Some results on eigenvalues of the Cartan matrices for finite groups

東京農工大学・工 和田倶幸 (Tomoyuki Wada)

$G$: a finite group
$F$: an algebraically closed field of characteristic $p > 0$
$B$: a block of the group algebra $FG$ with defect group $D$ of order $p^d$
$C_B = (c_{ij})$: the Cartan matrix of $B$ i.e. $c_{ij}$ is the multiplicity of an irreducible $FG$-module $S_j$ in a projective cover $P_i$ of $S_i$ as a composition factor, where $S_j$ and $P_i$ belong to $B$.

The following are well known properties of the Cartan matrix $C_B$.

- nonnegative (integral) indecomposable symmetric
- positive definite
- all elementary divisors are a power of $p$, the largest one is $p^d = |D|$ and the others are smaller than $p^d$

$\rho(B)$: the Perron-Frobenius (i.e. the largest) eigenvalue of $C_B$

We note the following.

- eigenvalues and elementary divisors are not equal in general
- $G = A_5$ (the alternating group of degree 5), $p = 2$, $B = B_0$ (the principal block)
  $\Rightarrow \rho(B) = (7 + \sqrt{33})/2 > |D| = 4$

1. Known properties of $\rho(B)$

The following are known about lower and upper bounds for $\rho(B)$ in [K-W].

(1) $|O_p(G)| \leq \rho(B) \leq u$ for any block $B$ of $FG$, where $u := \dim_F P(F_G)$ and $P(F_G)$ is a projective cover of the trivial $FG$-module $F_G$. 
(2) If $G$ is $p$-solvable, then $\rho(B) \leq |D|$, and the equality holds if and only if the height of $\varphi = 0$ for all $\varphi \in IBr(B)$.

(3) If $D$ is cyclic, then $\frac{|D|}{p} + 1 \leq \rho(B) \leq |D|$.

(4) If $D \triangleleft G$, then $\rho(B) = |D|$.

We have a lower bound and an upper bound of $\rho(B)$ in (1) in terms of $G$, but it should be given in terms of $B$ for any block $B$ and any group $G$. In this talk we showed a lower bound of $\rho(B)$ in terms of $B$.

2. A lower bound of $\rho(B)$

$\text{Irr}(B) :=$ the set of all ordinary (complex) irreducible characters in $B$,

$\text{IBr}(B) :=$ the set of all irreducible Brauer characters in $B$,

$k(B) := |\text{Irr}(B)|, \ l(B) := |\text{IBr}(B)|$.

Let $\sigma$ be a permutation on $\{1, 2, \ldots, l\}$, where $l = l(B)$. Then we have the following:

**Theorem 1** ([W1]). Let $C_B = (c_{ij})$ be the Cartan matrix of any block $B$ of $FG$ for any finite group $G$. For $l = l(B)$, we set $l \setminus t := \{1, 2, \ldots, l\} - \{t\}$ for $1 \leq t \leq l$. Then we have

$$k(B) \leq \sum_{i=1}^{l} c_{ii} - \sum_{j \in l \setminus t} c_{j\sigma(j)}$$

for any cycle $\sigma$ of length $l$ and any choice of $1 \leq t \leq l$.

**Proof.** By the fact $C_B = ^t D_B D_B$ for the decomposition matrix $D_B$ of $B$, we write the right hand side of the above inequality by using decomposition numbers for $B$ and we can show a contribution for it of any $\chi \in \text{Irr}(B)$ is larger than or equal to 1.

**Corollary 2.** Let $B$ be a block of $FG$ with defect group $D$. Then $k(B) \leq \rho(B) l(B)$, and the equality holds if and only if $l(B) = 1$ and $k(B) = |D|$.

**Proof.** It is clear that $k(B) \leq \sum_{i=1}^{l(B)} c_{ii}$ even if we do not use Theorem 1. Combine it with the fact that $c_{ij} \leq \rho(B)$ for any $i, j$. 

Question 1. There must be sharper inequalities than Corollary 2. For example, does it hold that $k(B) \leq \rho(B)$?

The answer is no. Let $G = \text{SL}(2,p), p$ an odd prime, and $B$ be any one of blocks of defect 1. Then $l(B) = (p - 1)/2$, $k(B) = l(B) + 2$ and

$$C_B = \begin{pmatrix}
2 & 1 & 0 & \ldots & 0 \\
1 & 2 & 1 & \ddots & \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & 1 & 2 & 1 \\
0 & \ldots & 0 & 1 & 3
\end{pmatrix}.$$

Therefore $3 < \rho(C_B) < 4$ by Lemma 3.1 in [K-W], but $k(B) \geq 4$ if $p \geq 5$.

