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BERNSTEIN-TYPE APPROXIMATION PROCESSES FOR VECTOR-VALUED FUNCTIONS

TOSHIHIKO NISHISHIRAHO (西白保敏彦)
Faculty of Science, University of the Ryukyus (琉球大学理学部)

ABSTRACT. A sequence of the Bernstein-type operators for vector-valued functions is provided and its uniform convergence is considered by making use of a theorem of Korovkin type under certain requirements.

1. Introduction

Let f be a real-valued continuous function on the unit r-cube

$$\mathbb{I}_r = \{x = (x_1, x_2, \cdots, x_r) \in \mathbb{R}^r : 0 \leq x_i \leq 1, i = 1, 2, \cdots, r\},$$

where $\mathbb{R}^r$ is the r-dimensional Euclidean space and let n be a positive integer. Then the n-th Bernstein polynomial of f is defined by

$$B_n(f)(x) = \sum_{k_1=0}^{n} \cdots \sum_{k_r=0}^{n} \prod_{i=1}^{r} \binom{n}{k_i} x_i^{k_i}(1-x_i)^{n-k_i} f(k_1/n, \cdots, k_r/n). \quad (1)$$

It is well-known that $\{B_n(f)\}$ converges uniformly to f on $\mathbb{I}_r$ (cf. [6]). This result also remains true for a continuous function f taking values in a normed linear space ([8]).

In this paper, we give a generalization of (1) and consider its uniform convergence in the context of normed vector lattices. For this we have to establish a theorem of Korovkin type for vector-valued functions (cf. [8], [9], [10]). For the background of the Korovkin-type approximation theory, see the book of Altomare and Campiti [2], in which an excellent source and a vast literature of this theory can be found (cf. [3], [4], [5]).

2. A theorem of Korovkin type

Let X be a compact Hausdorff space and let E be a normed vector lattice with its positive cone $E_+ = \{a \in E : a \geq 0\}$. For the general notions and terminology needed from the theory of normed vector lattices, we refer to [12] (cf. [1], [7]). Let $B(X, E)$ denote the normed vector lattice of all E-valued norm bounded functions on X with the usual pointwise addition, scalar multiplication, ordering and the supremum norm $\|\cdot\|$. We shall use the same symbol $\|\cdot\|$ for the underlying norms. $C(X, E)$ denotes the closed sublattice of $B(X, E)$ consisting of all E-valued continuous functions on X. In the case when E is equal to $\mathbb{R}$, we simply write $B(X)$ and $C(X)$ instead of $B(X, E)$ and $C(X, E)$, respectively.

Throughout this paper we suppose that E always contains an element e such that $e > 0$, $\|e\| = 1$ and $|a| \leq \|a\|e$ for all $a \in E$. We call e the normal order unit of E. We define $\rho(x) = e$ and $1_X(x) = 1$ for all $x \in X$. Notice that ρ and 1_X are the normal order units of $C(X, E)$ and $C(X)$, respectively. For any $a \in E$ and $v \in B(X)$, the function $v \otimes a$ is defined by $(v \otimes a)(x) = v(x)a$ for all $x \in X$. Also, for any $v \in B(X)$ and $f \in B(X, E)$, we define $(vf)(x) = v(x)f(x)$ for all $x \in X$. Clearly, $v \otimes a$ and vf belong to $B(X, E)$, and $\|v \otimes a\| = \|v\|\|a\|$, $\|vf\| \leq \|v\|\|f\|$ and $\rho = 1_X \otimes e$. We shall denote by $C(X) \otimes E$ the linear subspace of $C(X, E)$ consisting of all finite sums of functions of the form $v \otimes a$, where $v \in C(X)$ and $a \in E$. A bounded linear operator L of $C(X, E)$ into $B(X, E)$ is said to be quasi-positive if $v, w \in C(X)$ and $|v| \leq w$, then $\|L(v \otimes a)(x)\| \leq \|L(w \otimes a)(x)\|$ for all $a \in E_+$ and all $x \in X$. (cf. [8], [9]). A typical example of such an operartor is given by

$$T(f) = hf \quad \text{for every } f \in C(X, E), \tag{2}$$

where h is an arbitrary fixed function in $B(X)$.

Lemma 1. *If L is a positive linear operator of $C(X, E)$ into $B(X, E)$, then it is quasi-positive and $\|L\| = \|L(\rho)\|$.*

Proof. Let $v, w \in C(X), |v| \leq w$ and $a \in E_+$. Then we have $|v \otimes a| \leq w \otimes a$, and so $|L(v \otimes a)| \leq L(w \otimes a)$. Thus for all $x \in X$, $|L(v \otimes a)(x)| \leq L(w \otimes a)(x)$, which implies $\|L(v \otimes a)(x)\| \leq \|L(w \otimes a)(x)\|$. Since, for all $f \in C(X, E)$, $|f| \leq \|f\|\rho$, we have $|L(f)| \leq \|f\|L(\rho)$, and so $\|L(f)\| \leq \|f\|\|L(\rho)\|$. Therefore, $\|L\| \leq \|L(\rho)\|$. On the other hand, $\|L(\rho)\| \leq \|L\|$ because of $\|\rho\| = 1$. □

Lemma 2. *([8; Lemma 2) $C(X) \otimes E$ is dense in $C(X, E)$.*

In fact, this is an immediate consequence of [11; Theorem 1.15], since $C(X)$ separates the points of X.

