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GROWTH AND COEFFICIENT ESTIMATES FOR UNIFORMLY LOCALLY UNIVALENT FUNCTIONS ON THE UNIT DISK

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ABSTRACT. In this note, we shall give a sharp growth estimate for a uniformly locally univalent holomorphic function on the unit disk. As applications, we shall investigate the growth of coefficients of such a function and mention the connection with Hardy spaces. We also give norm estimates for typical classes of univalent functions.

1. INTRODUCTION

We will call a holomorphic function f on the unit disk $\mathbb{D}$ *uniformly locally univalent* if f is univalent on each hyperbolic disk $D(a,\rho)=\{z\in\mathbb{D};|\frac{z-a}{1-\bar{a}z}|<\tanh\rho\}$ with radius ρ and center $a\in\mathbb{D}$ for a positive constant ρ. In particular, a holomorphic universal covering map of a plane domain D is uniformly locally univalent if and only if the boundary of D is uniformly perfect (cf. [12] or [17]). Also it is well-known (cf. [20]) that a holomorphic function f on the unit disk is uniformly locally univalent if and only if the pre-Schwarzian derivative (or nonlinearity) $T_f=f''/f'$ of f is hyperbolically bounded, i.e., the norm

$$\|T_f\|=\sup_{z\in\mathbb{D}}(1-|z|^2)|T_f(z)|$$

is finite. This quantity can be regarded as the Bloch norm of the function $\log f'$. Remark that a holomorphic function f is locally univalent at the point z if and only if $T_f=f''/f'$ is a well-defined holomorphic function near z. Roughly speaking, the quantity T_f measures the deviation of f from orientation-preserving similarities (non-constant linear functions).

Because T_f is invariant under the post-composition by a non-constant linear function, we may assume that a holomorphic function f on the unit disk is normalized so that $f(0)=0$ and $f'(0)=1$. We denote by $\mathcal{A}$ the set of such normalized holomorphic functions on the unit disk. And we denote by $\mathcal{B}$ the set of normalized uniformly locally univalent functions: $\mathcal{B}=\{f\in\mathcal{A};\|T_f\|<\infty\}$. The space $\mathcal{B}$ has a structure of non-separable complex Banach space under the Hornich operation ([19]).

For a non-negative real number λ we set

$$\mathcal{B}(\lambda)=\{f\in\mathcal{A};\|T_f\|\le 2\lambda\},$$

here the number 2 is due to some technical reason. The functions in $\mathcal{B}(\lambda)$ can be characterized as the following.

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Proposition 1.1. *Let a non-negative constant λ be given. A locally univalent function $f \in A$ belongs to $\mathcal{B}(\lambda)$ if and only if for any pair of points z_1, z_2 in $\mathbb{D}$ it holds that*

$$|g(z_1) - g(z_2)| \leq 2\lambda d_{\mathbb{D}}(z_1, z_2), \tag{1.1}$$

where $g(z) = \log f'(z)$ and $d_{\mathbb{D}}(z_1, z_2) = \tanh^{-1}|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}|$ stands for the hyperbolic distance between z_1 and z_2 in the unit disk $\mathbb{D}$.

Proof. First of all, note that we can take a holomorphic branch g of $\log f'$ for a locally univalent holomorphic function f on the unit disk. The "only if " part is shown by integrating the inequality $|g'(z)| = |T_f(z)| \leq 2\lambda/(1 - |z|^2)$ along the hyperbolic geodesic joining z_1 and z_2. The "if " part directly follows from the observation:

$$\lim_{z' \to z} \frac{|g(z') - g(z)|}{d_{\mathbb{D}}(z', z)} = (1 - |z|^2)|g'(z)|.$$

□

The following theorem is significant in connection with univalent function theory.

Theorem A (Becker and Pommerenke [3], [4]). *The set $\mathcal{S}$ of normalized univalent holomorphic functions on the unit disk is contained in $\mathcal{B}(3)$ and contains $\mathcal{B}(\frac{1}{2})$. The result is sharp.*

We note that the Schwarzian derivative S_f of f can be written as $S_f = (T_f)' - (T_f)^2/2$. Thus the space $\mathcal{B}$ has a close connection with (the Bers embedding of) the Teichmüller spaces. Especially, it is expected to be useful when considering the Bers boundary of the Teichmüller spaces since the quantity T_f is much easier to treat than S_f. In fact, the space $\mathcal{T}_1 := \{T_f; f \in \mathcal{S}$ has a quasiconformal extension to the Riemann sphere $\}$ can be regarded as a model of the universal Teichmüller space (cf. [1] and [23]).

Here, as a result in this direction, we mention the following.

Corollary . *For a constant $k \in [0,1)$, let $\mathcal{S}_k$ be the subset of $\mathcal{S}$ consisting of those functions which can be extended to k-quasiconformal self-mappings of the Riemann sphere $\widehat{\mathbb{C}}$. Then, we have*

$$\mathcal{B}(k/2) \subset \mathcal{S}_k.$$

This implication is easily obtained by the λ-lemma (see, for example, [13, p. 121]). This already (implicitly) appeared in the paper [3] by Becker.

