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Applications of a summation formula

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1 Introduction and statement of results

Let Δ be the Laplacian on the compact Riemann manifold M. Then Δ has a discrete spectrum

$$0=\lambda_{0}<\lambda_{1}\leq\lambda_{2}\leq\cdots,$$

for which we introduce the zeta-function

$$Z(s)=\sum_{n=1}^{\infty}\frac{1}{\lambda_{n}^{s}},\quad \mathrm{Re}(s)=\sigma>\alpha$$

(0-energy level excluded), absolutely convergent in a half-plane in view of the Weyl law. It is shown that $Z(s)$ can be continued to the region including 0, and we can interpret the (otherwise) divergent 'determinant'

$$\det'\Delta=\prod_{n=1}^{\infty}\lambda_{n}$$

as the zeta-regularized product (or the functional determinant)

$$\det\Delta=e^{-Z'(0)}$$

which is the Determinant of the Laplacian in the title, where we note that since

$$Z'(s)=-\sum_{n=1}^{\infty}\frac{\log\lambda_{n}}{\lambda_{n}^{s}},\quad \sigma>\alpha$$

$e^{-Z'(0)}$ is formally equal to the product $\det'\Delta$ of positive eigenvalues.

For compact Riemann surfaces with constant curvature the determinants of the Laplacian have recently been studied extensively by D'Hoker-Phong [6], [7], Sarnak [13], Voros [17](for non-compact case, see, e.g. Efrat [8]), in view of their relevance to superstring theory. The main feature is the computation of determinants in terms of values of the Selbarg zeta-function, where multiple gamma functions play important roles.

For compact Riemann manifolds of higher dimensions, such as the unit n-sphere

$$S^{n-1}=\{(x_{1},\cdots,x_{n})\in\mathbf{R}^{n}|x_{1}^{2}+\cdots+x_{n}^{2}=1\}$$

the regularized determinants have been also studied, notably by Weisberger [18], [19], Vardi [15] and Choi [1], [2].

They computed the determinant of the Laplacian of the unit n-sphere S^{n-1} with standard metric in terms of the values of the derivative of the Hurwitz (Riemann) zeta-function at 0. The unit 2-sphere case which was computed in all above papers have interest again due to its relationship to superstring theory (see Vardi [15], Osgood, Phillips, Sarnak [11], Weisberger [18], [19]).

Our purpose is to give a closed form evaluation of $\det \Delta_n$ for any n, and give a corrected version of Vardi's Theorem 1.1 and 1.2 [15], thus compiling all existing special cases to higher dimension.

We note that our elementary method applies to any dimension, while Weisberger's method seems to be restricted to the 2-dimensional case, and Choi's method seems too complicated to modify it to higher (even 3-) dimension.

We now set out to state our theorems. We recall from [16] that the eigenvalues of the standard Laplacian on the n-sphere are $k(k+n-1)$ with multiplicity

$$\binom{k+n}{n} - \binom{k+n-2}{n} \quad (k = 0, 1, 2, \cdots).$$

We form the zeta-function

$$Z(s) = Z_n(s) = \sum_{k=1}^{\infty} \frac{\binom{k+n}{n} - \binom{k+n-2}{n}}{(k(k+n-1))^s}$$

(zero mode excluded), which is absolutely convergent for $\mathrm{Re}\, s := \sigma > \frac{n}{2}$, and we shall prove in Lemma 3 that it can be continued to a half-plane including the origin. Thus we can define the (regularized) determinant $\det \Delta_n$ of the Laplacian of the n-sphere by

$$\det \Delta_n = e^{-Z'(0)}.$$

We shall prove the following closed form for $\det \Delta_n$.

