguti in [11] show that if u_0 satisfies $\|(x\cdot\nabla)^j u_0\| \le C^{j+1}j!^s$ for $j = 0, 1.2...$, then the solution belongs to Gevrey $\gamma^{s/2}$ with respect to x for $t \ne 0$. Theorem 1 is proved by Kajitani in [8] and [10], when $\sigma = \kappa = 1$.

1 Weighted Sobolev spaces

We introduce some Sobolev spaces with weights. Let ρ, δ be real numbers and $\kappa \in (0,1]$. Define

$$\hat{H}^\kappa_\delta = \{u \in L^2_{loc}(R^n); e^{\delta<x>^\kappa}u(x) \in L^2(R^n)\}.$$

For $\rho \ge 0$ let define

$$H^\kappa_\rho = \{u \in L^2(R^n); Fu(\xi) \in \hat{H}_\rho(R^n_\xi)\},$$

where Fu stands for the Fourier transform of u. For $\rho < 0$ we define H^κ_ρ as the dual space of $H^\kappa_{-\rho}$. Then the Fourier transform F becomes bijective from H^κ_ρ to $\hat{H}_{\rho^\kappa}$. We define the operator $e^{\rho<D>^\kappa}$ mapping continuously from $H^\kappa_{\rho_1}$ to $H^\kappa_{\rho_1-\rho}$ as follows;

$$e^{\rho<D>^\kappa}u(x) = F^{-1}(e^{\rho<\xi>^\kappa}Fu(\xi))(x),$$

for $u \in H^\kappa_{\rho_1}$ and $e^{\delta<x>^\kappa}$ maps continuously from $\hat{H}^\kappa_{\delta_1}$ to $\hat{H}^\kappa_{\delta_1-\delta}$. We define for $\delta \ge 0$ and $\rho \in R$

(1.1) $$H^\kappa_{\rho,\delta} = \{u \in H_\rho; e^{\rho<D>^\kappa}u \in \hat{H}^\kappa_\delta\}.$$

For $\delta < 0$ we define $H^\kappa_{\rho,\delta}$ as the dual space of $H^\kappa_{-\rho,-\delta}$. We note that $H^\kappa_{\rho,0} = H^\kappa_\rho, H^\kappa_{0,\delta} = \hat{H}^\kappa_\delta$ and $H^\kappa_{0,0} = L^2(R^n)$. Furthermore we define for $\rho \ge 0$ and $\delta \in R$

(1.2) $$\tilde{H}^\kappa_{\rho,\delta} = \{u \in \hat{H}^\kappa_\delta; e^{\delta<x>^\kappa}u \in H^\kappa_\rho\}$$

and for $\rho < 0$ define $\tilde{H}^\kappa_{\rho,\delta}$ as the dual spase of $\tilde{H}^\kappa_{-\rho,-\delta}$. Denote by H' the dual space of a topological space H. Then $H^{\kappa'}_{\rho,\delta} = H^\kappa_{-\rho,-\delta}$ and $\tilde{H}^{\kappa'}_{\rho,\delta} = \tilde{H}^\kappa_{-\rho,-\delta}$ hold for any ρ and $\delta \in R$. We shall prove $H^\kappa_{\rho,\delta} = \tilde{H}^\kappa_{\rho,\delta}$ later on (see Proposition 3.8).

Lemma 1.1. *Let $\rho, \delta \in R$. Then*

(i) $$H^\kappa_{\rho,\delta} = e^{-\rho<D>^\kappa}e^{-\delta<x>^\kappa}L^2 = e^{-\rho<D>^\kappa}\hat{H}^\kappa_\delta.$$

(ii) $$\tilde{H}^\kappa_{\rho,\delta} = e^{-\delta<x>^\kappa}e^{-\rho<D>^\kappa}L^2 = e^{-\delta<x>^\kappa}H^\kappa_\rho.$$

Lemma 1.2 *Let $1 > \rho > 0, \delta \in R$ and $u \in \tilde{H}^\kappa_{\rho,\delta}$. Then*

(1.6) $$|D^\alpha_x u(x)| \le C_n(1-\epsilon)^{-n/2}\|u\|_{\tilde{H}^\kappa_{\rho,\delta}}(\epsilon\rho)^{-|\alpha|}|\alpha|!e^{\delta<x>^\kappa}$$

for $x \in R^n, \alpha \in N^n$ and $0 < \epsilon < 1$.

We can prove these lemmas analogously to the case of $\kappa = 1$ which is proved in [10].

2 Almost analytic extension of symbols

Following Hörmander's notation we define the symbol classes of pseudo-differential operators. Let $m(x,\xi), \varphi(x,\xi), \psi(x,\xi)$ a weight and $g = \varphi^{-2}dx^2 + \psi^{-2}d\xi^2$ a Riemann metric. We denote by $S(m,g)$ the set of symbols $a(x,\xi)$ satisfying

$$|a^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha\beta} m(x,\xi)\psi^{-\alpha|}\theta^{-|\beta|},$$

for $(x,\xi) \in R^{2n}, \alpha, \beta \in N^n$., where $a^{(\alpha)}_{(\beta)} = \partial^\alpha_\xi D^\beta_x a$. Let $d \ge 1$. Moreover we call that a function $a(x,\xi) \in S(m,g)$ belongs to $\gamma^d S(m,g)$, if $a(x,\xi)$ satisfies that there are $C_a > 0, \rho_a > 0$ such that

$$|a^{(\alpha)}_{(\beta)}(x,\xi)| \le C_a \rho_a^{-|\alpha+\beta|} |\alpha+\beta|!^d \psi^{-|\beta|}\varphi^{-|\alpha|} \tag{2.1}$$

for $(x,\xi) \in R^{2n}, \alpha, \beta \in N^n$. We denote $g_0 = dx^2 + d\xi^2$ and $g_1 =< x >^{-2} dx^2 + < \xi >^{-2} d\xi^2$. We remark that the symbol class $\gamma^1 S(m, g_i)(i = 0, 1)$ is introduced in [10] when $d = 1$. Here we consider the case of $d > 1$.

Let $d > 1$ and $\chi(t) \in C_0^\infty((0,\infty))$ satsfying that $\chi(t) = 0, t \le 1/2, \chi(t) = 1, t \le 1$, and

$$|D^k_t \chi(t)| \le C_0 \rho_0^{-k} k!^d, \tag{2.2}$$

for $t \in R, k \in N$. Then for a weight $w(x,\xi) \in \gamma^d S(m, g_1)$ and a parameter $b > 0$ we can see easily that $\chi(bw(x,\xi)) \in \gamma^d S(1, g_1)$ satisfying

$$|D^\beta_x D^\alpha_\xi \chi(bw(x,\xi)))| \le C_1 \rho_1^{-|\alpha+\beta|} |\alpha+\beta|!^d < x >^{-|\beta|} < \xi >^{-|\alpha|}, \tag{2.3}$$

for $(x,\xi) \in R^{2n}, \alpha, \beta \in N^n, b \ge 1$.