2. Growth estimate for the class $\mathcal{B}(\lambda)$

In the class $\mathcal{B}(\lambda)$ for $0 \leq \lambda < \infty$ the function

$$F_\lambda(z) = \int_0^z \left(\frac{1+t}{1-t}\right)^\lambda dt$$

is extremal as we shall see later. We remark that $F_\lambda \in \mathcal{A}$ can be defined for any complex number λ and satisfies $T_{F_\lambda} = 2\lambda(1-z^2)^{-1}$, thus $\|T_{F_\lambda}\| = 2|\lambda|$. F_λ may provide an example of a function with small pre-Schwarzian norm which does not belong to typical classes of univalent functions when λ is sufficiently small and $\lambda \notin \mathbb{R}$.

In practice, it is important to know the univalence of F_λ.

Lemma 2.1. *For a non-negative number λ, the function F_λ is univalent in the unit disk if and only if $0 \le \lambda \le 1$.*

Proof. First, we compute the Schwarzian derivative S_{F_λ} of F_λ. Then, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^2|S_{F_\lambda}(z)| = \sup_{z\in\mathbb{D}}(1-|z|^2)^2\frac{2\lambda|2z-\lambda|}{|1-z^2|^2} = 2\lambda(\lambda+2).$$

In particular, if $1 < \lambda$, then $2\lambda(\lambda+2) > 6$, thus the Nehari-Kraus theorem implies that F_λ is not univalent.

On the other hand, if $0 \le \lambda \le 1$, we have $\mathrm{Re}F_\lambda'(z) > 0$ in the unit disk, hence the Noshiro-Warschawski theorem ensures the univalence of F_λ in this case. □

The following result is elementary and might be known. But we shall include the proof because of its importance for our aim.

Theorem 2.2 (Distortion Theorem). *Let λ be a non-negative real number. For an $f \in \mathcal{B}(\lambda)$ it holds that*

$$F_\lambda'(-|z|) = \left(\frac{1-|z|}{1+|z|}\right)^\lambda \le |f'(z)| \le \left(\frac{1+|z|}{1-|z|}\right)^\lambda = F_\lambda'(|z|), \quad \text{and} \tag{2.1}$$

$$|f(z)| \le F_\lambda(|z|) \tag{2.2}$$

in the unit disk. Furthermore, if f is univalent then

$$-F_\lambda(-|z|) \le |f(z)| \le F_\lambda(|z|). \tag{2.3}$$

If the equality occurs in any of the above inequalities at some point $z_0 \ne 0$, then f must be a rotation of F_λ, i.e., $f(z) = \bar{\mu}F_\lambda(\mu z)$ for a unimodular constant μ.

Proof. Applying Proposition 1.1 in the case of $z_1 = z$ and $z_2 = 0$, we see

$$|\log f'(z)| \le \lambda \log\frac{1+|z|}{1-|z|}. \tag{2.4}$$

Taking the real part of $\log f'$, we obtain (2.1). And the integration of (2.1) yields (2.2). The inequality (2.3) can be shown by the same method as in the proof of the Koebe distortion theorem. The equality cases are obvious. (Note that the inequality (2.3) is sharp only for $\lambda \le 1$ by Lemma 2.1.) □

Since $\int_0^1(\frac{1+t}{1-t})^\lambda dt < \infty$ for $\lambda < 1$ and $\int_0^r(\frac{1+t}{1-t})^\lambda dt \le \frac{2^\lambda}{\lambda-1}(1-r)^{1-\lambda}$ for $\lambda > 1$, we have the following

Corollary 2.3. *For $\lambda > 1$ any $f \in \mathcal{B}(\lambda)$ satisfies the growth condition*

$$f(z) = O(1-|z|)^{1-\lambda}$$

as $|z| \to 1$. Furthermore, if f is univalent, then $f(\mathbb{D})$ contains the disk $\{|z| < -F_\lambda(-1)\}$. This constant $-F_\lambda(-1)$ is best possible for $0 \le \lambda \le 1$.

On the other hand, for $\lambda < 1$, a function $f \in \mathcal{B}(\lambda)$ is always bounded with a uniform bound $F_\lambda(1)$.

We note again that for $\lambda \le 1/2$ the function $f \in \mathcal{B}(\lambda)$ must be univalent. We also note that, for $0 \le \lambda \le 1$, we have $-F_\lambda(-1) \le -F_1(-1) = 2\log 2 - 1 = 0.38629\cdots$, therefore the result above is better than the Koebe one-quarter theorem.