Theorem 1. We have for arbitrary dimension n,

$$\det \Delta_n = \exp\Big(\sum_{d=0}^{n-1} T_{n,d} H'_{n-1,d}(0)\Big),$$

where $H'_{n-1,d}$ and $T_{n,d}$ are as given in Lemma 2, and Lemma 6, respectively:

$$\begin{aligned} H'_{n-1,d}(0) &= 2\zeta'(-d) + \sum_{l=0}^{d-1} \binom{d}{l} (1-n)^{d-l} \zeta'(-l) \\ &+ (-1)^d \sum_{l=2}^{n-1} (n-l-1)^d \log l \\ &- \frac{2}{d+1} \Big(-\frac{n-1}{2}\Big)^{d+1} \sum_{\substack{l=1 \\ 2 \nmid l}}^{d} \binom{d+1}{l+1} \sum_{\substack{j=1 \\ 2 \nmid j}}^{l} \frac{1}{j}, \end{aligned}$$

and

$$T_{n,d} = \frac{1}{n!}\sum_{r=d+1}^{n} s(n,r)\binom{r}{d}(n^{r-d} - (n-2)^{r-d}),$$

with $s(n,r)$ denoting the Stirling number of the first kind defined by the Newton expansion

$$(x)_n = \sum_{r=0}^{n} s(n,r)x^r.$$

Corollary. We have

$$\begin{array}{llcl} \text{(i)} & \det\Delta_1 & = & 4\pi^2 \\ \text{(ii)} & \det\Delta_2 & = & A^4 e^{\frac{1}{6}} \\ \text{(iii)} & \det\Delta_3 & = & \pi\exp(\frac{\zeta(3)}{\pi^2}) \\ \text{(iv)} & \det\Delta_4 & = & \frac{1}{3}e^{-\frac{2}{3}\zeta'(-3)}A^{\frac{13}{3}}e^{\frac{83}{144}}, \end{array}$$

and similarly for higher dimensions.

From Theorem 1 we immediately deduce

Theorem 2. For arbitrary dimension n, (i) there are computable rational numbers $\alpha_n, \beta_n, \gamma_n, \tau_{n,1}, \cdots, \tau_{n,n-1}$ with $\tau_{n,n-1} = -\frac{4}{(n-1)!}, \alpha_n \neq 0$, such that

$$\det\Delta_n = \alpha_n^{\beta_n} e^{\gamma_n} \prod_{m=0}^{n-1} e^{\tau_{n,m}\zeta'(-m)};$$

(ii) there are computable rational numbers $A_n, B_n, C_n, Q_{n,1}, \cdots, Q_{n,n}$ with $Q_{n,n} = \frac{2^{n+1}}{2^n-1}, A_n \neq 0$, such that

$$\det\Delta_n = A_n^{B_n} e^{C_n} \prod_{m=1}^{n} \Gamma_m(\frac{1}{2})^{Q_{n,m}},$$

where $\Gamma_n(x)$ denotes the multiple gamma function (cf. Choi [1], Vardi [15]).

We intentionally used the same notation as in Vardi [15], but they may have slightly different values.

2 Proofs

Lemma 1. For $\lambda = 0, 1, 2, \cdots$, $\alpha > 0$ and $|z| < \alpha$ we have

$$\begin{aligned} \sum_{m=2}^{\infty} \frac{z^m}{m+\lambda}\zeta(m,\alpha) &= \sum_{k=0}^{\lambda}\binom{\lambda}{k}\zeta'(-k,\alpha - z)z^{-k} - \zeta'(-\lambda,\alpha)z^{-\lambda} \\ &\quad - \sum_{m=0}^{\lambda-1}\frac{z^{-m}}{\lambda - m}\zeta(-m,\alpha) \\ &\quad - \frac{z}{\lambda+1}(\psi(\lambda+1) - \psi(\alpha) + \gamma), \end{aligned}$$

where $\psi = \frac{\Gamma'}{\Gamma}$ denotes the Euler digamma function.