Lemma 2.1. *Let $d \ge 1$ and $\{p_k(x,\xi)\}_{k=1}^\infty$ be a series of symbols satisfying*

$$|p^{(\alpha)}_{k(\beta)}(x,\xi)| \le m(x,\xi)(< x >< \xi >)^k \rho_p^{-|\alpha+\beta|-k} |\alpha+\beta|!^d k!^d \langle x\rangle^{-|\beta|} \langle \xi\rangle^{-|\alpha|}, \tag{2.4}$$

for $(x,\xi) \in R^{2n}, \alpha, \beta \in N^n$ and $k \ge 0$. Then there is $p(x,\xi) \in \gamma^{(d)} S(m, g_1)$ such that

$$p(x,\xi) - \sum_{k=0}^{N-1} p_k(x,\xi) \in \gamma^{(d)} S(m(\langle x\rangle\langle\xi\rangle\rho_p)^{-N} N!^d, g_1), \tag{2.5}$$

for any integer $N \ge 0$.

Proof This lemma is essentially a result of [2]. The case of $d = 1$ is explained in [10]. Here we prove the lemma in the case of $d > 1$. Let $b_k = \rho_p^{-1} k!^{\frac{d}{k}} M$ and $M \ge 2$.Define

$$p(x,\xi) = \sum_{k=0}^{\infty} p_k(x,\xi)\chi(b_k(\langle x\rangle\langle\xi\rangle)^{-1}), \tag{2.6}$$

Then we have

$$|p^{(\alpha)}_{(\beta)}(x,\xi)| = |\sum_k \sum_{\alpha',\beta'} \binom{\alpha}{\alpha'} \binom{\beta}{\beta-\beta'} p^{(\alpha')}_{k(\beta')} (\chi(b_k(\langle x\rangle\langle\xi\rangle)^{-1}))^{(\alpha=\alpha')}_{(\beta-\beta')}|$$

$$\le \sum_k \sum_{\alpha',\beta'} \binom{\alpha}{\alpha'} \binom{\beta}{\beta-\beta'} m(x,\xi)\rho_k^{-|\alpha'+\beta'|} |\alpha'+\beta'|!^d \langle x\rangle^{-|\beta|}\langle\xi\rangle^{-|\alpha|}$$

$$\times M^{-k} C_0 \rho_0^{-|\alpha-\alpha'+\beta-\beta'|} |\alpha-\alpha'+\beta=\beta'|!^d$$

$$\leq 2\frac{\rho_0}{\rho_0-\rho_p}m(x,\xi)\rho^{-|\alpha+\beta|}|\alpha+\beta|!^d\langle x\rangle^{-|\beta|}\langle\xi\rangle^{-|\alpha|},$$

for $(x,\xi)\in R^{2n}, \alpha,\beta\in N^n$. Here we used the following inequality

$$\sum_{\alpha'\leq\alpha}\binom{\alpha}{\alpha'}\rho_p^{-|\alpha'|}|\alpha'|!^d\rho_0^{-|\alpha-\alpha'|}|\alpha-\alpha'|!^d\leq\frac{\rho_0}{\rho_0-\rho_p}|\alpha|!^d, \tag{2.7}$$

for $\rho_0>\rho_p$. Moreover we can write

$$p(x,\xi)-\sum_{k=0}^{N-1}p_k(x,\xi)$$

$$=\sum_{k=N}^{\infty}p_k(x,\xi)\chi(b_k(\langle x\rangle\langle\xi\rangle)^{-1})+\sum_{k=0}^{N-1}p_k(x,\xi)(1-\chi(b_k(\langle x\rangle\langle\xi\rangle)^{-1})$$

$$=: I+II.$$

Noting that $\rho_p^{-k}k!^d(M\langle x\rangle\langle\xi\rangle)^{-N}\leq 1$ on $supp\chi(b_k(\langle x\rangle\langle\xi\rangle)^{-1})$for $k\geq N$ and $\rho_p^{-k}k!^d(M\langle x\rangle\langle\xi\rangle)^{-N}\geq 1/2$ on $supp(1-\chi(b_k(\langle x\rangle\langle\xi\rangle)^{-1}))$ for $k\leq N-1$ respectively, we can see that I and II belong to $\gamma^dS(m(\langle x\rangle\langle\xi\rangle\rho_p)^{-N}N!^d,g)$. Q.E.D.

Let $a(x,\xi)\in\gamma^d$(m,g$_1$), that is, $a(x,\xi)$ satisfies (2.1). Denote $b_\alpha(x)=B\rho_a^{-1}4^n\langle x\rangle^{-1}|\alpha|!^{\frac{d-1}{|\alpha|}}$ for $x\in R^n$. We define an almost analytic extension of $a(x,\xi)$ as follows,

$$a(x+iy,\xi+i\eta)=\sum_{\alpha,\beta}a^{(\alpha)}_{(\beta)}(x,\xi)(-y)^\beta(i\eta)^\alpha\chi(b_\beta(x)|y|)\chi(b_\alpha(\xi)|\eta|)(\alpha!\beta!)^{-1}, \tag{2.8}$$

for $x,y,\xi,\eta\in R^n$, where $a^{(\alpha)}_{(\beta)}(x,\xi)=\partial^\alpha_\xi(-i\partial_x)^\beta a(x,\xi)$. Then we can prove easily

Proposition 2.2 *Let $a(x,\xi)\in\gamma^dS(m,g_1)$. Then the function $a(x+iy,\xi+i\eta)$ defined by (2.8) satisfies the following properties.*

(i) $|D^\beta_x\partial^\alpha_\xi D^\gamma_y\partial^\delta_\eta a(x+iy,\xi+i\eta)|\leq Cm(x,\xi)(C\rho_a)^{-|\alpha+\beta+\gamma+\delta|}\langle x\rangle^{-|\beta|}\langle\xi\rangle^{-|\alpha|}\langle y\rangle^{-|\gamma|}\langle\eta\rangle^{-|\delta|}|\alpha+\beta+\gamma+\delta|!^d$.

