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ON STARLIKENESS AND CONVEXITY OF FUNCTIONS AND THE SCHWARZIAN DERIVATIVE

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ABSTRACT. The purpose of this paper is to generalize Miller and Mocanu's result [2].

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ defined by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C},\ \text{and } |z| < 1\}$. Also, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to the class $\mathcal{S}$ is said to be in the class $\mathcal{S}^*$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad \text{in} \quad \mathcal{U}$$

and is said to be in the class $\mathcal{C}$ if and only if

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{in} \quad \mathcal{U}.$$

We denote by $\{f, z\}$ the Schwarzian derivative, which is characterized by the equality

$$\{f, z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{1}$$

In [1], Nunokawa et al. obtained the following result:

Theorem A. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} + z^2\{f, z\}\right)\right] \geq -\frac{1}{2} \quad \textit{in} \quad \mathcal{U}. \tag{2}$$

Then $f(z) \in \mathcal{S}^*$.

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Remark. Theorem A is an extension of Miller and Mocanu [2], where the right hand side of (2) is improved from 0 to $-\frac{1}{2}$.

Further Miller and Mocanu [2] obtained the following results :

Theorem B. *Let $f(z) \in \mathcal{A}$ satisfy*

$$\operatorname{Re}\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^2 + z^2\{f,z\}\right] > 0 \qquad in \quad \mathcal{U}.$$

Then $f(z) \in \mathcal{C}$.

Theorem C. *Let $f(z) \in \mathcal{A}$ satisfy*

$$\operatorname{Re}\left[\left(1+\frac{zf''(z)}{f'(z)}\right) e^{z^2\{f,z\}}\right] > 0 \qquad in \quad \mathcal{U}.$$

Then $f(z) \in \mathcal{C}$.

Let us investigate improvements of these results in the next section.

2. Main Results

The following result will be required in our investigation :

Lemma. [3] *Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and suppose that there exists a point $z_0 \in \mathcal{U}$ such that* $\operatorname{Re}\{p(z)\} > 0$ *for* $|z| < |z_0|$ *and* $\operatorname{Re}\{p(z_0)\} = 0 \quad (p(z_0) \neq 0)$. *Then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where k is a real number and

$$k \geq \frac{1}{2}\left(a+\frac{1}{a}\right) \geq 1 \qquad when \quad p(z_0) = ia, \; a > 0,$$

and

$$k \leq \frac{1}{2}\left(a+\frac{1}{a}\right) \leq -1 \qquad when \quad p(z_0) = ia, \; a < 0.$$

Now we state our main result.

Theorem 1. *Let $f(z) \in \mathcal{A}$ and satisfy one of the following inequalities :*

$$\text{(3)} \qquad \operatorname{Re}\left[\left(\frac{zf'(z)}{f(z)}\right)^{4m-1}\left(1+\frac{zf''(z)}{f'(z)} + z^2\{f,z\}\right)\right] < \frac{1}{2}\left|\frac{zf'(z)}{f(z)}\right|^{4m-2}\left(3\left|\frac{zf'(z)}{f(z)}\right|^2 + 1\right) \qquad in \quad \mathcal{U},$$

$$\text{(4)} \qquad \operatorname{Re}\left[\left(\frac{zf'(z)}{f(z)}\right)^{4m-3}\left(1+\frac{zf''(z)}{f'(z)}+z^2\{f,z\}\right)\right]$$
$$> -\frac{1}{2}\left|\frac{zf'(z)}{f(z)}\right|^{4m-4}\left(3\left|\frac{zf'(z)}{f(z)}\right|^2+1\right) \qquad in \quad \mathcal{U},$$

where m is a positive integer. Then $f(z) \in \mathcal{S}^*$.

Proof. Let us put

$$p(z) = \frac{zf'(z)}{f(z)},$$

then we easily have

$$1+\frac{zf''(z)}{f'(z)} = p(z)+\frac{zp'(z)}{p(z)}$$

and from (1), by a simple calculation, we have

$$\text{(5)} \qquad z^2\{f,z\} = z^2\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{zf''(z)}{f'(z)}\right)^2$$
$$= \frac{zp'(z)}{p(z)}+\frac{z^2p''(z)}{p(z)}-\frac{3}{2}\left(\frac{zp'(z)}{p(z)}\right)^2+\frac{1}{2}\left\{1-p(z)^2\right\}.$$

