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A remark on the critical exponent of Kleinian groups

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In this note, we remark the following lemma on the critical exponent $\delta(G)$ of a non-elementary Kleinian group G, which is equal to the Hausdorff dimension of the conical limit set of G.

Lemma 1 *Let G be a Kleinian group, and let Γ be a subgroup of G such that the limit set $\Lambda(\Gamma)$ is a proper subset of $\Lambda(G)$. Suppose that the Poincaré series for Γ diverges at the critical exponent $\delta(\Gamma)$. Then $\delta(G)>\delta(\Gamma)$.*

Bowen [2] proved that the limit set of a quasifuchsian group Γ has Hausdorff dimension 1 if and only if Γ is a Fuchsian group. Generalizing this fact, Canary–Taylor [3] and Bishop–Jones [1] determined all the finitely generated Kleinian groups whose limit sets have Hausdorff dimension not greater than 1.

Theorem 2 *If the Hausdorff dimension of $\Lambda(G)$ for a finitely generated Kleinian group G is not greater than 1, then $\Lambda(G)$ is either a circle or a totally disconnected set.*

We prove Theorem 2 by using Lemma 1.

Proof of Theorem 2. By Maskit's classification, G has either a quasifuchsian subgroup, a totally degenerate subgroup or the totally disconnected limit set. However, it cannot have a totally degenerate subgroup because the Hausdorff dimension of its limit set is known to be 2. If G has a quasifuchsian subgroup Γ, it follows from Bowen's theorem that Γ is Fuchsian. Since the Poincaré series for Γ diverges at the critical exponent ($= 1$), Lemma 1 implies that $\Lambda(G) = \Lambda(\Gamma)$. ■

Proof of Lemma 1. By a result due to Patterson and Sullivan [4], there exists a $\delta(G)$-dimensional, G-invariant, conformal probability measure μ on the sphere at infinity. Then we can choose a disk $\Delta \subset \Omega(\Gamma)$ such that $\mu(\Delta) > 0$ and $\Delta \cap \gamma(\Delta) = \emptyset$ for any non-trivial element $\gamma \in \Gamma$. Since

$$1 > \sum_{\gamma\in\Gamma} \mu(\gamma(\Delta)) \approx \mu(\Delta) \cdot \sum_{\gamma\in\Gamma} |\gamma'(x)|^{\delta(G)},$$

where x is the center of Δ, the Poincaré series for Γ converges at the exponent $\delta(G)$. This implies that $\delta(G) > \delta(\Gamma)$. ■

References

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[3] R. Canary and E. Taylor, *Kleinian groups with small limit sets,* Duke Math. J. 73 (1994), 371–381.

[4] D. Sullivan, *The density at infinity of a discrete group of hyperbolic mot ions* , Publ. Math. IHES. 50 (1979), 172–202.