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Determination of Optimal Repair-Cost Limit Replacement Strategy by Lorenz Transform Method

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Abstract: In this paper, we consider a repair-cost limit replacement problem and develop a graphical method to determine the optimal repair-cost limit which minimizes the expected cost per unit time in the steady-state, using the Lorenz transform of the repair-cost distribution function. The method proposed can be applied to an estimation problem of the optimal repair-cost limit from empirical repair-cost data.

Keywords: repair-cost limit replacement policy, graphical method, Lorenz transform, non-parametric estimation, maintenance model.

1. INTRODUCTION

The repair-cost limit replacement policies can provide how to design the recovery mechanism of a system using two maintenance options: repair and replacement, in terms of cost minimization. That is, if the repair cost of a failed unit is greater than the replacement cost, one should replace a failed one, otherwise should repair it. First this problem was considered by Drinkwater and Hastings [1] and Hestings [2] for army vehicles. Especially, Hestings [2] proposed three methods of optimizing the repair-cost limit replacement policies by simulation, hill-climbing and dynamic programming. Since the seminal contributions above, a number of authors developed different probability models. For instance, Kaio and Osaki [3] reformulated the Hastings' original problem from the viewpoint of renewal reward argument and discussed both continuous and discrete models under the discounted cost criteria. Love, Rodger and Blazenko [4] examined the similar problem for vehicle replacement using Potal Canada data which is constructed by dividing the life of the vehicle into discrete ages. Park [5] considered a simple but interesting cost limit replacement policy under minimal repair. Love and Guo [6] extended the repair-limit analysis by incorporating a changing force of mortality as the unit ages in the framework of a Markov or semi-Markov decision process.

As Love and Guo [6] pointed out implicitly, it is often assumed that the repair-cost distribution function is arbitrary but known. Of course, this seems to be rather restrictive in many practical situations. To this end, practitioners have to determine the repair-cost limit under incomplete information on the repair-cost distribution in most cases. Dohi, Koshimae, Kaio and Osaki [7] and Dohi, Kaio and Osaki [8] proposed non-parametric estimators of the optimal repair-cost limit from the empirical cost data. More precisely, they applied the total time on test (TTT) statistics to those estimation problems in accordance with the graphical idea by Bergman [9] and Bergman and Klefsjö [10]. If the optimal repair-cost limit has to be estimated from the sample data with unknown repair-cost distribution, their method will be useful in practice, since one need not specify the repair-cost distribution.

However, it should be noted that the repair-cost limit replacement problems in [7, 8] were very interesting but somewhat different from existing ones. The main objective in [7, 8] was to derive the optimal cost-limit to retire the repair action, *i.e.* if the repair is not completed within a cost limit, the failed unit is scrapped and then a new spare is ordered. Such a policy seems to be plausible in many practical situations, but should be distinguished from the original repair-cost limit problem. In this paper, we consider a repair-cost limit replacement problem in the framework of renewal reward processes and propose an estimation method based on the Lorenz curve. Notice that the basic idea in this paper is similar to the graphical one used in [7, 8] but the statistical device employed here is different from the TTT statistics. The Lorenz curve was introduced first by Lorenz [11] into Economics to describe income distributions. Since the Lorenz curve is essentially equivalent to the Pareto curve used in the quality control, it will be one of the most important statistics in every social sciences.

The more general and tractable definition of the Lorenz curve was made by Gastwirth [12]. Goldie [13] proved the strong consistency of the empirical Lorenz curve and discovered its several convergence properties. Chandra and Singpurwalla [14] and Klefsjö [15] investigated the relationship between the TTT statistics and the Lorenz statistics and derived a few aging and partial ordering properties. Recently, the further results on two statistics were examined by Pham and Turkkan [16] and Perez-Ocon, Gamiz-Perez and Ruiz-Castro [17]. It is shown that the estimator of the optimal repair-cost limit derived in this paper has also several powerful properties proved in earlier contributions above.

The paper is organized as follows. In Section 2, we describe the repair-cost limit replacement problem under consideration. In Section 3 we develop a graphical method to calculate the optimal repair-cost limit which minimizes the expected cost per unit time in the steady-state. Then, it is seen that the Lorenz curve plays an important role to derive the optimal solution on the graph. In Section 4, the statistical estimation problem is discussed. We show that estimator of the optimal repair-cost limit has a strong consistency and examine its convergence property.