Lemma 2. We have the decomposition

$$
\begin{aligned}
Z(s) = Z_n(s) &= \sum_{k=1}^{\infty} \frac{\binom{k+n}{n} - \binom{k+n-2}{n}}{(k(k+n-1))^s} \\
&= \sum_{d=0}^{n-1} T_{n,d} H_{n-1,d}(s),
\end{aligned}
$$

where

$$
T_{n,d} = \frac{1}{n!} \sum_{r=d+1}^{n} s(n,r) \binom{r}{d} (n^{r-d} - (n-2)^{r-d}), \tag{1}
$$

$s(n,r)$ denoting Stirling numbers of the first kind and

$$
H_d(s) = H_{n,d}(s) = \sum_{k=1}^{\infty} \frac{k^d}{(k(k+n))^s}. \tag{2}
$$

Lemma 3. The expression

$$
\begin{aligned}
H_d(s) &= H_{n,d}(s) \\
&= \sum_{l=0}^{d} \binom{d}{l} (-\frac{n}{2})^{d-l} \\
&\quad \times \sum_{r=0}^{\infty} \frac{1}{r!} (\frac{n^2}{4})^r \frac{\Gamma(s+r)}{\Gamma(s)} \zeta(2s+2r-l, 1+\frac{n}{2}).
\end{aligned} \tag{3}
$$

provides us with an analytic continuation of $H_d(s)$ to the half-plane $\sigma > -1$ (at least) with a possible simple pole at $s = \frac{1}{2}$, when splitted into three parts :

$$
\begin{aligned}
H_d(s) &= (\sum_{\substack{l=0 \\ 2 \nmid l}}^{d} \binom{d}{l} (-\frac{n}{2})^{d-l} \frac{1}{(\frac{l+1}{2})!} (\frac{n^2}{4})^{\frac{l+1}{2}}) \zeta(2s+1, 1+\frac{n}{2}) \\
&+ \sum_{l=0}^{d} \binom{d}{l} (-\frac{n}{2})^{d-l} \\
&\quad \times \sum_{r > \frac{d+1}{2}+1} \frac{1}{r!} (\frac{n^2}{4})^r \frac{\Gamma(s+r)}{\Gamma(s)} \zeta(2s+2r-l, 1+\frac{n}{2}) \\
&+ \sum_{l=0}^{d} \binom{d}{l} (-\frac{n}{2})^{d-l} \\
&\quad \times \sum_{\substack{r \le \frac{d+1}{2}+1 \\ 2r-l \neq 1}} \frac{1}{r!} (\frac{n^2}{4})^r \frac{\Gamma(s+r)}{\Gamma(s)} \zeta(2s+2r-l, 1+\frac{n}{2}),
\end{aligned} \tag{4}
$$

where the first term gives the principal part which does not appear when $d = 0$.

Proof. First use the binomial theorem to expand $k^d = (k + \frac{n}{2} - \frac{n}{2})^d$ to get

$$
H_d(s) = \sum_{l=0}^{d} \binom{d}{l} (-\frac{n}{2})^{d-l} G_l(s), \tag{5}
$$

where

$$G_l(s) = \sum_{k=1}^{\infty} \frac{(k+\frac{n}{2})^l}{((k+\frac{n}{2})^2 - \frac{n^2}{4})^s}. \tag{6}$$

Factoring the power of $k+\frac{n}{2}$ out and applying the binomial theorem again to the remaining factor $(1-(\frac{n}{2k+n})^2)^{-s}$, we deduce that

$$G_l(s) = \sum_{k=1}^{\infty}(k+\frac{n}{2})^{-2s+l}\sum_{r=0}^{\infty}\frac{\Gamma(s+r)}{r!\Gamma(s)}(\frac{n}{2k+n})^{2r}. \tag{7}$$

Changing the order of summation, permissible by absolute convergence for $\sigma > \frac{l+1}{2}$, we obtain the desired form of $G_l(s)$.