Remark . By using the integral representation of the Gauss hypergeometric function (cf. Rainville [15] p.47, Theorem 16),

$$\begin{aligned}\frac{F_\lambda(z)}{z} &= \int_0^1 \left(\frac{1+tz}{1-tz}\right)^\lambda dt \\ &= \sum_{k=0}^{\infty} \binom{\lambda}{k} z^k \int_0^1 t^k(1-tz)^{-\lambda} dt \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\lambda+k)}{(k+1)!\Gamma(\lambda)} z^k F(-\lambda, k+1; k+2; -z),\end{aligned}$$

where $F(a,b;c;z)$ denotes the Gauss hypergeometric function. Also, the values $F_\lambda(1)$ and $-F_\lambda(-1)$ can be expressed in terms of the Gauss hypergeometric function. For example, by [14] p.491,

$$\begin{aligned}-F_\lambda(-1) &= \int_0^1 \left(\frac{1-t}{1+t}\right)^\lambda dt = \frac{1}{\lambda+1} F(1, \lambda; \lambda+2; -1) \\ &= \frac{1}{2^\lambda(\lambda+1)} F(\lambda, \lambda+1; \lambda+2; 1/2) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\lambda+k)}{k!(\lambda+k+1)\Gamma(\lambda)2^{\lambda+k}},\end{aligned}$$

which may also be rewritten in terms of the difference of two Digamma functions ([14], p.489, Eq.12) :

$$-F_\lambda(-1) = \lambda\left[\psi\left(\frac{\lambda+1}{2}\right) - \psi\left(\frac{\lambda}{2}\right)\right] - 1 \qquad \left(\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}\right).$$

Similarly, we have $F_\lambda(1) = \lambda[\psi(-\lambda/2) - \psi((1+\lambda)/2)] - 1$. It may be useful to note the following elementary estimate:

$$\frac{1}{(\lambda+1)2^\lambda} < -F_\lambda(-1) < \frac{1}{\lambda+1}.$$

In the above theorem, the case $\lambda = 1$ is critical. In this case, by Theorem 2.2, we can see that for $f \in \mathcal{B}(1)$

$$|f(z)| \le F_1(|z|) = 2\log\frac{1}{1-|z|} - |z|.$$

In particular, a function in $\mathcal{B}(1)$ need not be bounded (for instance, F_1). The next proposition gives a boundedness criterion for functions in $\mathcal{B}(1)$.

Proposition 2.4. *If a holomorphic function f on the unit disk satisfies that*

$$\beta(f) := \varlimsup_{|z|\to 1-0} \left\{(1-|z|^2)|T_f(z)| - 2\right\} \log\frac{1}{1-|z|^2} < -2 \tag{2.5}$$

then f is bounded. Here, the constant -2 in the right hand side is sharp.

Proof. By assumption, there exists a $\beta < -2$ such that the left-hand side in (2.5) is less than β. Thus, for some $0 < r_0 < 1$, $(1-|z|^2)|T_f(z)| - 2 \le \frac{\beta}{\log\frac{1}{1-|z|^2}}$, i.e.,

$$|T_f(z)| \le \frac{2}{1-|z|^2} + \frac{\beta}{(1-|z|^2)\log\frac{1}{1-|z|^2}} \tag{2.6}$$

for any $z \in \mathbb{C}$ with $r_0 < |z| < 1$. Here, we may choose r_0 sufficiently close to 1 so that $1 - r_0^2 < e^{-1}$ and that $\beta_1 := (1+r_0)\beta/2 < -2$.

Integrating the inequality (2.6), we see that, for $|z| > r_0$,

$$\begin{aligned} |\log f'(z)| &\le \log\frac{1+|z|}{1-|z|} + \int_{r_0}^{|z|} \frac{\beta dt}{(1-t^2)\log\frac{1}{1-t^2}} + C_1 \\ &\le \log\frac{1+|z|}{1-|z|} + \int_{r_0}^{|z|} \frac{\beta_1 dt}{2(1-t)\log\frac{1}{2(1-t)}} + C_1 \\ &= \log\frac{1-|z|}{1+|z|} + \frac{\beta_1}{2}\log\log\frac{1}{2(1-|z|)} + C_2, \end{aligned}$$

where C_1 and C_2 are constants depending only on f and r_0. In particular, we have

$$|f'(z)| \le e^{C_2}\frac{1+|z|}{1-|z|}\left(\log\frac{1}{2(1-|z|)}\right)^{\beta_1/2}$$

Since $\beta_1/2 < -1$ the function $\frac{1+t}{1-t}(\log\frac{1}{2(1-t)})^{\beta_1/2}$ is integrable on the interval $[r_0, 1)$. Thus f is bounded.

The sharpness follows from the example below. □

Example 2.1. Let a constant $\beta < 0$ be given. Choose a constant $c > 0$ so that $c\beta + 2 \ge 0$. Now we consider the function $f \in \mathcal{A}$ determined by

$$f'(z) = \frac{K}{1-z}\left(1 + c\log\frac{2}{1-z}\right)^{\beta},$$

where $K = (1 + c\log 2)^{-\beta}$. Then this function satisfies that $\|T_f\| = 2$. And moreover, f is bounded in the uint disk if and only if $\beta < -1$.