(ii)
$$|(\partial_{x_j}+i\partial_{y_j})D^\beta_x\partial^\alpha_\xi D^\gamma_y\partial^\delta_\eta a(x+iy,\xi+i\eta)|$$
$$\leq Cm(x,\xi)(C\rho_a)^{-|\alpha+\beta+\gamma+\delta|}e^{-c_0(\frac{\langle x\rangle}{|y|})^{\frac{1}{d-1}}}\langle x\rangle^{-|\beta|}\langle\xi\rangle^{-|\alpha|}\langle y\rangle^{-|\gamma|}<\eta>^{-|\delta|}|\alpha+\beta+\gamma+\delta|!^d.$$

(iii)
$$|(\partial_{\xi_j}+i\partial_{\eta_j})D^\beta_x\partial^\alpha_\xi D^\gamma_y\partial^\delta_\eta a(x+iy,\xi+i\eta)|$$
$$\leq Cm(x,\xi)(C\rho_a)^{-|\alpha+\beta+\gamma+\delta|}e^{-c_0(\frac{\langle\xi\rangle}{|\eta|})^{\frac{1}{d-1}}}\langle x\rangle^{-|\beta|}\langle\xi\rangle^{-\alpha|}<y>^{-|\gamma|}\langle\eta\rangle^{-|\delta|}|\alpha+\beta+\gamma+\delta|!^d.$$

For simplicity denote $\gamma^{1/\kappa}S(e^{\delta\langle x\rangle^\kappa+\rho\langle\xi\rangle^\kappa},g_0)$ by $A^\kappa_{\rho,\delta}$, where $g_0=dx^2+d\xi^2$. For $a_i\in A^\kappa_{\rho_i,\delta_i}(i=1,2)$ we define a product of $a_1$and a_2 as follows,

$$(a_1\circ a_2)(x,\xi)=os-\int\int_{R^{2n}}e^{-iy\eta}a_1(x,\xi+\eta)a_2(x+y,\xi)dy\bar{d}\eta, \tag{2.9}$$

$$=\lim_{\epsilon\to0}\int\int_{R^{2n}}e^{-iy\eta-\epsilon(|y|^2+|\eta|^2)}a_1(x,\xi+\eta)a_2(x+y,\xi)dy\bar{d}\eta,$$

where $\bar{d}\eta=(2\pi)^{-n}d\eta$. Then we can show the proposition below.

Proposition 2.3. *(i) Let* $\kappa \leq 1$ *and* $a_i \in A^{\kappa}_{\rho_i,\delta_i}, i = 1,2$. *Then there is* $\epsilon_0 > 0$ *such that if* $|\rho_1|, |\delta_2| \leq \epsilon_0$, *the product* $a_1 \circ a_2$ *belongs to* $A^{\kappa}_{\rho_1+\rho_2,\delta_1+\delta_2}$.
(ii) Let $a_i \in A^{\kappa}_{\rho_i,\delta_i}, i = 1,2,3$. *Then if* $|\rho_i|(i = 1,2), |\delta_i|(i = 2,3) \leq \epsilon_0/2$, *we have* $(a_1 \circ a_2) \circ a_3 = a_1 \circ (a_2 \circ a_3)$.

Proposition 2.4 *Let* $d \geq 1$ *and* $a_i \in \gamma^d S(\langle x\rangle^{m_i}\langle\xi\rangle^{\ell_i}, g_1), i = 1,2$. *Then* $a_1 \circ a_2$ *belongs to* $S(\langle x\rangle^{m_1+m_2}\langle\xi\rangle^{\ell_1+\ell_2}, g_1)$ *and moreover we can decompose*

$$a_1 \circ a_2(x,\xi) = p(x,\xi) + r(x,\xi), \tag{2.10}$$

where $p(x,\xi) \in \gamma^d S(\langle x\rangle^{m_1+m_2}\langle\xi\rangle^{\ell_1+\ell_2}, g_1)$ *satisfies that there are* $C > 0$ *and* $\varepsilon_0 >$ *such that*

$$p(x,\xi) - \sum_{|\gamma|<N} \gamma!^{-1} a_1^{(\gamma)}(x,\xi) a_{2(\gamma)}(x,\xi) \in \gamma^d S(C^{1+N} N! \langle x\rangle^{m_1+m_2-N}\langle\xi\rangle^{\ell_1+\ell_2-N}, g), \tag{2.11}$$

for any non negative integer N, *and* $r(x,\xi)$ *belongs to* $A^{1/d}_{-\varepsilon_0,-\varepsilon_0}$.

3 Pseudo-differential operators

Let $< \kappa \leq 1$. Now we want to define a pseudo differential operator $a(x,D)$ for a symbol $a(x,\xi) \in A^{\kappa}_{\rho,\delta}$, which operates from $H^{\kappa}_{\rho',\delta'}$ to $H^{\kappa}_{\rho'-\rho,\delta'-\delta}$. When ρ and δ are non positive, since $A^{\kappa}_{\rho.\delta}$ is contained in the usual symbol class $S^0_{0,0}$(denote by $S^m_{\rho,\delta}$ the Hörmander's class), we can define

$$a(x,D)u(x) = \int e^{ix\xi} a(x,\xi)\hat{u}(\xi) \bar{d}\xi, \tag{3.1}$$

for $u \in L^2(R^n)$ and for $a \in A^{\kappa}_{\rho,\delta}$. Moreover for $a_i \in A^{\kappa}_{\rho_i,\delta_i}, i = 1,2$ (ρ_i and δ_i non positive) the symbol $\sigma(a_1(x,D)a_2(x,D))(x,\xi)$ of the product of $a_1(x,D)$ and $a_2(x,D)$ can be written as follows,

$$\sigma(a_1(x,D)a_2(x,D))(x,\xi) = (a_1 \circ a_2)(x,\xi) \tag{3.2}$$

and we have

$$a_1(x,D)(a_2(x,D)u)(x) = (a_1 \circ a_2)(x,D)u(x) \tag{3.3}$$

for $u \in L^2(R^n)$, where $a_1 \circ a_2$ is defined by (2.9). Next we shall show that (3.2) and (3.3) are valid for any ρ_i, δ_i. To do so, we need some preparations. Let $a \in A^{\kappa}_{\rho,\delta}$ and $u \in H^{\kappa}_{\rho}$. Then we can define $a(x,D)u(x)$ which belongs to $\hat{H}^{\kappa}_{\delta}$. In fact, put $\tilde{a}(z,\eta) = e^{-\delta\langle x\rangle^{\kappa}+\rho\langle\xi\rangle^{\kappa}} a(x,\xi)$. Then $\tilde{a}(z,\xi) \in A^{\kappa}_{0,0}$. Noting that $e^{\rho\langle\xi\rangle^{\kappa}}\hat{u}(\xi)$ we can define

$$e^{-\delta\langle x\rangle^{\kappa}} a(x,D)u(x) = \int e^{ix\xi}\tilde{a}(x,\xi) e^{\rho\langle\xi\rangle^{\kappa}}\hat{u}(\xi)\bar{d}\xi, \tag{3.4}$$

which is in L^2, that is, $a(x,D)u \in \hat{H}^{\kappa}_{\delta}$. For $\epsilon > 0$ we denote $\chi_\epsilon(x) = e^{-\epsilon\langle x\rangle^2}$ and $\chi_\epsilon(D) = e^{-\epsilon\langle D\rangle^2}$.