Once (3) is established, it is enough to show that the infinite series in the second term of (4) is absolutely convergent for $\sigma > -1$. This is the case because $\zeta(2s+2r-l; 1+\frac{n}{2}) \ll (1+\frac{n}{2})^{-2r}$ and so $\sum_{r>\frac{d+1}{2}+1} \ll \sum_r \{(\frac{n}{n+2})^2\}^r < \infty$ uniformly in $\sigma > -1$.

(7) can be proved by quite elementary means, similar to the spirit of proof of analytic continuation of $Z(s)$ by Egami.

Lemma 4 (cf. Proposition 3.1 of Vardi [15]).

$$\begin{aligned} H_d'(0) = H_{n,d}'(0) &= \sum_{k=1}^{n}(k-n)^d \log k - \frac{1}{2^d}\frac{(-n)^{d+1}}{d+1}\sum_{\substack{l=1 \\ 2\nmid l}}^{d}\binom{d+1}{l+1}\sum_{\substack{j=1 \\ 2\nmid j}}^{l}\frac{1}{j} \\ &\quad +\zeta'(-d) + (-n)^d\sum_{r=0}^{d}\binom{d}{r}\frac{\zeta'(-r)}{(-n)^r}. \end{aligned}$$

We split the proof of Lemma 4 into a few sublemmas. Lemma 5 gives a more detailed decomposition of $H_d(s)$ than that in Lemma 3, and as its corollary gives a handy formula for $H_d'(s)$. Then in Lemma 6 we obtain by limit process a closed form evaluation of $H_d'(0)$, and in Lemma 7 we collect auxiliary formulas that enable us to simplify the formula in Lemma 6.

Lemma 5. For $\sigma > -1$, we have

$$\begin{aligned} \text{(i)}\quad H_d(s) &= \sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}(\zeta(2s-l, 1+\frac{n}{2}) \\ &\qquad + \sum_{r=1}^{\infty}\frac{(\frac{n}{2})^r}{r!}\frac{\Gamma(s+r)}{\Gamma(s)}\zeta(2s-l+r, 1+\frac{n}{2})) \\ &+ \sum_{l=0}^{d}\binom{d}{l}\sum_{m=1}^{\infty}(-\frac{n}{2})^{d-l+m}\frac{\Gamma(s+m)}{m!\Gamma(s)}\zeta(2s-l+m, 1+\frac{n}{2}) \\ &+ \sum_{l=0}^{d}\binom{d}{l}\sum_{m=1}^{\infty}(-\frac{n}{2})^{d-l+m}\frac{\Gamma(s+m)}{m!\Gamma(s)} \\ &\qquad \times\sum_{r=1}^{\infty}\frac{(\frac{n}{2})^r}{r!}\frac{\Gamma(s+r)}{\Gamma(s)}\zeta(2s-l+r+m, 1+\frac{n}{2}). \end{aligned}$$

$$
\begin{aligned}
\text{(ii)}\quad H_d'(s) &= \sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}(2\zeta'(2s-l,1+\frac{n}{2})\\
&\quad+\sum_{r=1}^{\infty}\frac{(-\frac{n}{2})^r}{r!}\frac{\Gamma(s+r)}{\Gamma(s)}((\psi(s+r)-\psi(s))\zeta(2s-l+r,1+\frac{n}{2})\\
&\quad+2\zeta'(2s-l+r,1+\frac{n}{2})))\\
&+\sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}\\
&\quad\times\sum_{r=1}^{\infty}\frac{(-\frac{n}{2})^r}{r!}\frac{\Gamma(s+r)}{\Gamma(s)}((\psi(s+r)-\psi(s))\zeta(2s-l+r,1+\frac{n}{2})\\
&\quad+2\zeta'(2s-l+r,1+\frac{n}{2}))\\
&+\sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}\sum_{m=1}^{\infty}\frac{(-\frac{n}{2})^m}{m!}\frac{\Gamma(s+m)}{\Gamma(s)}(\psi(s+m)-\psi(s))\\
&\quad\times\sum_{r=1}^{\infty}\frac{(\frac{n}{2})^r}{r!}\frac{\Gamma(s+r)}{\Gamma(s)}\zeta(2s-l+r+m,1+\frac{n}{2})\\
&+\sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}\sum_{m=1}^{\infty}\frac{(-\frac{n}{2})^m}{m!}\frac{\Gamma(s+m)}{\Gamma(s)}\\
&\quad\times\sum_{r=1}^{\infty}\frac{(\frac{n}{2})^r}{r!}\frac{\Gamma(s+r)}{\Gamma(s)}((\psi(s+r)-\psi(s))\\
&\quad\times\zeta(2s-l+r+m,1+\frac{n}{2})+2\zeta'(2s-l+r+m,1+\frac{n}{2})).
\end{aligned}
$$