2. MODEL DESCRIPTION

Consider a single-unit repairable system, where each spare is provided only by an order after a lead time L (> 0) and each failed unit is repairable. The original unit begins operating at time 0 and the mean time to failure for each unit is m_f (> 0). When the unit has failed, the decision maker wishes to determine whether he or she should repair it or order a new spare. If the decision maker estimates that the repair is completed within a prespecified cost limit $v_0 \in [0, \infty)$, then the repair is started immediately at the failure time. The mean repair time is m_s (> 0) when the repair cost does not exceed v_0. On the other hand, if the decision maker estimates that the repair cost exceeds the cost limit v_0, then the failed unit is scrapped immediately and a new spare unit is ordered. Then the spare unit is delivered after the lead time L. Without any loss of generality, it is assumed that the unit once repaired is presumed as good as new and that the time required for replacement is negligible.

The repair cost for each unit is an i.i.d. random variable and unknown. The decision maker has a *subjective* probability distribution function $H(v)$ on the repair cost, with density $h(v)$ and finite mean m_m (> 0). Suppose that the distribution function $H(v)$ is arbitrary, continuous and strictly increasing in $v \in [0, \infty)$, and has an inverse function, *i.e.* $H^{-1}(\cdot)$. Under these model assumptions, define the interval from the start of the operation to the following start as one cycle. Figure 1 depicts the configuration of the model under consideration. The costs considered in this paper are the following:

k_f (> 0): a cost per unit shortage time.

c (> 0): a cost for each order.

We make the following additional assumptions:

(A-1) $m_s > L$.

(A-2) $k_f m_s < k_f L + c$.

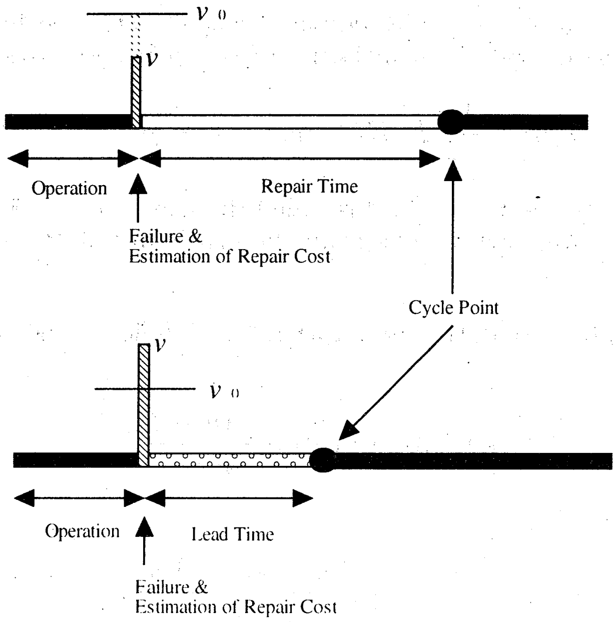

Figure 1: Configuration of Model 1.

The assumption (A-1) implies that the mean repair time m_s is strictly longer than the lead time. In the assumption (A-2), the shortage cost when the repair cost does not exceed v_0 is less than the total cost when the new spare is ordered. It is noticed that these assumptions motivate the underlying problem to determine the optimal repair-cost limit.

Let us formulate the expected cost during one cycle. If the decision maker judges that a new spare unit should be ordered, then the ordering cost for one cycle is $c\overline{H}(v_0)$, where $\overline{H}(\cdot) = 1 - H(\cdot)$. In this case, the expected shortage cost for one cycle is $k_f L\overline{H}(v_0)$. On the other hand, if he or she selects the repair option, the expected repair cost for one cycle is $\int_0^{v_0} v dH(v)$ and the expected shortage cost for one cycle is $k_f m_s H(v_0)$. Thus the total expected cost for one cycle is

$$E_C(v_0) = \int_0^{v_0} v dH(v) + k_f\{m_s H(v_0) + L\overline{H}(v_0)\} + c\overline{H}(v_0). \tag{1}$$