To prove $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ in $\mathcal{U}$, we show $\operatorname{Re}\{p(z)\} > 0$ in $\mathcal{U}$. If there exists a point $z_0 \in \mathcal{U}$ such that

$$\operatorname{Re}\{p(z)\} > 0 \qquad \text{for} \quad |z| < |z_0|$$

and

$$\operatorname{Re}\{p(z_0)\} = 0 \quad (p(z_0) \neq 0),$$

then from Lemma we have

$$\frac{z_0p'(z_0)}{p(z_0)} = ik,$$

and (3), (4) and (5) imply

$$\text{(6)} \qquad \left(\frac{z_0f'(z_0)}{f(z_0)}\right)^l\left(1+\frac{z_0f''(z_0)}{f'(z_0)}+z_0^2\{f,z_0\}\right)$$
$$= (ia)^l\left[ia+ik+ik+\frac{z_0^2p''(z_0)}{p(z_0)}-\frac{3}{2}(ik)^2+\frac{1}{2}\left\{1-(ia)^2\right\}\right]$$
$$= (ia)^l\left[i\left\{a+k+k\left(1+\frac{z_0p''(z_0)}{p'(z_0)}\right)\right\}+\frac{3}{2}k^2+\frac{1}{2}\left(1+a^2\right)\right],$$

where l is a positive integer. Let J be the right hand side of (6). For the case $l = 2n-1$,

$$J = (ia)^{2n-1}\left[i\left\{a+k+k\left(1+\frac{z_0p''(z_0)}{p'(z_0)}\right)\right\}+\frac{3}{2}k^2+\frac{1}{2}\left(1+a^2\right)\right].$$

Therefore we have

$$\operatorname{Re}\{J\} = (-1)^n\, a^{2n-1}\left[a + k + k\left(1 + \operatorname{Re}\left\{\frac{z_0 p''(z_0)}{p'(z_0)}\right\}\right)\right].$$

Considering the geometrical property, we notice that the tangential vector of the curve $p(z) = p(z_0)$ moves to positive direction near the point $p(z_0)$. In short, $p(z)$ is convex in the neighborhood of the point $p(z_0)$, or

$$1 + \operatorname{Re}\left\{\frac{z_0 p''(z_0)}{p'(z_0)}\right\} \geq 0.$$

(i) Case $n = 2m$:

$$\begin{aligned}
\operatorname{Re}\{J\} &= (-1)^{2m}\, a^{4m-1}\left[a + k + k\left(1 + \operatorname{Re}\left\{\frac{z_0 p''(z_0)}{p'(z_0)}\right\}\right)\right] \\
&\geq a^{4m-1}(a + k) \\
&= -a^{4m-2}\left(-a^2 - ak\right) \\
&\geq -a^{4m-2}\left\{-a^2 - \frac{1}{2}\left(a^2 + 1\right)\right\} \\
&= -a^{4m-2}\left(-\frac{3}{2}a^2 - \frac{1}{2}\right) \\
&= \frac{1}{2}\left|\frac{z_0 f'(z_0)}{f(z_0)}\right|^{4m-2}\left(3\left|\frac{z_0 f'(z_0)}{f(z_0)}\right|^2 + 1\right).
\end{aligned}$$

(ii) Case $n = 2m - 1$:

$$\begin{aligned}
\operatorname{Re}\{J\} &= (-1)^{2m-1}\, a^{4m-3}\left[a + k + k\left(1 + \operatorname{Re}\left\{\frac{z_0 p''(z_0)}{p'(z_0)}\right\}\right)\right] \\
&\leq -a^{4m-3}(a + k) \\
&= a^{4m-4}\left(-a^2 - ak\right) \\
&\leq a^{4m-4}\left\{-a^2 - \frac{1}{2}\left(a^2 + 1\right)\right\} \\
&= a^{4m-4}\left(-\frac{3}{2}a^2 - \frac{1}{2}\right) \\
&= -\frac{1}{2}\left|\frac{z_0 f'(z_0)}{f(z_0)}\right|^{4m-4}\left(3\left|\frac{z_0 f'(z_0)}{f(z_0)}\right|^2 + 1\right).
\end{aligned}$$

These contradict (3) and (4), respectively. Hence we must have

$$\operatorname{Re}\{p(z)\} > 0 \quad \text{in } \mathcal{U}$$

or

$$\operatorname{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > 0 \quad \text{in } \mathcal{U},$$

which means $f(z) \in \mathcal{S}^*$. This completes our proof.

Setting $\alpha = 1$ in Theorem 1, we obtain

Corollary 1. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}+z^2\{f,z\}\right)\right] > -\frac{1}{2}\left(1+3\left|\frac{zf'(z)}{f(z)}\right|^2\right) \qquad in \quad \mathcal{U}.$$

Then $f(z) \in \mathcal{S}^*$.

Corollary 1 is better than Theorem A.