Lemma 3.1. *(i) Let* $a \in A^{\kappa}_{\rho,\delta}(\rho,\delta \in R), u \in L^2$ *and* $\epsilon_0 > 0$ *chosen in Proposition 2.3. Then for any* $\epsilon > 0$

$$a(x,D)(\chi_\epsilon(D)\chi_\epsilon(x)u)(x) = (a(x,\xi)\chi_\epsilon(\xi)) \circ \chi_\epsilon(x))(x,D)u(x) \tag{3.5}$$

and

$$(a\chi_\epsilon(\xi)) \circ \chi_\epsilon(x) \in A^{\kappa}_{\rho-\epsilon_0,\delta-\epsilon_0}. \tag{3.6}$$

(ii) Let $u \in L^2$ and $\epsilon_0 > 0$ chosen in Proposition 2.3. Then there is $\epsilon_1 > 0$ such that for any $\varepsilon > 0$

$$e^{-\rho<D>^\kappa}(e^{-\delta<x>^\kappa}\chi_\epsilon(x)\chi_\epsilon(D)u)(x) = a_\epsilon(x,D)u(x), \tag{3.7}$$

where

$$a_\epsilon(x,\xi) = e^{-\rho<\xi>^\kappa} \circ (e^{-\delta<x>^\kappa}\chi_\epsilon(x)\chi_\epsilon(\xi)) \in A^\kappa_{-\rho-\epsilon_0,-\delta-\epsilon_0}, \tag{3.8}$$

for $|\rho| \le \epsilon_0$ and $\rho < \epsilon_1$. We can prove the following lemma by use of Lemma 3.1.

Lemma 3.2. *Let $u \in H^\kappa_{\rho,\delta}$ and $|\rho|, |\delta| \le \epsilon_0/2$ (ϵ_0 is given in Proposition 2.3). Then for any $\epsilon > 0$ there is $u_\epsilon \in H^\kappa_{\epsilon_0/2,\epsilon_0/2}$ such that*

$$\|u - u_\epsilon\|_{H^\kappa_{\rho,\delta}} < \epsilon. \tag{3.9}$$

Lemma 3.3. *Let $a \in A^\kappa_{\rho,\delta}, 0 < \epsilon_0', \tilde{\epsilon}_0 \le \epsilon_0$($\epsilon_0$ is given in Proposition 2.3) and $u \in H^\kappa_{\epsilon_0',\tilde{\epsilon}_0}$. Then there is $\epsilon_2 > 0$ independent of a, ρ and δ such that $a(x,D)u(x)$belongs to $H^\kappa_{\epsilon_0'-\rho,\tilde{\epsilon}_0-\delta}$ if $0 < \epsilon_0' - \rho \le min\{\epsilon_0, \epsilon_2$ $rho_a\}$ and $0 < \tilde{\epsilon}_0 - \delta \le \epsilon_0$.*

Lemma 3.4. *Let $a_i \in A^\kappa_{\rho_i,\delta_i}(i = 1,2)$ and $u \in H^\kappa_{\epsilon_0',\tilde{\epsilon}_0}(\epsilon_0', \tilde{\epsilon}_0 > 0)$. Then if $|\rho_1| \le \epsilon_0, |\delta_2| \le \epsilon_0, 0 < \epsilon_0' - \rho_2 \le \epsilon_0 min\{1, \rho_{a_2}\}, 0 < \tilde{\epsilon}_0 - \delta_2 \le \epsilon_0, 0 < \epsilon_0' - \rho_2 - \rho_1 \le \epsilon_0 min\{1, \rho_{a_1}\}$ and $0 < \tilde{\epsilon}_0 - \delta_2 - \delta_1 \le \epsilon_0$ are valid (ϵ_0 is given in Proposition 2.3), we have*

$$a_1(x,D)(a_2(x,D)u)(x) = (a_1 \circ a_2)(x,D)u(x), \tag{3.10}$$

which is in $H^\kappa_{\epsilon_0'-\rho_1-\rho_2,\tilde{\epsilon}_0-\delta_1-\delta_2}$.

Let $a \in A^\kappa_{\rho,\delta}(|\rho|, |\delta| \le \epsilon_0/4), u \in H^\kappa_{\epsilon_0/2,\epsilon_0/2}$ and $|\rho_1|, |\delta_1| < \epsilon_0/4$. Put $w = e^{\delta_1<x>^\kappa}e^{\rho_1<D>^\kappa}u$, which is in $H^\kappa_{\epsilon_0/2-\rho_1,\epsilon_0/2-\delta_1}$. Since we can write $u = e^{-\rho_1<D>^\kappa}(e^{-\delta_1<x>^\kappa}w)$, we get by use of Lemma 3.4 with $\epsilon_0' = \epsilon_0/2 - \rho_1, \tilde{\epsilon}_0 = \epsilon_0/2 - \delta_1, a_1 = a(x,\xi)e^{-\rho_1<\xi>^\kappa}$ and $a_2 = e^{-\delta_1<x>^\kappa \le k}, \epsilon_{a_2} = 1$,

$$a(x,D)u(x) = a(x,D)(e^{-\rho_1<D>^\kappa}(e^{-\delta_1<x>^\kappa}w) = ((a(x,\xi)e^{-\rho_1<\xi>^\kappa}) \circ e^{-\delta_1<x>^\kappa})(x,D)w(x).$$

Noting that $a_1(x,\xi) := (e^{(\delta_1-\delta)<x>\le k}e^{(\rho_1-\rho)<\xi>^\kappa}) \circ (a(x,\xi)e^{-\rho_1<\xi>^\kappa}) \circ e^{-\delta_1<x>^\kappa} \in A^\kappa_{0,0}$, we obtain

$$\|au\|_{H^\kappa_{\rho_1-\rho,\delta_1-\delta}} = \|a_1(x,D)w\|_{L^2} \le C\|w\|_{L^2} = C\|u\|_{H^\kappa_{\rho_1,\delta_1}} \tag{3.11}$$

for any $u \in H^\kappa_{\epsilon_0/2,\epsilon_0/2}$. Since $H^\kappa_{\epsilon_0/2,\epsilon_0/2}$ is dense in $H^\kappa_{\rho_1,\delta_1}$ from Lemma 3.2, we get the following theorem.

Theorem 3.5 *Let $a \in A^\kappa_{\rho,\delta}(|\rho|, |\delta| \le \epsilon_0/4), |\rho_1|, |\delta_1| < \epsilon_0/4$, where ϵ_0 are given in Proposition 2.3. Then $a(x,D)$ maps from $H^\kappa_{\rho_1,\delta_1}$ to $H^\kappa_{\rho_1-\rho,\delta_1-\delta}$ and satisfies the following inequality*

$$\|au\|_{H^\kappa_{\rho_1-\rho,\delta_1-\delta}} \le C\|u\|_{H^\kappa_{\rho_1,\delta_1}} \tag{3.12}$$

for any $u \in H^\kappa_{\rho_1,\delta_1}$.

For $a \in A^\kappa_{\rho,\delta}$, we difine

$$a^t(x,\xi) = os - \int\int e^{iy\eta}a(x+y,\xi+\eta)dyd\bar{\eta}, \tag{3.13}$$

and $a^*(x,\xi) = a^t(\bar{x},\xi)$. Then we can prove the following lemma, by the same way as that of the proof (i) of Proposition 2.3.

