On the cohomology of finite Chevalley groups and free loop spaces of classifying spaces

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Abstract

1 notations

Let $p$ be a prime and $\mathbb{F}_q$ be the finite field with $q$ elements. Let $G_\mathbb{C}$ be a Chevalley group scheme such that $\mathbb{C}$-rational points $G_\mathbb{C}(\mathbb{C})$ is a simply connected complex Lie group when we change its topology. Hereafter we denote $G_\mathbb{C}(K)$ (resp. $G_\mathbb{C}(\mathbb{C})$) by $G(K)$ (resp. $G$) for a field $K$. We also denote its classifying space by $BG$ and define the free loop space $\mathcal{L}BG$ of $BG$ and the loop space $\Omega BG$ of $BG$ by

$$\mathcal{L}BG = \{l \mid l: S^1 \to BG\} \quad \text{and} \quad \Omega BG = \{l \mid l(1) = *, l \in \mathcal{L}BG\},$$

where $S^1$ is the unit circle on the complex number $\mathbb{C}$ and $*$ is a base point of $BG$. It is well known that $\Omega BG$ is weakly homotopy equivalent to $G$.

2 results and comments

Theorem. Let $\mathbb{F}_q$ be a finite field with $q = p^n$ elements and $l$ be a prime number that divides $q - 1$ but does not divide the order of the Weyl group of $G$. Then we have an ring isomorphism

$$H^*(\mathcal{L}BG, \mathbb{Z}/l) \cong H^*(G(\mathbb{F}_q), \mathbb{Z}/l) \cong H^*(BG, \mathbb{Z}/l) \otimes H^*(G, \mathbb{Z}/l).$$

We can prove the theorem immediately from Kleinerman [3] and Kono-Kozima [4].
Remark. Our theorem is partial. Here we indicate an example.

Theorem (Fong-Milgram [1], Kono-Kozima [4]). Let $G_2$ be an exceptional Lie type $G_2$. Then we have a ring isomorphism

$$H^*(\mathcal{L}BG_2, \mathbb{Z}/2) \cong H^*(G_2(\mathbb{F}_q), \mathbb{Z}/2)$$

for $4|q - 1$.

We propose a question: Let $l$ be a prime number such that $l$ (resp. 4) divides $q - 1$ if $l$ is odd (resp. even). Then we have a ring isomorphism

$$H^*(\mathcal{L}BG, \mathbb{Z}/l) \cong H^*(G(\mathbb{F}_q), \mathbb{Z}/l)$$

Acknowledgment The author is grateful to professors A.Kono, K.Kuribayashi and N.Yagita for their many suggestions.

References