Proof. To prove (i) we proceed as in the proof of Lemma 3. After factoring out the factor $(k+\frac{n}{2})^{l-2s}$, we decompose the remaining factor $(1-(\frac{n}{2}\frac{1}{k+\frac{n}{2}})^2)^{-s}$ as $(1-\frac{n}{2}\frac{1}{k+\frac{n}{2}})^{-s}(1-(-\frac{n}{2})\frac{1}{k+\frac{n}{2}})^{-s}$ and apply the binomial theorem to each factor to obtain

$$
(8)\qquad G_l(s)=\sum_{r=0}^{\infty}\frac{\Gamma(s+r)}{r!\Gamma(s)}(\frac{n}{2})^r\frac{1}{(k+\frac{n}{2})^r}\sum_{m=0}^{\infty}\frac{\Gamma(s+m)}{m!\Gamma(s)}(-\frac{n}{2})^m\frac{1}{(k+\frac{n}{2})^m}
$$

in place of (7). Substituting this in (3) and changing the order of summation, we deduce that

$$
\begin{aligned}
(9)\qquad H_d(s) &= \sum_{l=0}^{d}\binom{d}{l}\sum_{m=0}^{\infty}(-\frac{n}{2})^{d-l+m}\frac{\Gamma(s+m)}{m!\Gamma(s)}\\
&\quad\times\sum_{r=0}^{\infty}\frac{(\frac{n}{2})^r}{r!}\frac{\Gamma(s+r)}{\Gamma(s)}\zeta(2s-l+r+m,1+\frac{n}{2}),
\end{aligned}
$$

the process being legitimate by absolute convergence.

The term with $m=0$ on the RHS of (9) gives the first term of (i) after extracting the term with $r=0$, while the remaining sum $\sum_{m=1}^{\infty}$ gives the second and third terms of (i) when rewriting it as the sum with $r=0$ and one with $r\geq 1$.

The formula (ii) follows directly from (i) by applying Leibniz's rule and noting the identity

$$(\frac{\Gamma(s+m)}{\Gamma(s)})' = \frac{\Gamma(s+m)}{\Gamma(s)}(\psi(s+m)-\psi(s)),$$

thereby completing the proof.

Lemma 6. It holds that

$$\begin{aligned} H_d'(0) &= \sum\nolimits^+ + \sum\nolimits^- \\ &+ \sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}\frac{1}{2}(\frac{n}{2})^{l+1}\sum_{m=1}^{l}\frac{(-1)^m}{m(l+1-m)}, \end{aligned}$$

where

$$\begin{aligned} \sum\nolimits^+ &= \sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}(2\zeta'(-l,1+\frac{n}{2}) \\ &+ \sum_{\substack{r=1\\ r\neq l+1}}^{\infty}\frac{(\frac{n}{2})^r}{r}\zeta(r-l,1+\frac{n}{2}) + \frac{(\frac{n}{2})^{l+1}}{l+1}\frac{1}{2}\sum_{j=1}^{l}\frac{1}{j}), \end{aligned}$$

and

$$\begin{aligned} \sum\nolimits^- &= \sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}(\sum_{\substack{m=1\\ m\neq l+1}}^{\infty}\frac{(-\frac{n}{2})^m}{m}\zeta(m-l,1+\frac{n}{2}) \\ &+\frac{(-\frac{n}{2})^{l+1}}{l+1}\frac{1}{2}\sum_{j=1}^{l}\frac{1}{j}) \end{aligned}$$

with $\gamma(\alpha) = -\psi(\alpha)$ denoting the first generalized Euler constant for the Hurwitz zeta-function.