Lemma 3.6. *Let $a \in A^{\kappa}_{\rho,\delta}$ and $|\rho|, |\delta| \le \epsilon_0$. Then $a^t(x,\xi)$ defined in (2.29) belongs to $A^{\kappa}_{\rho,\delta}$. Moreover it holds*

$$(3.14),\qquad (a^t(x,D)u,\varphi)_{L^2} = (u, a(\bar{x},D)\varphi)_{L^2},$$

$$(a^*(x,D)u,\varphi)_{L^2} = (u, a(x,D)\varphi)_{L^2},$$

for any $u, \varphi \in H^{\kappa}_{\epsilon_0}$.

The relation (3.14) and the inequality (3.12) yield

$$|(a^t u,\varphi)| \le \|u\|_{H^{\kappa}_{\rho-\rho_1,\delta-\delta_1}} \|\bar{a}\varphi\|_{H^{\kappa}_{\rho_1-\rho,\delta_1-\delta}} \le C\|u\|_{H^{\kappa}_{\rho-\rho_1,\delta-\delta_1}} \|\varphi\|_{H^{\kappa}_{\rho_1,\delta_1}},$$

if $|\rho|, |\delta| \le \epsilon_0/4$ and $|\rho_1|, |\delta_1| < \epsilon_0/4$. Therefore taking account that $H^{\kappa}_{\epsilon_0/2,\epsilon_0/2}$ is dense in $H^{\kappa}_{\rho_1,\delta_1}$, we get from (3.14)

$$(3.15)\qquad \|a^t u\|_{H^{\kappa}_{-\rho_1,-\delta_1}} \le C\|u\|_{H^{\kappa}_{\rho-\rho_1,\delta-\delta_1}},$$

for any $u \in H^{\kappa}_{\rho_1,\delta_1}$. Thus we get the following proposition.

Propostion 3.7. *Let $a \in A^{\kappa}_{\rho,\delta}$ and $|\rho|, |\delta| \le \epsilon_0/4$ and $|\rho_1|, |\delta_1| < \epsilon_0/4$. Then the pseudodifferential operators $a^t(x,D)$ and $a^*(x,D)$ satisfy (3.15).*

Noting that $(e^{\delta<x>^{\kappa}} e^{\rho<D>^{\kappa}})^t = e^{\rho<D>^{\kappa}} e^{\delta<x>^{\kappa}}$, we have for $u \in H^{\kappa}_{\rho,\delta}$

$$e^{\rho<D>^{\kappa}} e^{\delta<x>^{\kappa}} u(x) = (e^{\delta<x>^{\kappa}} e^{\rho<D>^{\kappa}})^t (e^{-\rho<D>^{\kappa}} e^{-\delta<x>^{\kappa}} e^{\delta<x>^{\kappa}} e^{\rho<D>^{\kappa}} u)(x)$$
$$= (e^{\delta<x>^{\kappa}} e^{\rho<D>^{\kappa}})^t \circ (e^{-\delta<x>^{\kappa}} e^{-\rho<D>^{\kappa}})^t e^{\delta<x>^{\kappa}} e^{\rho<D>^{\kappa}} u(x).$$

Moreover we can see from Proposition 2.3 and Lemma 2.9 that$(e^{\delta<x>^{\kappa}} e^{\rho<\xi>^{\kappa}})^t \circ (e^{-\delta<x>^{\kappa}} e^{-\rho<\xi>^{\kappa}})^t$ is in $A^{\kappa}_{0,0}$. Hence we obtain the fact below.

Proposition 3.8. *Let $|\rho|, |\delta| \le \epsilon_0/4$. Then u belongs to $H^{\kappa}_{\rho,\delta}$ if and only if $u \in \tilde{H}^{\kappa}_{\rho,\delta}$.*

The following result on the multiple symbols of pseudodifferential operators is a special case of Lemma 2.2 of Chapter 7 in Kumanogo's book [12].

Lemma 3.9. *Let $r_j(x,\zeta) \in A^{\kappa}_{0,0} (j = 1,2,...,v)$ and put*

$$q_v(x,D) = r_1(x,D) r_2(x,D) \cdots r_v(x,D).$$

Then the symbol $q_v(x,\zeta)$ belongs to $A^{\kappa}_{0,0}$ and satisfies

$$(3.16)\qquad |q^{(\alpha)}_{v(\beta)}(x,\zeta)| \le C^v \prod_{j=1}^{v} C_{r_j} \bar{\varepsilon}_v^{-|\alpha+\beta|} |\alpha+\beta|!,$$

for $(x,\zeta) \in R^{2n}$, $\alpha, \beta \in N^n$, where C is independent of v and $\bar{\varepsilon}_v = min\{\varepsilon_{r_j}/4\}$.

We can prove easily the following lemma as a corollary of Lemma 3.9, by using the Neumann series method.

Lemma 3.10. *Let $r(x,\xi)$ be in $A^{\kappa}_{0,0}$. If $C_r > 0$ is sufficiently small, then there is the inverse $(I + r(x,D))^{-1}$ which is a pseudodifferential operator with its symbol contained in $A^{\kappa}_{0,0}$.*

Lemma 3.11. *Let $j(x,\xi) \in \gamma^d S(\varepsilon_1, g_1)$. Then if $\varepsilon_1 > 0$ is small enough, there are $k_1(x,\xi) \in \gamma^d S(\varepsilon_1 < x >^{-1} < \xi >^{-1}, g_1)$, $\varepsilon_0 > 0$ independent of ε_1 and $r_\infty(x,\xi) \in A^{1/d}_{-\varepsilon_0,-\varepsilon_0}$ such that $(I + j(x,D))^{-1} = k(x,D) + k_1(x,D) + r_\infty(x,D)$, where $k(x,\xi) = (1 + j(x,\xi))^{-1}$.*

4 Fourier Integral Operators

For $\vartheta \in AS(\rho_\vartheta < \xi > + \delta_\vartheta < x >, g)(\rho_\vartheta, \delta_\vartheta \geq 0)$,where $d\kappa \leq 1$, we denote

$$\phi(x,\xi) = x\xi - i\vartheta(x,\xi).$$

For $a \in A^{\kappa}_{0,0}$ we define a Fourier integral operator with a phase function $\phi(x,\xi)$ as follows,

$$a_\phi(x,D)u(x) = \int_{R^n} e^{i\phi(x,\xi)} a(x,\xi)\hat{u}(\xi)\bar{d}\xi, \tag{4.1}$$

for $u \in H_{\varepsilon_0,\varepsilon_0}$. Putting $p(x,\xi) = a(x,\xi)e^{\vartheta(x,\xi)}$, we can see $p(x,\xi) \in A^{\kappa}_{\rho_\vartheta,\delta_\vartheta}$. Therefore we can regard $a_\phi(x,D)$ as a pseudo differential operator with its symbol $p = ae^{\vartheta}$ defined in §2 and consequently it follows from Theorem 3.5 that $a_\phi(x,D)$ acts continuously from $H^{\kappa}_{\rho,\delta}$ to $H^{\kappa}_{\rho-\rho_\vartheta,\delta-\delta_\vartheta}$. However in order to construct the inverse operator of $p(x,D)$ it is better to regard $p(x,D)$ as a Fourier integral operator. In paticular for $a = 1$ we denote

$$I_\phi(x,D)u(x) = \int e^{i\phi(x,\xi)}\hat{u}(\xi)\bar{d}\xi, \tag{4.2}$$

$$I^R_\phi(x,D)v(x) = \int e^{ix\xi}\bar{d}\xi \int e^{i\phi(y,\xi)}v(y)dy. \tag{4.3}$$