In fact, first observe that

$$T_f(z) = \frac{1}{1-z} + \frac{c\beta}{(1-z)(1+c\log\frac{2}{1-z})} = \frac{1}{1-z}\left[1 + \frac{\beta}{\frac{1}{c} + \log\frac{2}{1-z}}\right].$$

By the fact that $\mathrm{Re}\frac{2}{1-z} > 1$, one can conclude that $\mathrm{Re}w > \frac{1}{c} \ge -\beta/2$ and $|\mathrm{Im}w| < \pi/2$, where $w = \frac{1}{c} + \log\frac{2}{1-z}$. Noting that $|1+\beta/w|^2 = 1 + \beta(2\mathrm{Re}w + \beta)/|w|^2 \le 1$, one can see that $|T_f(z)| \le \frac{1}{|1-z|} \le \frac{1}{1-|z|}$. In particular, it holds that $(1-|z|^2)|T_f(z)| \le 1 + |z| < 2$. On the other hand, it is easy to see that $\lim_{x\to 1-0}(1-x^2)|T_f(x)| = 2$, thus $\|T_f\| = 2$.

Next, we shall show that $\beta(f) = 2\beta$. Since $|1+\beta/w| = [1 + \beta(2\mathrm{Re}w + \beta)/|w|^2]^{1/2} \sim 1 + \beta(\mathrm{Re}w + \beta/2)/|w|^2 \sim 1 + \beta/\mathrm{Re}w \sim 1 - \beta/\log|1-z|$ as $z \to 1$ and since the function

$t(1+\beta/\log t)$ of t is monotonically increasing for sufficiently large t, we have

$$\begin{aligned}
\beta(f) &= \overline{\lim_{\mathbb{D}\ni z\to 1}} \{(1-|z|^2)|T_f(z)| - 2\} \log \frac{1}{1-|z|^2} \\
&= \overline{\lim_{\mathbb{D}\ni z\to 1}} \left\{ \frac{(1-|z|^2)}{|1-z|}\left(1+\frac{\beta}{\log 1/|1-z|}\right) - 2 \right\} \log \frac{1}{1-|z|^2} \\
&= \overline{\lim_{\mathbb{D}\ni z\to 1}} \left\{ (1+|z|)\left(1+\frac{\beta}{\log 1/(1-|z|)}\right) - 2 \right\} \log \frac{1}{1-|z|^2} \\
&= \overline{\lim_{x\to 1-0}} \left\{ -(1-x)\log\frac{1}{1-x^2} + (1+x)\beta \frac{\log\frac{1}{1-x^2}}{\log\frac{1}{1-x}} \right\} = 0 + 2\beta.
\end{aligned}$$

In particular, we can conclude that f is bounded if $\beta < -1$ by Proposition 2.4.

On the other hand, in the case that $\beta \geq -1$, noting that $\int_{r_0}^1 \frac{1}{1-x}(\log\frac{1}{1-x})^\beta = \infty$, we can directly see $\overline{\lim}_{x\to 1-0} f(x) = +\infty$, thus f is unbounded.

3. Applications

As applications of the results in the previous section, we will derive various properties of the functions in the class $\mathcal{B}(\lambda)$. We begin with the Hölder continuity of those functions. Recall the following fundamental fact due to Hardy-Littlewoood.

Theorem B (cf. [6]). *Let α be a constant such that $0 < \alpha \leq 1$. A holomorphic function f on the unit disk is Hölder continuous of exponent α if and only if $f'(z) = O(1-|z|)^{\alpha-1}$ as $|z| \to 1$.*

Combining this with Theorem 2.2, we have

Theorem 3.1. *Let $0 \leq \lambda < 1$. Then any function $f \in \mathcal{B}(\lambda)$ is Hölder continuous of exponent $1-\lambda$ on the unit disk.*

Remark . We can directly see that $|f(z_1) - f(z_2)| \leq \frac{C}{1-\lambda}|z_1 - z_2|^{1-\lambda}$ for any pair of points $z_1, z_2 \in \mathbb{D}$, where C is an absolute constant, owing to the estimate $\int_r^s (\frac{1+t}{1-t})^\lambda dt \leq \frac{2^\lambda}{1-\lambda}((1-r)^{1-\lambda} - (1-s)^{1-\lambda}) \leq \frac{2^\lambda}{1-\lambda}(s-r)^{1-\lambda}$ for $0 < r < s < 1$.

Second we consider coefficient estimates for the class $\mathcal{B}(\lambda)$. Let $f(z) = z + a_2z^2 + \cdots \in \mathcal{B}(\lambda)$. Then, by definition, $|T_f(0)| \leq 2\lambda$, which implies $|a_2| \leq \lambda$. Of course, this is sharp because the equality holds for the function F_λ. But, a function in $\mathcal{B}(\lambda)$ essentially different from F_λ may attain this maximum. For instance, consider the function $f(z) = (e^{2\lambda z} - 1)/2\lambda$.