Proof. This can be proved in principle by substituting (then most terms vanish because of the zero of $\Gamma(s)^{-1}$ at $s=0$) the value $s=0$ in Lemma 5 (ii), but in some terms involving $\zeta(2s+1,1+\frac{n}{2})$, we cannot substitute the value and we are to take limit as $s\to 0$. Using the expansions near $s=0$

$$\begin{aligned} \zeta(s+1,\alpha) &= \frac{1}{s}+\gamma(\alpha)+O(s), \\ \zeta'(s+1,\alpha) &= -\frac{1}{s^2}+O(1), \\ \Gamma(s)^{-1} &= s+O(s^2), \end{aligned}$$

we see that

$$\begin{aligned} &\frac{1}{\Gamma(s)}((\psi(s+l)-\psi(s))\zeta(2s+1,1+\frac{n}{2})+2\zeta'(2s+1,1+\frac{n}{2})) \\ =\ &(s+O(s^2))((\frac{1}{s+l}+\cdots+\frac{1}{s+1})(\frac{1}{2s}+\gamma(1+\frac{n}{2})+O(s)) \end{aligned}$$

$$
\begin{aligned}
& +2(-\frac{1}{4s^2}+O(1))) \\
= & \frac{1}{2}(\frac{1}{s+l}+\cdots+\frac{1}{s+1})+O(s) \\
\to & \frac{1}{2}\sum_{j=1}^{l}\frac{1}{j}, \quad s\to 0.
\end{aligned}
$$

Denoting the 4 terms on the RHS of (ii) by $S_1 - S_4$, we see that in S_1, S_2, and S_4 the above process applies to the terms with $r = l+1$, while other terms in S_1 and S_2 are of the form

$$\frac{(\frac{n}{2})^r}{r}\zeta(r-l,1+\frac{n}{2}),\quad \frac{(-\frac{n}{2})^r}{r}\zeta(r-l,1+\frac{n}{2}),$$

in view of

$$\lim_{s\to 0}\frac{\Gamma(s+r)}{\Gamma(s)}(\psi(s+r)-\psi(s)) = (r-1)!\ .$$

And S_4 vanishes since the inner infinite series has a definite value as $s\to 0$ and there is a factor $\Gamma(s)^{-1}$, which is zero for $s=0$.

Regarding S_3, we note that the non-zero contributions come only from those terms with $r = l+1-m$, where $m\le l$. Take the limit as $s\to 0$ of the sum

$$
\begin{aligned}
& \sum_{m=1}^{l}\frac{(-\frac{n}{2})^m}{m!}\frac{\Gamma(s+m)}{\Gamma(s)}(\psi(s+m)-\psi(s)) \\
\times & \frac{(\frac{n}{2})^{l+1-m}}{(l+1-m)!}\frac{\Gamma(s+l+1-m)}{\Gamma(s)}\zeta(2s+1,1+\frac{n}{2}),
\end{aligned}
$$

using (ii) and $\frac{1}{\Gamma(s)}\zeta(2s+1,1+\frac{n}{2})\to\frac{1}{2}$ as $s\to 0$. Then we see that this is

$$\frac{1}{2}\sum_{m=1}^{l}\frac{(-\frac{n}{2})^m}{m}\frac{(\frac{n}{2})^{l+1-m}}{(l+1-m)},$$

so that this gives the third term for $H_d'(0)$. This completes the proof.