Theorem 4.1. *Let $a \in \gamma^d S(\langle x\rangle^m \langle\xi\rangle^\ell, g_1)$, $\vartheta \in \gamma^d S(\rho_\vartheta\langle\xi\rangle^\kappa + \delta_\vartheta\langle x\rangle^\kappa, g_1)$ and $\phi = x\xi - i\vartheta(x,\xi)$. Assume $d\kappa \leq 1$. Then if $\rho_\vartheta, \delta_\vartheta$ are sufficiently small, $\tilde{a}(x,D) = I_\phi(x,D)a(x,D)I_\phi^{-1}$ and $\tilde{a}'(x,D) = I_\phi(x,D)^{-1}a(x,D)I_\phi(x,D)$ are pseudodifferential operators of which symbols are given by*

$$\tilde{a}(x,\xi) = p(x,\xi) + r(x,\xi), \tag{4.4}$$

$$a'(x,\xi) = p'(x,\xi) + r'(x,\xi), \tag{4.5}$$

where

$$p(x,\xi) - a(x - i\nabla_\xi\vartheta(x,\Phi), \xi + i\nabla_x\vartheta(x,\Phi)) \in \gamma^d S(< x >^{m-1} < \xi >^{\ell-1}, g_1), \tag{4.6}$$

$$\tilde{p}'(x,\xi) - a(x + i\nabla_\xi\vartheta(\Phi',\xi), \xi - i\nabla_x\vartheta(\Phi',\xi)) \in \gamma^d S(< x >^{m-1} < \xi >^{\ell-1}, g_1), \tag{4.7}$$

where $\Phi = \Phi(x,x,\xi)$ and $\Phi' = \Phi'(x,\xi,\xi)$ are given by (4.6) and (4.19) respectively and r, r' belong to $A^{\kappa}_{-\varepsilon_0,-\varepsilon_0}$ for an $\varepsilon_0 > 0$ independent of ρ_ϑ.

This theorem is proved in [10] in the case of $d = \kappa = 1$. We can prove it similar way as that of [10].

Next we consider a phase function $\vartheta \in \gamma^d S(\langle x\rangle^\sigma\langle\xi\rangle^\delta, g_1)$. When $\sigma + \delta = \kappa = 1/d < 1$ or $\sigma + \delta = 1$ and $d = min(\delta^{-1,\sigma^{-1}})$, Theorem 4.1 holds aiso, that is, we can prove Theorem 4.6 below. So far we consider only $d, \sigma, \delta, \kappa$ above. We note that $d > 1$.

Lemma 4.2. *Let* $a(x,\xi) \in \gamma^d S(<x>^m<\xi>^\ell, g_1)$ *and* $\vartheta \in \gamma^d S(\rho_\vartheta <\xi>^\delta<x>^\sigma, g_1)(\rho_\vartheta \geq 0)$. *Put* $\phi = x\xi - i\vartheta(x,\xi)$ *and* $\tilde{a}(x,D) = a_\phi(x,D)I^R_{-\phi}(x,D)$. *If* ρ_ϑ *is sufficiently small, then* $\tilde{a}(x,\xi)$ *belongs to* $S(<x>^m<\xi>^\ell, g)$ *and moreover satisfies*

$$\tilde{a}(x,\xi) = \tilde{p}(x,\xi) + r(x,\xi), \tag{4.8}$$

for $x, \xi \in R^n$, *and*

$$\tilde{p}(x,\xi) - \sum_{|\gamma|<N} \gamma!^{-1} D_y^\gamma \partial_\eta^\gamma \{a(x,\Phi(x,y,\eta))J(x,y,\eta)\}_{y=x,\eta=\xi}$$

$$\in \gamma^d S(C^{1+N}N!^d <x>^{m-N}<\xi>^{\ell-N}, g_1) \tag{4.9}$$

for any N, *where* $\Phi(x,y,\xi)$ *is a solution of the following equation,*

$$\Phi(x,y,\xi) - i\tilde{\nabla}_x\vartheta(x,y,\Phi(x,y,\xi)) = \xi, \tag{4.10}$$

$$\tilde{\nabla}_x\vartheta(x,y,\xi) = \int_0^1 \nabla\vartheta(y + t(x-y),\xi)dt, \tag{4.11}$$

$J(x,y,\xi) = \frac{D\Phi(x,y,\xi)}{D\xi}$ *is the Jacobian of* $\Phi, r(x,\xi) \in A^{1/d}_{-\varepsilon_0,-\varepsilon_0}$, *and* $C > 0, \varepsilon_0 > 0$ *are independent of* ρ_ϑ.

Lemma 4.3. *Let* $a(x,\xi)$ *and* ϑ *be satisfied with the same condition as one of Lemma 4.2. For* $\phi = x\xi - i\vartheta(x,\xi)$ *put* $a'(x,\xi) = I^R_{-\phi}(x,D)a_\phi(x,D)$. *Then if* ρ_ϑ *and* δ_ϑ *are sufficiently small,* $a'(x,\xi)$ *belongs to* $S(<x>^m<\xi>^\ell, g)$ *and moreover satisfies*

$$a'(x,\xi) = p'(x,\xi) + r'(x,\xi), \tag{4.12}$$

$$p'(x,\xi) - \sum_{|\gamma|<N} \gamma^{-1} D_y^\gamma \partial_\eta^\gamma \{a(\Phi'(y,\xi,\eta),\xi)J'(y,\xi,\eta)\}_{y=x,\eta=\xi} \tag{4.13}$$

$$\in \gamma^d S(C^{1+N}N!^d <x>^{m-N}<\xi>^{\ell-N}, g_1),$$

for any non negative integer N, *where* $\Phi'(y,\xi,\eta)$ *is a solution of the equation*

$$\Phi'(y,\xi,\eta) - i\tilde{\nabla}_\xi\vartheta(\Phi'(y,\xi,\eta),\xi,\eta) = y, \tag{4.14}$$

$$\tilde{\nabla}_\xi\vartheta(y,\xi,\eta) = \int_0^1 \nabla_\xi\vartheta(y,\eta + t(\xi-\eta))dt, \tag{4.15}$$

and $J'(y,\xi,\eta) = \frac{D\Phi'(y,\xi,\eta)}{Dy}$, *and* $r'(x,\xi) \in A^{1/d}_{-\varepsilon_0,-\varepsilon_0}(\varepsilon_0 > 0$ *is independent of* ρ_ϑ.