If the origin is a critical point of the function $(1-|z|^2)|T_f(z)|$ then $(T_f)'(0) = 6a_3 - (2a_2)^2 = 0$ though this condition need not be sufficient for $|a_2| = \lambda$.

As for the growth of coefficients of a holomorphic function $f(z) = a_0 + a_1z + a_2z^2 + \cdots$ in the unit disk, it is convenient to consider the integral mean of exponent $p \in \mathbb{R}$:

$$I_p(r,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

In fact, we have the following elementary

Lemma 3.2. *If $I_1(r, f) = O(1-r)^{-\alpha}$ as $r \to 1$ for a constant $\alpha \geq 0$, then we have $a_n = O(n^\alpha)$ as $n \to \infty$.*

Proof. Suppose that $I_1(r, f) \leq M(1-r)^{-\alpha}$ for $0 \leq r < 1$. Then, for $n > 1$ and $r = 1-1/n$, it follows from Cauchy's integral formula that

$$\begin{aligned}|a_n| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})(re^{i\theta})^{-n} d\theta \right| \leq r^{-n} I_1(r, f) \leq M r^{-n}(1-r)^{-\alpha} \\ &= M\left(1 - \frac{1}{n}\right)^{-n} n^\alpha < \frac{eMn}{n-1} n^\alpha.\end{aligned}$$

thus $|a_n| < 2eMn^\alpha$. □

In particular, for a function $f(z) = z + a_2 z^2 + \cdots$ in $\mathcal{B}(\lambda)$, by Theorem 2.2, we have $I_1(r, f') = O(1-r)^{-\lambda}$, thus an estimate $|a_n| = O(n^{\lambda-1})$ as $n \to \infty$. But we can improve the exponent of this order. For $\lambda > 0$, we set

$$\alpha(\lambda) = \frac{\sqrt{1+4\lambda^2} - 1}{2}.$$

Noting $\alpha(\lambda) = 2\lambda^2/(\sqrt{1+4\lambda^2} + 1)$, then we have

$$\frac{\lambda^2}{\lambda+1} < \alpha(\lambda) < \min\left\{\lambda^2, \frac{2\lambda^2}{2\lambda+1}\right\} \leq \min\{\lambda^2, \lambda\}.$$

We also note that

$$\alpha(\lambda) = \lambda - \frac{1}{2} + \frac{1}{8\lambda} + O\left(\frac{1}{\lambda^3}\right) \quad (\lambda \to \infty).$$

For this number, we have the next result.

Theorem 3.3. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be in $\mathcal{B}(\lambda)$. Then, for any $\varepsilon > 0$ and a real number p, we have $I_p(r, f') = O(1-r)^{-\alpha(|p|\lambda)-\varepsilon}$, in particular, $a_n = O(n^{\alpha(\lambda)-1+\varepsilon})$.*

This immediately follows from the next result.

Theorem C ([10, Lemma 5.3]). *Let h be a holomorphic function in the unit disk such that*

$$(1-|z|)\left|\frac{h'(z)}{h(z)}\right| \leq c \quad (r_0 \leq |z| < 1)$$

for constants $c > 0$ and $r_0 < 1$. Then, $I_p(r, h) = O(1-r)^{-\beta}$, where $\beta = (\sqrt{1+4p^2c^2} - 1)/2$ and $p \in \mathbb{R}$.

We note that this is a consequence of the Fuchsian differential inequality:

$$I_p''(r, h) \leq \frac{p^2}{2\pi} \int_0^{2\pi} |h(z)|^p \left|\frac{h'(z)}{h(z)}\right|^2 d\theta \leq \frac{p^2 c^2}{(1-r)^2} I_p(r, h).$$

Moreover if f is univalent, we may have a better growth estimate for the coefficients. First we remind the reader of the following result due to Littlewood, Paley, Clunie, Pommerenke and Baernstein II (see [2], [13, Theorem 8.8] and [8, Theorem 3.7]).

Theorem D. *Suppose that $f(z) = z + a_2z^2 + \cdots \in \mathcal{S}$ satisfies $f(z) = O(1-|z|)^{-\alpha}$. If $0.491 < \alpha \le 2$, then $\int_0^{2\pi} |f'(re^{i\theta})|d\theta = O(1-r)^{-\alpha}$ and $a_n = O(n^{\alpha-1})$. If $\alpha = 0$, in other words, if f is bounded, then $\int_0^{2\pi} |f'(re^{i\theta})|d\theta = O(1-r)^{-0.491}$ and $a_n = O(n^{0.491-1})$.*

In view of Corollary 2.3 we have the following result as a corollary.

Theorem 3.4. *Let $f(z) = z + a_2z^2 + \cdots \in \mathcal{S}$. If $f \in \mathcal{B}(\lambda)$ with $1.491 < \lambda \le 3$, then it holds that $a_n = O(n^{\lambda-2})$ as $n \to \infty$. This order estimate is best possible.*

In order to see the sharpness, we may consider the function $f(z) = (1-z)^{1-\lambda} = 1 + a_1z + a_2z^2 + \cdots$ for $1 < \lambda$. We note that f is univalent in the unit disk if $1 < \lambda \le 3$. For this function, we can see that $\|T_f\| = 2\lambda$ and $a_n = \Gamma(\lambda+n-1)/n!\Gamma(\lambda-1) \sim n^{\lambda-2}$ as $n \to \infty$ by Stirling's formula.