Lemma 7. We have the evaluation

$$
\begin{aligned}
\text{(i)}\quad & \sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}\sum_{m=0}^{l}\binom{l}{m}\zeta'(-m)(\frac{n}{2})^{l-m} = \zeta'(-d) \\
\text{(ii)}\quad & \sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}\sum_{m=0}^{l}\binom{l}{m}\zeta'(-m,1+n)(-\frac{n}{2})^{l-m} \\
= & \sum_{m=0}^{d}\binom{d}{m}(-n)^{d-m}\zeta'(-m,1+n) \\
\text{(iii)}\quad & -\frac{1}{2}\sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}\frac{1}{l+1}((\frac{n}{2})^{l+1}+(-\frac{n}{2})^{l+1})\sum_{j=1}^{l}\frac{1}{j} \\
= & \frac{1}{2}(-\frac{n}{2})^{d+1}\sum_{l=1}^{d}\frac{1}{l+1}\binom{d}{l}((-1)^l-1)\sum_{j=1}^{l}\frac{1}{j}
\end{aligned}
$$

$$\text{(iv)} \qquad -\frac{1}{2}(-\frac{n}{2})^{d+1}\sum_{l=1}^{d}(-1)^l\binom{d}{l}\sum_{m=1}^{l}\frac{(-1)^m}{m(1+l-m)}$$
$$= \frac{1}{2}(-\frac{n}{2})^{d+1}\sum_{l=1}^{d}\frac{1}{l+1}\binom{d}{l}((-1)^l-1)\sum_{j=1}^{l}\frac{(-1)^{j+1}}{j}$$

Proof of Lemma 4. Substituting from Lemma 1, we see that

$$\begin{aligned}\sum^+ + \sum^- &= \sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}(-\frac{1}{2}\frac{(\frac{n}{2})^{l+1}+(-\frac{n}{2})^{l+1}}{l+1}\sum_{j=1}^{l}\frac{1}{j})\\ &+\sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}\sum_{m=0}^{l}\binom{l}{m}\zeta'(-m)(\frac{n}{2})^{l-m}\\ &+\sum_{l=0}^{d}\binom{d}{l}(-\frac{n}{2})^{d-l}\sum_{m=0}^{l}\binom{l}{m}\zeta'(-m,1+n)(\frac{n}{2})^{l-m}.\end{aligned}$$

Using Lemma 7, we get

$$\begin{aligned}\sum^+ + \sum^- &= \zeta'(-d)+\sum_{m=0}^{d}\binom{d}{l}(-n)^{d-m}\zeta'(-m,1+n)\\ &+\frac{1}{2}(-\frac{n}{2})^{d+1}\sum_{l=1}^{d}\frac{1}{l+1}\binom{d}{l}((-1)^l-1)\sum_{j=1}^{l}\frac{1}{j},\end{aligned}$$

the last term of which together with the last sum on the RHS of the formula in Lemma 6 gives the second term on the RHS of the formula in Lemma 4. Hence

$$\begin{aligned}H_d'(0) &= \zeta'(-d)+\sum_{m=0}^{d}\binom{d}{m}(-n)^{d-m}\zeta'(-m,1+n)\\ &-\frac{2}{d+1}(-\frac{n}{2})^{d+1}\sum_{\substack{l=1\\ 2\nmid l}}^{d}\binom{d+1}{l+1}\sum_{\substack{j=1\\ 2\nmid j}}^{l}\frac{1}{j}.\end{aligned}$$

Expressing $\zeta'(-m,1+n)=\zeta'(-m)+\sum_{r=2}^{n}r^m\log r$ completes the proof.

Proof of Theorem 2. (i) is a restatement of Theorem 1 and is a corrected form of Theorem 1.4 of Vardi [15], while (ii) follows from (i) and Theorem 1.1 of Vardi [15], and is a corrected form of Theorem 1.3 of Vardi [15].