Also, the mean time of one cycle is

$$E_T(v_0) = m_f + m_s H(v_0) + L\overline{H}(v_0). \tag{2}$$

It may be appropriate to adopt an expected cost per unit time in the steady-state over an infinite planning horizon. The total expected cost per unit time in the steady-state is, from the renewal reward argument [18],

$$TC(v_0) = \lim_{t\to\infty} \frac{\text{E[the total cost on } (0, t]]}{t} = E_C(v_0)/E_T(v_0). \tag{3}$$

Then the problem is to determine the optimal repair-cost limit $v_0{}^*$ such as

$$TC(v_0^*) = \min_{0\le v_0<\infty} TC(v_0). \tag{4}$$

3. GRAPHICAL METHOD

In stead of differentiating $TC(v_0)$ with respect to v_0 directly, we here employ the following graphical method. Define the Lorenz transform of the repair-cost distribution $p \equiv H(v)$ by

$$\phi(p) = \frac{1}{m_m} \int_0^p H^{-1}(v)dv, \quad (0 \leq p \leq 1). \tag{5}$$

Then the curve $\mathcal{L} = (p, \phi(p)) \in [0, 1] \times [0, 1]$ is called the *Lorenz curve* [12-17]. It should be noted that the curve $\mathcal{L}$ is absolutely continuous from the continuity of $H(v)$. From simple algebraic manipulations, we have

THEOREM 3.1: Suppose that the assumption (A-1) holds. The minimization problem in Eq.(4) is equivalent to

$$\min_{0 \leq p \leq 1} : M(p, \phi(p)) \equiv \frac{\phi(p) + \xi}{p + \eta}, \tag{6}$$

where

$$\xi = \frac{cm_s - \{k_f(m_s - L) - c\}m_f}{m_m(m_s - L)}, \tag{7}$$

$$\eta = \frac{m_f + L}{m_s - L}. \tag{8}$$

The proof is omitted for brevity. Consequently, the optimal repair-cost limit is determined by $p^* = H(v_0^*)$ which minimizes the tangent slope from the point $\mathrm{B} = (-\eta, -\xi) \in (-\infty, 0) \times (-\infty, 0)$ to the curve $\mathcal{L}$ in the plane $(x, y) \in (-\infty, +\infty) \times (-\infty, +\infty)$ under the assumption (A-2).
More precisely, we prove the uniqueness of the optimal repair-cost limit.

THEOREM 3.2: Suppose that the assumptions (A-1) and (A-2) hold. Then there exists a unique optimal solution $p^* = H(v_0^*)$ $(0 < v_0^* < \infty)$ minimizing $M(p, \phi(p))$, where p^* is given by the x-coordinate at the point of contact for the curve $\mathcal{L}$ from the point B.

PROOF: From (A-1) and (A-2), it can be seen that $\xi > 0$ and $\eta > 0$. Differentiating $M(p, \phi(p))$ with respect to p and setting it equal to zero implies

$$q(p) \equiv (d\phi(p)/dp)(p + \eta) - (\phi(p) + \xi). \tag{9}$$

where $d\phi(p)/dp = H^{-1}(p)/m_m$. Further, we have

$$dq(p)/dp = d^2\phi(p)/dp^2(p + \eta) > 0 \tag{10}$$

and the function $M(p, \phi(p))$ is strictly convex in p, since $d^2\phi(p)/dp^2 = 1/\{m_m h(H^{-1}(p))\} > 0$. From $q(0) = -\xi < 0$ and $q(1) \to \infty$, the proof is completed. (Q.E.D.)

EXAMPLE 3.3: We give an example for the graphical method proposed above. Suppose that the repair-cost distribution $H(v)$ is known and obeys the Weibull distribution:

$$H(v) = \exp\{-(\frac{t}{\theta})^\beta\} \tag{11}$$

with the shape parameter $\beta = 4.0$ and the scale parameter $\theta = 0.9$. The other model parameters are $c = 0.4500$ (\$), $L = 0.4000$ (day), $k_f = 0.4000$ (\$/day) $m_f = 0.3000$ (day), $m_s = 1.5000$ (day) and $m_m = 0.8157$ (\$). The determination of the optimal repair-cost limit is presented in Fig. 2.