On the other hand, in the case that f is univalent with $\|T_f\| < 3$, the situation seems rather complicated. Given a holomorphic function $f(z) = z + a_2z^2 + \cdots$ in the unit disk, let $\gamma(f)$ denote the infimum of exponents γ such that $a_n = O(n^{\gamma-1})$ as $n \to \infty$, i.e.,

$$\gamma(f) = \overline{\lim_{n\to\infty}} \frac{\log n|a_n|}{\log n}.$$

And, for a subset X of $\mathcal{A}$, we denote by $\gamma(X)$ the supremum of $\{\gamma(f); f \in X\}$. As for $\gamma(\mathcal{S}_b)$, where $\mathcal{S}_b$ denotes the class of normalized bounded univalent functions in the unit disk, it has been shown ([5] and [9]) that $0.24 < \gamma(\mathcal{S}_b) < 0.4886$, and conjectured by Carleson and Jones that $\gamma(\mathcal{S}_b) = 0.25$. We also remark that the growth of coefficients seems to involve an irregurality of the boundary of image under f when f is bounded univalent (see [13, Chapter 10]) and, recently, Makarov and Pommerenke observed a remarkable phenomenon of phase transition of the functional $\gamma(f)$ with respect to the Minkowski dimension of the boundary curve [9].

Now we turn to our case. Theorem 3.3 and the above example $(1-z)^{1-\lambda}$ (or, $-\log(1-z)$ when $\lambda = 1$) yield

$$\lambda - 1 \le \gamma(\mathcal{B}(\lambda)) \le \alpha(\lambda) = \frac{\sqrt{1+4\lambda^2}-1}{2}. \tag{3.1}$$

By standard calculations, we can see that the extremal function F_λ also satisfies $\gamma(F_\lambda) = \lambda - 1$.

For $0 < \lambda \le 1/2$, we note that $\alpha(\lambda) \le \lambda^2 - 2\lambda^4/3 \le 5/24 = 0.2083\cdots$, because $\sqrt{1+x} < 1 + x/2 - x^2/(6+4\sqrt{2}) < 1 + x/2 - x^2/12$ for $0 < x \le 1$. Remark again that $\mathcal{B}(1/2) \subset \mathcal{S}_b$.

Remark . Actually, by Theorem C, for any $f \in \mathcal{A}$, we have the estimate

$$\gamma(f) \le \frac{1}{2}\left(\sqrt{1 + 4\left(\overline{\lim_{|z|\to1}}(1-|z|)|T_f(z)|\right)^2} - 1\right).$$

Next we consider the relationship between the class $\mathcal{B}(\lambda)$ and Hardy spaces. The following are fundamental results in the univalent function theory.

Theorem E (cf. [13]). *Let β be a constant with $0 \le \beta \le 2$. If a univalent function $f \in \mathcal{S}$ satisfies that $f(z) = O(1-|z|)^{-\beta}$ as $|z| \to 1$, then the following holds.*

For $0 < p < 1/\beta$, we have $f \in H^p$. For $1/\beta < p$, we have $M_p(r,f) = O(1-r)^{1/p-\beta}$ $(r \to 1)$.

Where $M_p(r,f)$ denotes L^p-integral mean of f, i.e., $M_p(r,f) = I_p(r,f)^{1/p}$.

Theorem F (Pommerenke [11]). *Let f be a univalent holomorphic function on the unit disk. Then, $f \in BMOA$ if and only if f is Bloch, i.e., $\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty$.*

Combining these theorems with Theorem 2.2, we have the following results.

Theorem 3.5. *Let $f \in \mathcal{S}$ and set $\|T_f\| = 2\lambda$.*
If $\lambda < 1$ then $f \in H^\infty$.
If $\lambda > 1$ then $f \in H^p$ for any $0 < p < 1/(\lambda - 1)$.
If $\lambda = 1$ then $f \in BMOA$.

Note that $H^\infty \subset BMOA \subset \bigcap_{0<p<\infty} H^p$.

Remark . Most of the above results can be extended to the case of p-valent, more generally, mean p-valent functions with $p < \infty$ (see Hayman [8]).

We shall mention a connection with integral means for univalent functions. For a univalent function $f \in \mathcal{S}$ and a real number p, we set (cf. [13, Chapter 8])

$$\beta_f(p) = \overline{\lim_{r\to 1-0}} \frac{\log \int_0^{2\pi} |f'(re^{i\theta})|^p d\theta}{\log \frac{1}{1-r}} = \overline{\lim_{r\to 1-0}} \frac{\log I_p(r,f')}{\log \frac{1}{1-r}}.$$

The Brennan conjecture asserts that $\beta_f(-2) \le 1$ for every univalent holomorphic function f.