Question 2. Does it hold that $k(B) \leq \rho(B)$, in $p$-solvable groups?

Now we assume $G$ is $p$-solvable, then we have the following.

**Proposition 3.** Let $G$ be a $p$-solvable group and $B$ a block of $FG$ with $l(B) = 2$. Assume the $p'$-part $f'_i$ of the degree $f_i$ of two irreducible Brauer characters $\varphi_i$ for $i = 1, 2$ are equal. Then $k(B) \leq \rho(C_B)$.

**Proof.** The explicit form of $C_B$ in this case is known in [N-W]. Theorem 1 shows that $k(B) \leq c_{11} + c_{22} - c_{12}$. We can verify that the right hand side of the above inequality $\leq \rho(B)$ by the form of $C_B$.

Remark 4. We added an assumption in the above proposition, but it is conjectured in [N-W, p.329] that $f_1' = f_2'$ for $p$-solvable groups. Isaacs showed this is true if $G$ is solvable in [I], and it is also proved to be true in some cases in [N-W]. Therefore, $k(B) \leq \rho(C_B)$ for $B$ with $l(B) = 2$ in $p$-solvable groups, for example, if $G$ is solvable, $B$ is the principal block, or $B$ has an abelian defect group.

Remark 5. Proposition 3 does not hold in general. K. Erdmann determined the shape of the Cartan matrix of tame blocks in [E] (i.e. $p=2$ and a defect group $D$ is dihedral, generalized quaternion or semidihedral). For example, it actually fails in the following cases.

Let $G = \text{PGL}(2,31)$ and $B$ be the principal block. Then $D$ is a dihedral group of order
$2^6$, $l(B) = 2$, $C_B = \begin{pmatrix} 4 & 2 \\ 2 & 17 \end{pmatrix}$, $k(B) = 19$ (Erdmann's list D(2B)), but $\rho(C_B) < 19$ by Lemma 3.1(2) in [K-W].

We saw in the proof of Proposition 3 that Theorem 1 works well. So the diagonal entries of $C_B$ for $p$-solvable groups seem to be not so extremely larger than the other entries, while it does not hold in general as is shown in the examples above.

**Conjecture.** If $G$ is $p$-solvable, then $k(B) \leq \rho(B)$.

If Conjecture is true, then Brauer's $k(B)$ conjecture (that is $k(B) \leq |D|$ for any finite group) is true in $p$-solvable groups, because [K-W] has showed $\rho(B) \leq |D|$ in $p$-solvable groups. Since Brauer's $k(B)$ conjecture is not yet proved to be true even if $G$ is a solvable group, it must be quite difficult to show directly that Conjecture is true. There sure is a possibility of the existence of a counter example for it. But we raise some more evidences for the conjecture.

1. If $G$ is of $p$-length 1, or $D$ is abelian, then Conjecture can be reduced to the case that $D < G$ by Külshammer [Kü].

2. If $B$ is tame, then Conjecture is true by [E-M, Kü, Ko1, B-W].

3. If $p = 3$ and $D \simeq M(3)$ (i.e. extra special 3-group of order 27 with exponent 3), then Conjecture is true by [Ko2].

4. Assume Brauer's $k(B)$ conjecture is true for $p$-solvable groups. If $k(B) = |D|$, then $k(B) = \rho(B)$ by [M].

3. The Cartan matrix of a certain class of finite solvable groups

If there exists a counter example for Conjecture, Theorem 1 seems to assert that the non diagonal entries of its Cartan matrix must be extremely smaller than the diagonal ones. So first we should find $p$-solvable groups (blocks) whose Cartan matrix has many zero entries and $l(B)$ is large like $\text{SL}(2,p)$ because $\rho(B)$ is small and $k(B)$ is large. Here by making use of Ninomiya's result in [N] we give an explicit form of the Cartan matrix of a certain class of solvable groups. The author owes to Professor Tetsuro Okuyama who taught him the following type of groups whose Cartan matrix has zero entries.

$GF(p^n)$ : the finite field with $p^n$ elements
$A(p^n)$: the additive group of $GF(p^n)$
$M(p^n)$: the multiplicative group of $GF(p^n)$
$X(p^n)$: the affine group of $GF(p^n)$ i.e. $M(p^n) \rtimes A(p^n)$ by ordinary scalar multiplication, then $X(p^n)$ is a complete Frobenius group whose Frobenius kernel is a Sylow $p$-subgroup, and it is known that the Cartan matrix of $FX(p^n)$ is of the form

\[
\begin{pmatrix}
2 & 1 & \cdots & 1 \\
1 & 2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 2
\end{pmatrix}
\]

$<\sigma>$: the Galois group of $GF(p^n)$ over $GF(p)$ of order $n$
$G(p^n):=\langle \sigma \rangle \rtimes X(p^n)$.