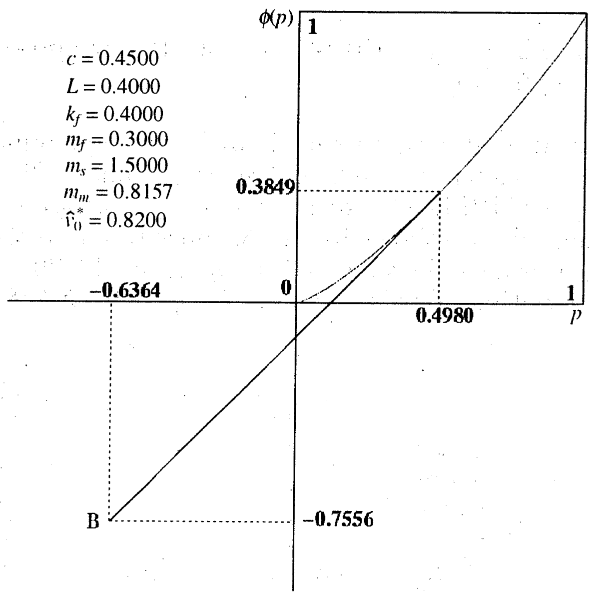

Figure 2: Determination of optimal repair-cost limit on the Lorenz curve.

In this case, we have B $= (-0.6364, -0.7556)$ and the optimal point with minimum slope from B is $(p^*, \phi(p^*)) = (0.4980, 0.3849)$. Thus, the optimal repair-cost limit is $r_0^* = H^{-1}(0.4980) = 0.8200$.

4. STATISTICAL ESTIMATION METHOD

Based on the graphical idea in Section 3, we propose a non-parametric method to estimate the optimal repair-cost limit replacement policy. Suppose that the optimal repair-cost limit has to be estimated from an ordered complete sample $0 = x_0 \le x_1 \le x_2 \le \cdots \le x_n$ of repair cost data from an absolutely continuous repair-cost distribution H, which is unknown. The estimator of $H(x) = p$ is the empirical distribution given by

$$H_{in}(x) = \begin{cases} i/n & \text{for} \quad x_i \le x < x_{i+1}, \\ 1 & \text{for} \quad x_n \le x. \end{cases} \tag{12}$$

where $i = 0, 1, 2, \cdots, n-1$. Then the *sample Lorenz curve* [13] is defined as

$$\phi_{in} \equiv \sum_{i=1}^{[np]} x_i / \sum_{i=1}^{n} x_i, \tag{13}$$

where $[a]$ is the greatest integer in a. Plotting the point $(i/n, \phi_{in})$, $(i = 0, 1, 2, \cdots, n)$, and connecting them by line segments, we obtain the sample Lorenz curve $\mathcal{L}_n \in [0,1] \times [0,1]$.

As empirical counterpart of THEOREM 3.1, we propose a non-parametric estimator of the optimal repair-cost limit.

THEOREM 4.1: (i) The optimal repair-cost limit can be estimated by $\hat{r_{0n}}^* = x_{i^*}$, where

$$\left\{ i^* \mid \min_{0 \le i \le n} \frac{\phi_{in} + \xi}{i/n + \eta} \right\}. \tag{14}$$

(ii) The estimator $\hat{v_{0n}}^* = x_{i^*}$ in Eq.(14) is strongly consistent, *i.e.* $\hat{v_n} = x_{i^*} \to v_0^*$ as $n \to \infty$.

The result in (i) is trivial. The proof of (ii) is based on the asymptotic property $\phi_{in} \to \phi(p)$ as $n \to \infty$, which is due to Goldie [13].

EXAMPLE 4.2: The repair-cost data were made by the random number following the Weibull distribution with shape parameter $\beta = 4.0$ and scale parameter $\theta = 0.9$. The other model parameters are same as EXAMPLE 3.3 except that $m_m = 0.8567$ ($). The empirical Lorenz curve based on the 30 sample data is shown in Fig. 3. Since B = $(-0.6364, -0.7195)$, the optimal point with minimum slope from B becomes $(i^*/n, \phi_{i^*,n}) = (14/30, \phi_{14,30}) = (0.4667, 0.3711)$. Thus, the estimator of the optimal repair-cost limit is $\hat{v}_{0n}^* = 0.8393$ ($).