For $f \in \mathcal{B}(\lambda)$, as a corollary of Theorem 3.3, we have the next

Theorem 3.6. *For $f \in \mathcal{B}(\lambda)$ amd $p \in \mathbb{R}$ the inequality*

$$\beta_f(p) \le \alpha(|p|\lambda) = \frac{\sqrt{1+4p^2\lambda^2} - 1}{2}$$

holds. In particular, the Brennan conjecture is true for any univalent function f with $\|T_f\| \le \sqrt{2}$.

4. Norm estimate for various classes of univalent functions

In this section, we provide several norm estimates for well-known classes of univalent functions. These enable us to obtain growth and coefficient estimates for those classes, which agree with known results in many cases.

The following is due to S. Yamashita. (The case of strongly starlike functions was first shown by [18].)

Theorem G (Yamashita [22]). *Let $0 \le \alpha < 1$ and $f \in \mathcal{S}$.*
If f is starlike of order α, i.e., $\mathrm{Re}(zf'(z)/f(z)) > \alpha$, then $\|T_f\| \le 6 - 4\alpha$.
If f is convex of order α, i.e., $\mathrm{Re}(1 + zf''(z)/f'(z)) > \alpha$, then $\|T_f\| \le 4(1-\alpha)$.

If f is strongly starlike of order α, i.e., $\arg(zf'(z)/f(z)) < \pi\alpha/2$, *then* $\|T_f\| \le M(\alpha) + 2\alpha$, *where* $M(\alpha)$ *is a specified constant depending only on* α *satisfying* $2\alpha < M(\alpha) < 2\alpha(1+\alpha)$.

All of the bounds are sharp.

Remark . For the equality cases and more detailed and greatly general results, consult the paper [22] by S. Yamashita. For information about the constant $M(\alpha)$ see [18] or [22].

Now we state general and useful principles for estimation of the norm of T_f. A holomorphic function f on the unit disk is said to be *subordinate* to another g if f can be written as $f = g \circ \omega$, where ω is a holomorphic self-mapping of the unit disk with $\omega(0) = 0$. Remark that the Schwarz lemma implies that $|\omega(z)| \le |z|$ and also Pick's version of the Schwarz lemma does that

$$\frac{|\omega'(z)|}{1-|\omega(z)|^2} \le \frac{1}{1-|z|^2} \tag{4.1}$$

for any point $z \in \mathbb{D}$.

We also note that if $g \in \mathcal{S}$, then f is subordinate to g if and only if $f(0) = 0$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

The following always generates a sharp result for fixed g. The idea is due to Littlewood.

Theorem 4.1 (Subordination Principle I). *Let* $g \in \mathcal{B}$ *be given. For* $f \in \mathcal{A}$, *if* f' *is subordinate to* g' *then we have* $\|T_f\| \le \|T_g\|$. *In particular, f is uniformly locally univalent on the unit disk.*

Proof. By assumption, there exists a holomorphic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $f' = g' \circ \omega$. Therefore, $T_f = T_g \circ \omega \cdot \omega'$. Thus (4.1) implies the following:

$$(1-|z|^2)|T_f(z)| = (1-|z|^2)|T_g(\omega)||\omega'| \le (1-|\omega|^2)|T_g(\omega)| \le \|T_g\|,$$

which leads to the conclusion. □

As a typical application of the Subordination Principle, we exhibit the following.

Theorem 4.2. *If* $f \in \mathcal{A}$ *satisfies that* $\mathrm{Re} f' > 0$ *on the unit disk, then* $\|T_f\| \le 2$. *The bound is sharp.*

Remark . The Noshiro-Warschawski theorem says that such an f must be univalent.

Proof. The condition $\mathrm{Re} f' > 0$ is equivalent to the statement that f' is subordinate to the function $F_1'(z) = \frac{1+z}{1-z}$. Thus we have $\|T_f\| \le \|T_{F_1}\| = 2$. □

We note that f' is a Gelfer function if $\mathrm{Re} f' > 0$, where a holomorphic function g on the unit disk with $g(0) = 1$ is called *Gelfer* when $g(z) + g(w) \ne 0$ for all $z, w \in \mathbb{D}$.

Therefore the next result can be viewed as a natural generalization of the above theorem.

Theorem 4.3. *Suppose that* f' *is a Gelfer function for an* $f \in \mathcal{A}$. *Then we have* $\|T_f\| \le 2$. *This bound is sharp.*

Proof. For a Gelfer function $g(z) = f'(z)$ it is known to hold that

$$\left|\frac{g'(z)}{g(z)}\right| \le \frac{2}{1-|z|^2}$$

(see [21]). Hence, the result immediately follows. □

The next is a variant of the subordination principle.