We consider the case $n=pq$, where $q$ is a prime number different from $p$. Let us set $G = G(p^{pq})$, then since $O_{p'}(G)$ is trivial, $G$ has only the principal block by a theorem of Fong and $G$ is of $p$-length 2.

**Theorem 6.** Under the above notation (see [W2] for more detailed notation), the Cartan matrix $C(G)$ of $FG$ is the following.

\[
\begin{array}{c|cccc|cccc|c}
\alpha_1 & \alpha_2 & \cdots & \alpha_{p-1} & \gamma_1 & \gamma_2 & \cdots & \gamma_n & \theta \\
\hline
2pl_q & pl_q & \cdots & pl_q & pl_q & \cdots & \cdots & \cdots & pl_q \\
pI_q & 2pl_q & \cdots & \cdots & pI_q & \cdots & \cdots & \cdots & pJ'_1 \\
\vdots & \ddots & \cdots & \cdots & \vdots & \ddots & \cdots & \cdots & \vdots \\
pI_q & \cdots & pl_q & 2pl_q & pl_q & \cdots & \cdots & \cdots & pI_q \\
\hline
p^iJ'_1 & B_1 & pJ'_3 & pqJ'_4 \\
\hline
pl_q & \cdots & pl_q & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
pI_q & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \ddots & \cdots & \cdots & \vdots & \ddots & \cdots & \cdots & \vdots \\
pI_q & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\hline
p^iJ'_2 & pq^iJ'_4 & p^iJ'_5 & B_2 \\
\end{array}
\]

where $I_s$ is the unit matrix of degree $s$, $J'_1, J'_2, J'_3, J'_4, J'_5$ is the $(p-1)q \times m, (p-1)q \times (r-m)/p, m \times nq, m \times (r-m)/p, nq \times (r-m)/p$ matrix all of whose entries are 1, respectively. Furthermore, $B_1 = pl_m + pqJ_m$ and $B_2 = I_{r-m} + pqJ_{r-m}$, where $J_s$ is the $s \times s$ matrix all of whose entries are 1.

It is known in general that $\Sigma_{i,j=1}^{l(B)} c_{ij}/l(B) \leq \rho(B)$ for any block $B$ of $FG$ for any finite group $G$, and now when $G = G(p^{pq})$ we can verify $k(FG) \leq \Sigma_{i,j=1}^{l(FG)} c_{ij}/l(FG)$. So we have
When $G = G(p^q)$ and $G(p^p)$, we have also $k(FG) \leq \rho(FG)$.

4. Eigenvalues and elementary divisors of $C_B$

Elementary divisors of $C_B$ are invariant under elementary operations i.e. $C_B$ and $SC_BT$ for unimodular matrices $S, T$ have the same elementary divisors, while eigenvalues of them are different in general. So elementary divisors and eigenvalues of $C_B$ do not coincide in general. When do they coincide? We have an answer to it in $p$-solvable groups as follows. This is a part of joint work with A. Hanaki, M. Kiyota and M. Murai [H, K, M, W].

Theorem 7. Let $G$ be a $p$-solvable group, $B$ a block of $FG$ with defect group $D$. Then the following are equivalent.

(a) Elementary divisors and eigenvalues of $C_B$ coincide.

(b) $\rho(B) = |D|$.

(c) The height of $\varphi = 0$ for all $\varphi \in \text{IBr}(B)$.

Proof. We have the following two results for $p$-solvable groups.

(1) Let $G$ be a $p$-solvable group and $\eta_G$ the character afforded by the principal indecomposable $FG$-module corresponding to the trivial $FG$-module $F_G$. Then $\eta_G(x)$ is a power of $p$ for any $p$-regular element $x \in G$.

(2) Let $G$ be a $p$-solvable group and $B$ a block of $FG$ of full defect. Suppose the height of $\varphi = 0$ for all $\varphi \in \text{IBr}(B)$. Then elementary divisors and eigenvalues of $C_B$ coincide.

Then Fong's two reduction theorem works well, and we have the result.

In this case Conjecture is equivalent to Brauer's $k(B)$ conjecture as $\rho(B) = |D|$.

References


Tomoyuki Wada
Department of Mathematics
Tokyo University of Agriculture and Technology
Saiwai-cho 3-5-8, Fuchu, Tokyo 183-0054, Japan
E-mail address: wada@cc.tuat.ac.jp