Lemma 4.4. *Let* $\vartheta(x,\xi) \in \gamma^d S(\rho_\vartheta\langle x\rangle^\sigma\langle\xi\rangle^\delta, g_1)$. *If* ρ_ϑ *and* δ_ϑ *are sufficently small, there is the inverse of* $I_\phi(x,D)$, *which maps continuously from* H_{ρ_1,δ_1} *to* $H_{\rho_1-\rho_\vartheta,\delta_1-\delta_\vartheta}$ *for* $|\rho_1|, |\delta_1|$ *small enough and satisfies*

$$I_\phi(x,D)^{-1} = I^R_{-\phi}(x,D)(I + j(x,D))^{-1} = (I + j'(x,D))^{-1}I^R_{-\phi}(x,D) \tag{4.16}$$

$$= I^R_{-\phi}(x,D)(k(x,D) + k_1(x,D) + r(x,D)) = (k'(x,\xi) + k_1'(x,D) + r'(x,D)))I^R_{-\phi}(x,D),$$

where $j(x,\xi) = J(x,0,\xi) - 1 + r_1(x,\xi), j'(x,\xi) = J'(x,\xi,0) - 1 + r_2(x,\xi), k(x,\xi) = J(x,0,\xi)^{-1}, k'(x,\xi) = J'(x,\xi,0)^{-1}$ *and* $k_1, k_1' \in \gamma^d S(< x >^{-1} < \xi >^{-1}, g_1)$ *and* $r, r' \in A^{1/d}_{-\varepsilon_0,-\varepsilon_0}$.

Lemma 4.5. *Let* $a(x,\xi)$ *and* ϑ *be satisfied with the same condition as one of Lemma 3,3. Let* $\phi = x\xi - i\vartheta$. *Then we have*

$$\sigma(I_\phi(x,D)a(x,D))(x,\xi) = I_\phi \circ a(x,\xi) = e^{\vartheta(x,\xi)}(q(x,\xi) + r(x,\xi)), \tag{4.17}$$

$$\sigma(a(x,D)I_\phi(x,D)(x,\xi) = a \circ I_\phi(x,\xi) = e^{\vartheta(x,\xi)}(q'(x,\xi) + r'(x,\xi)), \tag{4.18}$$

where r, r' *is in* $A^{1/d}_{-\varepsilon_0,-\varepsilon_0}$, *if* ρ_ϑ *is sufficiently small, and* q, q' *satisfies*

$$q(x,\xi) - \sum_{|\gamma|<N} \gamma!^{-1} D_y^\delta \partial_\eta^\gamma \{a(x + y - i\tilde{\nabla}_\xi \vartheta(x,\xi,\eta),\xi)\}_{y=\eta=0} \in \gamma^{1/d} S(C^{1+N} N!^d < x >^{m-N} < \xi >^{\ell-N}, g_1), \tag{4.19}$$

$$q'(x,\xi) - \sum_{|\gamma|<N} \gamma^{-1} D_y^\gamma \partial_\eta^\gamma \{a(x,\xi + \eta - i\tilde{\nabla}_x \vartheta(x,y,\xi))\}_{y=\eta=0} \in \gamma^d S(C^{1+N} N!^d < x >^{m-N} < \xi >^{\ell-N}, g_1), \tag{4.20}$$

for any positive integer N, *and* $C > 0$ *and* $\varepsilon_0 > 0$ *are independent of* ρ_ϑ, *where* $\tilde{\nabla}_\xi \vartheta(x,\xi,\eta) = \int_0^1 \nabla_\xi \vartheta(x,\xi + t\eta)dt$ *and* $\tilde{\nabla}_x \vartheta(x,y,\xi) = \int_0^1 \nabla_x \vartheta(x + ty,\xi)dt$.

Summing up Lemma 4.2-Lemma 4,5, we obtain the following theorem.

Theorem 4.6. *Let* $a \in \gamma^d S(< x >^m < \xi >^\ell, g_1), \vartheta \in \gamma^d S(\rho_\vartheta < \xi >^\delta < x >^\sigma, g_1)$ *and* $\phi = x\xi - i\vartheta(x,\xi)$. *Assume that* $\sigma + \delta = \kappa = 1/d < 1$ *or* $\sigma + \delta = \kappa = 1, d = min(\delta^{-1}, \sigma^{-1})$. *Then if* $\rho_\vartheta, \delta_\vartheta$ *are sufficiently small,* $\tilde{a}(x,D) = I_\phi(x,D)a(x,D)I_\phi^{-1}$ *and* $\tilde{a}'(x,D) = I_\phi(x,D)^{-1}a(x,D)I_\phi(x,D)$ *are pseudodifferential operators of which symbols are given by*

$$\tilde{a}(x,\xi) = p(x,\xi) + r(x,\xi), \tag{4.21}$$

$$a'(x,\xi) = p'(x,\xi) + r'(x,\xi), \tag{4.22}$$

where

$$p(x,\xi) - a(x - i\nabla_\xi \vartheta(x,\Phi), \xi + i\nabla_x \vartheta(x,\Phi)) \in \gamma^d S(< x >^{m-1} < \xi >^{\ell-1}, g_1), \tag{4.23}$$

$$\tilde{p}'(x,\xi) - a(x + i\nabla_\xi \vartheta(\Phi',\xi), \xi - i\nabla_x \vartheta(\Phi',\xi)) \in \gamma^d S(< x >^{m-1} < \xi >^{\ell-1}, g_1), \tag{4.24}$$

where $\Phi = \Phi(x,x,\xi)$ *and* $\Phi' = \Phi'(x,\xi,\xi)$ *are given by (4.6) and (4.10) respectively and* r, r' *belong to* $A^{1/d}_{-\varepsilon_0,-\varepsilon_0}$ *for an* $\varepsilon_0 > 0$ *independent of* ρ_ϑ.

5 Criterion to L^2−well posed Cauchy problem

For $T > 0$ let consider the following Cauchy problem,

$$\partial_t u(t,x) - i\Delta u(t,x) - b(t,x,D)u(t,x) = 0, \tag{5.1}$$

(5.2) $$u(0,x) = u_0(x),$$

for $(t,x) \in (0,T) \times R^n$. We assume that $b(t,x,\xi)$ is in $C^0([0,T]; S^1_{1,0})$. Moreover we suppose that there are $C \in R, K > 0$ such that

(5.3) $$Reb(t,x,\xi) \le C,$$

for $x, \xi \in R^n$ with $|x|, |\xi| \ge K$ and $t \in [0,T]$. Then we can prove the following theorem by use of the same method as that of [3] and [7].