Theorem 4.4 (Subordination Principle II). *Let $g \in \mathcal{B}$ be given. For $f \in \mathcal{A}$, if $zf'(z)/f(z)$ is subordinate to g' then we have*

$$\|T_f\| \le \sup_{z\in\mathbb{D}}(1-|z|^2)\left(\left|\frac{g'(z)-1}{z}\right| + |T_g(z)|\right) \tag{4.2}$$

$$\le \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{g'(z)-1}{z}\right| + \|T_g\|. \tag{4.3}$$

Proof. By assumption, there exists a holomorphic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $zf'(z)/f(z) = g'(\omega(z))$. By taking logarithmic derivative, we have the following formula.

$$\begin{aligned} T_f &= \frac{f'}{f} - \frac{1}{z} + \frac{g''(\omega)}{g'(\omega)}\omega' \\ &= \frac{\omega}{z}\frac{g'(\omega)-1}{\omega} + T_g(\omega)\omega'. \end{aligned}$$

From this, we can easily have the desired estimate. □

The following is a simple application of this principle.

Theorem 4.5. *If $f \in \mathcal{A}$ satisfies that $|zf'(z)/f(z) - 1| < 1$, then we have an estimate $\|T_f\| \le 2.25$. The equality holds if and only if f is a rotation of the function ze^z.*

Remark . In this case, f satisfies $\mathrm{Re} zf'(z)/f(z) > 0$ thus f is starlike, in particular, univalent in the unit disk.

Proof. We have only to apply the esitimate (4.2) with $g(z) = z + z^2/2$. Then, we have $\|T_f\| \le \sup(2+|z|-|z|^2) = 9/4$, where the supremum is attained only by $|z| = 1/2$. Thus, if $\|T_f\| = 9/4$, then $|\omega|$ must be the constant 1, whence f is a rotation of ze^z. Conversely, it is clear that the function $f(z) = ze^{\mu z}$ with $|\mu| = 1$ satisfies $\|T_f\| = 9/4$. □

Finally, we consider uniformly convex functions:

$$\mathrm{UCV} = \left\{f \in \mathcal{S}; \mathrm{Re}\left(1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right) \ge 0, \forall z, \forall \zeta \in \mathbb{D}\right\}.$$

For the geometric meaning of this class, see [7]. Rønning gave a simple characterization for this class.

Theorem H (Rønning [16]). *A function $f \in \mathcal{A}$ is uniformly convex if and only if $zT_f(z) \in W$ for any $z \in \mathbb{D}$, where W is the domain $\{w = u + iv; v^2 < 2u + 1\}$.*

We note that a conformal map $g : \mathbb{D} \to W$ with $g(0) = 0$ is given by

$$g(z) = \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 = \frac{8z}{\pi^2}\left(1+\frac{z}{3}+\frac{z^2}{5}+\frac{z^3}{7}+\cdots\right)^2.$$

Therefore, $f \in \mathcal{A}$ is uniformly convex if and only if $zT_f(z)$ is subordinate to the function g, i.e., there exists a holomorphic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $zT_f(z) = g(\omega(z))$. Since g has positive Taylor coefficients, we see that $|zT_f(z)| \le g(|\omega(z)|) \le g(|z|)$. Hence, we have

$$\|T_f(z)\| \le \sup_{0<x<1}(1-x^2)\frac{g(x)}{x} = \sup_{0<t<\infty} h(t),$$

where

$$h(t) = \frac{8t^2}{\pi^2}\frac{\cosh t}{\sinh^2 t}$$

and $\frac{1+\sqrt{x}}{1-\sqrt{x}} = e^t$. By the logarithmic differentiation, we have

$$\frac{h'(t)}{h(t)} = \frac{2\sinh 2t - t(\cosh 2t + 3)}{t\sinh 2t} = \frac{N(t)}{t\sinh 2t}.$$

Since $N''(t) = \frac{4(\tanh 2t - t)}{\cosh 2t}$ has the unique zero t_0 in $(0,\infty)$, the function $N'(t) = 3(\cosh 2t - 1) - 2t\sinh 2t$ attains its maximum at t_0. Since $N'(0) = 0$ and $N'(t) \to -\infty$ as $t \to \infty$, the function $N'(t)$ has the unique zero $t_1 > t_0$ in $(0,\infty)$. By exactly same reason, the function $N(t)$ has the unique zero $t_2 > t_1$ in $(0,\infty)$. Thus, $h(t)$ assumes its maximum at the point $t = t_2$. By a numerical calculation, we have $t_2 = 1.6061152988\cdots$, and $h(t_2) = 0.94774221287\cdots$. Therefore, we summalize as follows.

Theorem 4.6. *If $f \in \mathcal{A}$ is uniformly convex, then we have*

$$\|T_f\| \le h(t_2) = 0.94774\cdots,$$

where the equality occurs only when f is a rotation of the function $F \in \mathcal{A}$ determined by $T_F(z) = g(z)/z$.

Remark . By the corollary of Theorem A, we see that a uniformly convex function can be extended to a quasiconformal self-homeomorphism of the Riemann sphere with maximal dilatation at most $K_0 = 37.2718\cdots$.

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