Squares of Characters in Finite Groups

Masao KIYOTA
College of Liberal Arts and Sciences
Tokyo Medical and Dental University
Kounodai, Ichikawa, 272 JAPAN

e-mail : g38098@m-unix.cc.u-tokyo.ac.jp

This is a report of my joint paper \([K,S]\) with Hiroshi Suzuki (Department of Mathematics, International Christian University).

Let \(G\) be a finite group and let \(\chi\) be a real valued character of \(G\). The representation diagram of \(G\) with respect to \(\chi\), denoted by \(D(G,\chi)\), is a graph with \(\text{Irr}(G)\) as the vertex set such that vertices \(\chi_i\) and \(\chi_j\) are adjacent if and only if \((\chi \chi_i, \chi_j) > 0\). \(D(G,\chi)\) is undirected as \(\chi\) is real valued, but \(D(G,\chi)\) may have some loops. Note that \(D(G,\chi)\) is connected if and only if \(\chi\) is faithful. The problem we are interested in is that if we know the graph structure of the representation diagram \(D(G,\chi)\), then what can be said about the group structure of \(G\). Here we consider the simplest case i.e. the case \(D(G,\chi)\) is a path (open polygon) possibly with some loops. The following lemma is fundamental but easy to prove.

Lemma 1. Let \(\chi\) be a real valued character of a finite group \(G\). Let the representation diagram \(D(G,\chi)\) be a path possibly with some loops. Then \(\chi = a \chi_1 + b \chi\), for some faithful real valued \(\chi_1\) in \(\text{Irr}(G)\) and for some integers \(a \geq 0, b > 0\). In particular the diagrams \(D(G,\chi)\) and \(D(G,\chi_1)\) are identical modulo loops, i.e. neglecting loops.

If \(D(G,\chi)\) is a path then we may assume \(\chi\) is irreducible, and so we have

\[
\chi^2 = 1_G + a \chi + b \psi,
\]

for some \(\psi\) in \(\text{Irr}(G)\) and for some integers \(a \geq 0, b \geq 0\), since in the diagram \(\chi\)
is adjacent to $1_G$ and $\psi$ and possibly $\chi$ itself (loop). The groups with irreducible characters $\chi$ and $\psi$ satisfying (*) are completely determined in the next theorem.

Theorem 2. Let $\chi$ and $\psi$ be irreducible characters of a finite group $G$. Suppose that the equation (*) holds. If $\chi$ is faithful and real valued, then one of the following holds.

1. $\chi(1) = 1$ and $G$ is cyclic of order at most two.
2. $\chi(1) = 2$ and $G$ is the symmetric group of degree 3.
3. $\chi(1) = 2$ and $G$ is one of the binary polyhedral groups of order 24, 48 or 120.
4. $\chi(1) = 3$ and $G$ is the alternating group of degree 5.

For the proof we refer to [K,S]. By inspection of the representation diagram of each group listed in Theorem 2, we have the following

Corollary 3. Let $\chi$ be a real valued character of a finite group $G$ of order at least two. Let the representation diagram $D(G,\chi)$ be a path possibly with some loops. Then $G$ is the cyclic group of order two, or the symmetric group of degree 3.

If you are familiar with some terminology in algebraic combinatorics (for example in [B, I]), you may find that Corollary 3 is equivalent to the following

Corollary 4. Let $G$ be a finite group of order at least two. Suppose that the group association scheme $X(G)$ is $Q$-polynomial. Then $G$ is the cyclic group of order two, or the symmetric group of degree 3.

Here, we state some open problems.

Problem 5. Study the structure of finite groups $G$ when $D(G,\chi)$ is a tree
possibly with some loops.

Problem 6. Determine all finite groups whose group association scheme is \( P \)-polynomial. In other words, prove the dual statement of Corollary 4.

Problem 7. Study the structure of finite groups \( G \) with \( \chi \) and \( \psi \) in \( \text{Irr}(G) \) satisfying

\[
\chi \overline{\chi} = 1_G + a(\chi + \overline{\chi}) + b\psi.
\]

There are many interesting examples such as \( GL(2, 3), PSL(2, 7) \) and \( PSU(4, 2^2) \).

references