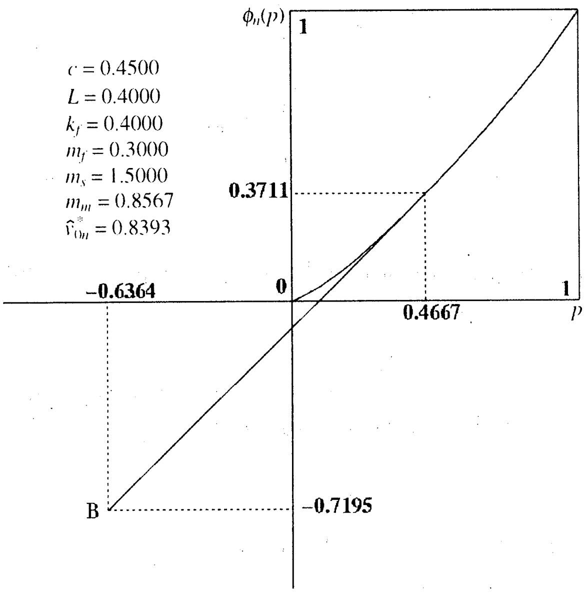

Figure 3: Estimation of optimal repair-cost limit on the empirical Lorenz curve.

If the estimator $\hat{v_{0n}}^* = x_{i^*}$ is obtained, it is easy to calculate the minimum expected cost. That is, differentiating of Eq.(3) with respect to v_0 yields the first condition of optimality:

$$\frac{dE_C(v_0)/dv_0}{\overline{H}(v_0)} E_T(v_0) = \frac{dE_T(v_0)/dv_0}{\overline{H}(v_0)} E_C(v_0). \tag{15}$$

This equation leads to

$$TC(v_0^*) = \frac{v_0^* + k_s m_s - k_f L - c}{m_s - L}. \tag{16}$$

Substituting $\hat{v_{0n}}^*$ into v_0^*, we can derive an estimator of the minimum expected cost $TC(v_0^*)$ which may be strongly consistent.

Of our next interest is the convergence speed of the estimators $\hat{v}_{0n}$ and $TC(\hat{v}_{0n})$. We examine numerically the strong consistency of the estimator derived in THEOREM 4.1.

EXAMPLE 4.3: Suppose that the repair-cost distribution and model parameters are similar to those in EXAMPLE 4.2. Then the real optimal repair-cost limit and the minimum expected cost become $v_0^* = 0.8200$ (\$) and $TC(v_0^*) = 0.7364$ (\$), respectively. On the other hand, the asymptotic behaviours of estimators for the optimal repair-cost limits and their associated minimum expected cost are depicted in Figs. 4 and 5. From these figures, we observe that the estimators converge to the corresponding real values around which the number of data is 20. In other words, without specifying the repair-cost distribution, the proposed nonparametric method may function to estimate the optimal repair-cost limit precisely.

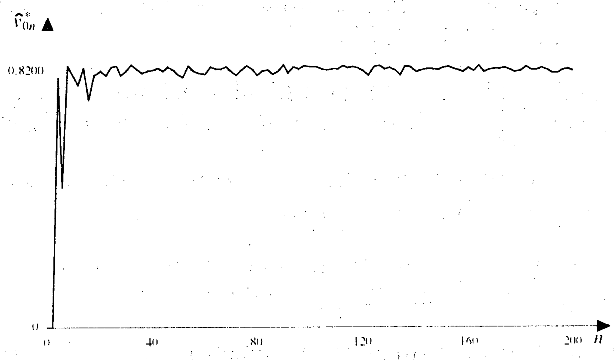

Figure 4: Asymptotic property of the estimator for the optimal repair-cost limit.

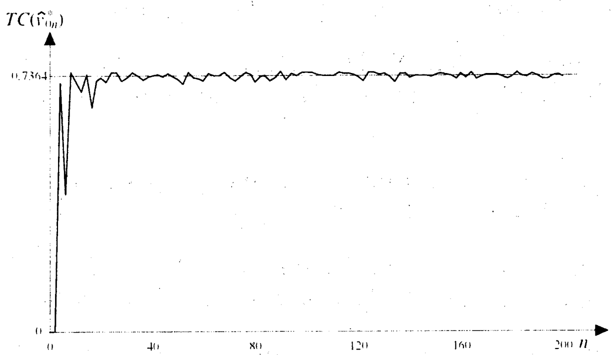

Figure 5: Asymptotic property of the estimator for the associated minimum expected cost.

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