Theorem 5.1. *Assume that the above conditions (4.3)-(4.5) are valid. For any $u_0 \in L^2$ and $f \in C^0([0,T]; L^2)$ there exists a unique solution $u \in C^0([0,T]; L^2) \cap C^1([0,T]; H^{-2})$ of the Cauchy problem (5.1)-(5.2).*

6 Proof of Theorem

Assume that $u(t,x)$ satisfies (1)-(2) in the introduction. Put $v(t,x) = e^{\rho t\langle D\rangle^\kappa} u(t,x)$. Then v satiesfies the following Cauchy problem,

(6.1) $$\frac{\partial}{\partial t} v(t,x) = (i\Delta + c(t,x,D))v(t,x),$$

(6.2) $$v(0,x) = u_0(x),$$

where

$$c(t,x,D) = \rho\langle D\rangle^\kappa + e^{\rho\langle D\rangle^\kappa} b(t,xD) e^{-\rho\langle D\rangle^\kappa}$$

(6.3) $$= \rho\langle D\rangle^\kappa + b(t,x,D) + b_1(t,x,D) + r_2(t,x,D),$$

where $b_1(x,\xi) \in \gamma^d S(<\xi><x>^{-1}, g_1), r_1(t,z,\zeta) \in A^\kappa_{-\varepsilon_0 + c\rho_0 T, -\varepsilon_0}$ from Theorem 4.1. Oncemore we change the unknown function v to w as follows,

(6.4) $$w(t,x) = I_\phi(x,D)v(t,x),$$

where $\phi = x\xi - i\epsilon\vartheta(t,x,\xi)$ and ϑ is given by

$$\vartheta(t,x,\xi) = \vartheta_0(x,\xi)\phi_0(\frac{\langle x\rangle}{M\langle\xi\rangle}) + t\langle\xi\rangle^{\sigma+\delta}(1 - \phi_0(\frac{\langle x\rangle}{M\langle\xi\rangle}),$$

$$\vartheta_0(x,\xi) = \frac{x\cdot\xi}{\langle x\rangle^{1-\sigma}\langle\xi\rangle^{1-\delta}\varepsilon_1})\phi_0(\frac{x\cdot\xi}{\langle x\rangle\langle\xi\rangle\varepsilon_1}) + \langle\xi\rangle^{\delta-\sigma} f(|x\cdot\xi|)[\phi_+(\frac{x\cdot\xi}{\langle x\rangle\langle\xi\rangle\varepsilon_1}) - \phi_-(\frac{x\cdot\xi}{\langle x\rangle\langle\xi\rangle\varepsilon_1})],$$

$$f(t) = \int_0^t (1+s^2)^{\frac{\sigma-1}{2}} ds,$$

and $\phi_\pm(t) = \chi(\pm t), \phi_0(t) = 1 - \phi_+(t) - \phi_-(t)$ and $\chi(t) \in \gamma^d(R)$ such that $\chi(t) = 1$ for $t \ge 1, \chi(t) = 0$ for $t \le 1/2, \chi'(t) \ge 0$ and $0 \le \chi(t) \le 1$. Then we can see that $\vartheta(t,x,\xi)$ belongs to $\gamma^d S(\langle x\rangle^\sigma\langle\xi\rangle^\delta, g_1)$ and that there are $\varepsilon_1 > 0, M > 0, K > 0, c_0 > 0$ such that ϑ satisfies

(6.5) $$(\partial_t + \xi\cdot\nabla_x)\vartheta(t,x,\xi) \ge c_0(\langle\xi\rangle^{2\delta}\langle x\rangle^{2\sigma-2} + \langle\xi\rangle^{\sigma+\delta} + \langle\xi\rangle\langle x\rangle^{\sigma+\delta-1}) - c_1,$$

for $x, \xi \in R^n$ with $|x|, |\xi| \ge K, |t| \le T$.

It follows from Lemma 4.4 that if $|\epsilon|$ is sufficiently small, we have the inverse $I_\phi(x,D)^{-1}$. Therefore we get the following Cauchy problem of w from (6.1)-(6.2),

$$\text{(6.6)} \qquad \frac{\partial}{\partial t}w(t,x) = (\partial_t I_\phi)I_\phi(x,D)^{-1}w(t,x) + I_\phi(i\Delta + c(t,x,D))I_\phi(x,D)^{-1}w(t,x),$$

$$\text{(6.7)} \qquad w(0,x) = I_\phi(x,D)u_0(x).$$

Since $\vartheta(t,x,\xi) \in \gamma^d S(\langle x\rangle^\sigma \langle\xi\rangle^\delta, g_1)$, it follows from (4.10) that $\nabla_\xi\vartheta(x,\Phi(x,\xi)) \in \gamma^d S(< x >^\sigma < \xi >^{\delta-1}, g_1), \nabla_x\vartheta(x,\Phi(x,\xi)) \in \gamma^d S(< x >^{\sigma-1} < \xi >^\delta, g_1)$, and $\Phi(x,\xi) - \xi \in \gamma^d S(< x >^{\sigma-1} < \xi >^\delta, g_1)$. Hence we have from (4.16) in Theorem 4.6 and Proposition 2.3

$$\text{(6.8)} \qquad \sigma(I_\phi \Delta I_\phi^{-1})(x,\xi) = -|\xi + i\epsilon\nabla_x\vartheta(x,\Phi))|^2 + a_1(x,D) + r_2(x,\xi),$$

$$= -(|\xi|^2 + |\nabla x\vartheta(t,x,\xi)|^2 + 2i\varepsilon\xi\cdot\nabla_x\vartheta(t,x,\xi)) + a_1'(x,\xi) + r_2(x,D)$$

where $a_1 \in S(< \xi >< x >^{-1}, g), a_1' \in S(< x >^{2\sigma-2} < \xi >^{2\delta} + \langle\xi\rangle\langle x\rangle^{-1}, g_1)$ and $r_2 \in A^{1/d}_{-\varepsilon_0+|c|\varepsilon|,-\varepsilon_0+c|\varepsilon|}$ for some $c > 0$ (independent of ε). Here we choose ε such that r_2 belongs to $S(1,g)$. Thus we obtain the equation of w from (6.6)-(6.7),

$$\text{(5.10)} \qquad \frac{\partial w}{\partial t} = (i\Delta + \rho\langle D\rangle^{\sigma+\delta} + b(t,x,D) + \epsilon(\partial_t + \xi\cdot\nabla_x)\vartheta)(t,x,D)) + r_3(t,x,D))w(t,x),$$

$$\text{(5.11)} \qquad w(0) = I_{\phi(0)}(x,D)u_0(x),$$

where $r_4 \in S(< \xi >^{2\delta} < x >^{2\sigma-2} + < \xi >< x >^{-1}, g_1)$. Moreover taking account of the assumptions (5) in the introduction and (6.5) we can choose conviniently $K > 0, \epsilon$ and ρ such that we have

$$\rho p(x,\xi) + Reb(t,x,\xi) - \epsilon H_a\theta(x,\xi) + Rer_4(t,x,\xi) \le 0,$$

for $x,\xi \in R^n$ with $|x|, |\xi| \ge K$,where $K > 0$ is sufficiently large. Therefore we can solve the Cauchy problem (6.6)-(6.7) by use of Theorem 5.1, since $w(0) = I_{\phi(0)}u_0$ belongs to L^2, and cosequently we get the solution $u = e^{-\rho t\langle D\rangle^\kappa} I_\phi(x,D)^{-1}w(t,x) = e(t,x,D)^{-1}I_{\phi(t)}(x,D)^{-1}I_\phi(x,D)^{-1}w$, which satisfies (6) from Lemma 1.2. This completes the proof of Theorem.

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