Theorem 2. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\text{(7)} \qquad \operatorname{Re}\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^{2n}+z^2\{f,z\}\right]+(-1)^{n+1}\left|1+\frac{zf''(z)}{f'(z)}\right|^{2n} > 0 \qquad in \quad \mathcal{U}$$

for positive integer n. Then $f(z) \in \mathcal{C}$.

Proof. Let us put

$$q(z) = 1+\frac{zf''(z)}{f'(z)}.$$

Note that $q(0) = 1$. Then from (1), we easily have

$$\text{(8)} \qquad z^2\{f,z\} = zq'(z) - \frac{1}{2}q(z)^2 + \frac{1}{2}.$$

To prove $1+\operatorname{Re}\{zf''(z)/f'(z)\} > 0$ in $\mathcal{U}$, we show $\operatorname{Re}\{q(z)\} > 0$ in $\mathcal{U}$. If there exists a point $z_0 \in \mathcal{U}$ such that

$$\operatorname{Re}\{q(z)\} > 0 \qquad \text{for} \quad |z| < |z_0|$$

and

$$\operatorname{Re}\{q(z_0)\} = 0 \quad (q(z_0) \neq 0),$$

then from Lemma a real number k ($k \neq 0$) exists such that

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik.$$

From (7) and (8), we have

$$\begin{aligned}
&\operatorname{Re}\left[\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right)^{2n}+{z_0}^2\{f,z_0\}\right]+(-1)^{n+1}\left|1+\frac{z_0 f''(z_0)}{f'(z_0)}\right|^{2n} \\
&= \operatorname{Re}\left\{(q(z_0))^{2n} + z_0 q'(z_0) - \frac{1}{2}q(z_0)^2 + \frac{1}{2}\right\} + (-1)^{n+1}\left|q(z_0)\right|^{2n} \\
&= \operatorname{Re}\left\{(ia)^{2n} - ak - \frac{1}{2}(ia)^2 + \frac{1}{2}\right\} + (-1)^{n+1}\left|ia\right|^{2n} \\
&\leq (-1)^n a^{2n} - \frac{1}{2}\left(a^2+1\right) + \frac{1}{2}\left(a^2+1\right) + (-1)^{n+1}\left|a\right|^{2n} \\
&= 0.
\end{aligned}$$

This is in contradiction to (7). Hence we must have

$$\operatorname{Re}\{q(z)\} > 0 \quad \text{in} \quad \mathcal{U}$$

or

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{in} \quad \mathcal{U}.$$

Therefore $f(z) \in \mathcal{C}$ and our result is established.

Taking $n = 1$ in Theorem 2, we have

Corollary 2. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\operatorname{Re}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right)^2 + z^2\{f,z\}\right] + \left|1 + \frac{zf''(z)}{f'(z)}\right|^2 > 0 \quad \textit{in} \quad \mathcal{U}.$$

Then $f(z) \in \mathcal{C}$.

Corollary 2 is better than Theorem B.

Theorem 3. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\operatorname{Re}\left[\left(1 + \frac{zf''(z)}{f'(z)}\right)^{2n-1} e^{z^2\{f,z\}}\right] \neq 0 \quad \textit{in} \quad \mathcal{U}.$$

Then $f(z) \in \mathcal{C}$.

Proof. Let us take the same function $q(z)$ as in the proof of Theorem 2. Then from the assumption of theorem and (8), we find

$$\begin{aligned}
&\operatorname{Re}\left[\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right)^{2n-1} e^{z_0{}^2\{f,z_0\}}\right] \\
&= \operatorname{Re}\left[(q(z_0))^{2n-1}\, e^{z_0 q'(z_0) - \frac{1}{2}q(z_0)^2 + \frac{1}{2}}\right] \\
&= \operatorname{Re}\left[(ia)^{2n-1}\, e^{-ak + \frac{1}{2}a^2 + \frac{1}{2}}\right] \\
&= \operatorname{Re}\left[i\,(-1)^{n+1}\, a^{2n-1} e^{-ak + \frac{1}{2}a^2 + \frac{1}{2}}\right] \\
&= 0.
\end{aligned}$$

This is a contradiction to the assumption. Hence we must have

$$\operatorname{Re}\{q(z)\} > 0 \quad \text{in} \quad \mathcal{U}$$

or

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{in} \quad \mathcal{U},$$

which yields our result.

Putting $n = 1$ in Theorem 3, we have

Corollary 3. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\mathrm{Re}\left[\left(1+\frac{zf''(z)}{f'(z)}\right)e^{z^2\{f,z\}}\right] \neq 0 \qquad in \quad \mathcal{U}.$$

Then $f(z) \in \mathcal{C}$.

Corollary 3 is a revision of Theorem C.

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