3 Remarks

Remark 1. The unit circle S^0. The determinant $\det\Delta_1$ of S^0 with standard Laplacian $\Delta_1=\frac{d^2}{dx^2}$ is

$$\det\Delta_1=(2\pi)^2.$$

Remark 2. The unit disc S^1. Vardi's formula for $F_2'(0)$ in Proposition 4.4 is correct. As a matter of fact, the proof of Theorem 1.4 ($p.505$) gives the incorrect value of $F_2'(0)$:

$$F_2'(0)=4\zeta'(-1)+\zeta'(0)-\frac{1}{2},$$

but in the statement of Proposition 4.4, the author omitted $\zeta'(0)$ by mistake to give a correct value. Accordingly, the formulas containing $F_2'(0)$ would have been as follows. The second formula in Proposition 4.5 would read

$$e^{\zeta'(-1)} = (\det \Delta_2)^{-\frac{1}{4}} e^{\frac{1}{8}}, \tag{10}$$

the formula for $\Gamma_2(\frac{1}{2})$ in Theorem 1.1 would read

$$\Gamma_2(\frac{1}{2}) = (\det \Delta_2)^{\frac{3}{8}} (\det \Delta_1)^{\frac{1}{8}} 2^{-\frac{11}{16}} e^{-\frac{1}{16}} \pi^{-\frac{3}{16}} \tag{11}$$

and that in Theorem 1.2 would read

$$\det \Delta_2 = \Gamma_2(\frac{1}{2})^{\frac{8}{3}} 2^{\frac{1}{2}} (\frac{e}{\pi})^{\frac{1}{16}}. \tag{12}$$

The formulas for $\det \Delta_2$ as given in Vardi (except for the one in Theorem 1.2 in which the factor $\pi^{-\frac{2}{3}}$ is missing) are in conformity with the formulas of Weisberger [18], [19] and of Choi [1], [2] and Quine and Choi [12]. The error in Vardi arises from the erroneous argument in Proposition 3.1 which lacks the evaluation of the infinite series $B(0)$, which looks rather difficult. In the statement of Proposition 3.1 this term $B(0)$ is missing, but the value given there is very close to the correct one.

Choi's argument in the case $n = 2$ [1] follows exactly Voros' and gives the correct value of $\det \Delta_2$. His statement on $p.166$ [1] is rather misleading because he says there that his value coincides with that of Vardi. However, Choi's correct value does not coincide with (10), but rather with the value given in Theorem 1 (Corollary (ii)).

Remark 3. The unit sphere.

The only correct existing formula is Choi's main theorem [1], [2] and Quine and Choi [12]. Choi's method uses the shifted generating Dirichlet series process of Voros', which requires a considerable amount of calculation with sophisticated multiple gamma function, and it looks rather hopeless to go on further to higher dimensions with Voros' method. Actually, the proof occupies the main body of Choi's thesis.

Vardi's general closed formula in Theorems 1.1 and 1.2 are wrong. Choi's remark in [1] was again rather misleading in that he calculates the same value (unknown in the literature) in two ways using both his results and Vardi's results, and concludes that they give different values. This leaves a possibility that both might be wrong, but this defect was rescued in [9], and Choi's result for $\det \Delta_3$ is correct and coincides with ours.

Since our closed formula (Theorem) gives correct values for both $\det \Delta_2$ and $\det \Delta_3$, it is of considerable trust.

Remark 4. Higher dimensions.

After presenting our results at the Japan-Korea Number Theory Conference, Dec. 24-27,1997 held at Saga University, we were communicated the paper of Quine and Choi [12], which gives closed formula for $\det \Delta_n$ for any n. Their method avoids the computation of the infinite series involving the Hurwitz zeta-function (Lemma 1) by an ingenious trick of introducting a regularization lemma (Lemma 1), in which cancellation of terms in our

Lemma 6 is effected by showing $G'(0) = 0$, and the proof is subtler than ours. We believe, however, our method has its own right, clarifying how those cancel one another.

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