Now, we have the following Korovkin-type theorem (cf. [8; Corollary 4 (i) and Remark]), which can be useful for later applications.

Theorem 1. *Let $\{L_\alpha\}$ be a net of quasi-positive linear operators of $C(X, E)$ into $B(X, E)$ such that there exsits an element α_0 for which*

$$\sup\{\|L_\alpha\| : \alpha \geq \alpha_0\} < \infty \tag{3}$$

and let T be as in (2). Let G be a subset of $C(X)$ separating the points of X. Then the following statements are equivalent:

(a) For all $g \in G, a \in E_+$ and for $j = 0, 1, 2,$

$$\lim_\alpha \|L_\alpha(g^j \otimes a) - T(g^j \otimes a)\| = 0, \tag{4}$$

where $g^0 = 1_X$.

(b) For all $g \in G$ and all $a \in E_+$, (4) holds with $j = 0$ and $\lim_\alpha \mu_\alpha(g,a) = 0$, *where*

$$\mu_\alpha(g,a) = \sup\{\|L_\alpha((g - g(y)1_X)^2 \otimes a)(y)\| : y \in X\}.$$

(c) For all $f \in C(X,E)$,

$$\lim_\alpha \|L_\alpha(f) - T(f)\| = 0.$$

Proof. Since

$$L_\alpha((g - g(y)1_X)^2 \otimes a)(y) = L_\alpha(g^2 \otimes a)(y) - T(g^2 \otimes a)(y)$$

$$-2g(y)\{L_\alpha(g\otimes a)(y) - T(g\otimes a)(y)\} + g^2(y)\{L_\alpha(1_X\otimes a)(y) - T(1_X\otimes a)(y)\},$$

we have

$$\mu_\alpha(g,a) \leq \|L_\alpha(g^2 \otimes a) - T(g^2 \otimes a)\|$$

$$+\ 2\|g\|\|L_\alpha(g \otimes a) - T(g \otimes a)\| + \|g^2\|\|L_\alpha(1_X \otimes a) - T(1_X \otimes a)\|.$$

Therefore (a) implies (b). Next we suppose that (b) is valid. Let $v \in C(X), b \in E$ and $\epsilon > 0$ be given. Note that b has the representation

$$b = b^+ - b^-,$$

where b^+ and b^- are the positive part and the negative part of b, respectively. Since X is compact and G separates the points of X, the original topology on X is identical with the weak topology on X induced by G. Therefore, there exists a finite subset $\{g_1, g_2, \cdots, g_m\}$ of G and a costant $K > 0$ such that

$$|v(x) - v(y)| \leq \epsilon + K\sum_{i=1}^{m}(g_i(x) - g_i(y))^2$$

for all $x, y \in X$. Hence it follows that

$$\|L_\alpha((v - v(y)1_X) \otimes b^+)(y)\| \leq \epsilon\|L_\alpha(1_X \otimes b^+)(y)\|$$

$$+K\sum_{i=1}^{m}\|L_\alpha((g_i - g_i(y)1_X)^2 \otimes b^+)(y)\|$$

for all $y \in X$, and so we have

$$\|L_\alpha(v \otimes b^+) - T(v \otimes b^+)\|$$

$$\leq \|L_\alpha(v\otimes b^+) - vL_\alpha(1_X\otimes b^+)\| + \|v\|\|L_\alpha(1_X\otimes b^+) - T(1_X\otimes b^+)\|$$

$$\leq \epsilon\|L_\alpha(1_X\otimes b^+)\| + K\sum_{i=1}^m \mu_\alpha(g_i, b^+) + \|v\|\|L_\alpha(1_X\otimes b^+) - T(1_X\otimes b^+)\|,$$

which together with the assertion (b) yields $\lim_\alpha \|L_\alpha(v\otimes b^+) - T(v\otimes b^+)\| = 0$. Similarly, we have $\lim_\alpha \|L_\alpha(v\otimes b^-) - T(v\otimes b^-)\| = 0$. Now, we have

$$\|L_\alpha(v\otimes b) - T(v\otimes b)\| \leq \|L_\alpha(v\otimes b^+) - T(v\otimes b^+)\| + \|L_\alpha(v\otimes b^-) - T(v\otimes b^-)\|,$$

and so

$$\lim_\alpha \|L_\alpha(v\otimes b) - T(v\otimes b)\| = 0.$$

Hence, in view of (3), Lemma 2 and the theorem of Banach-Steinhaus establish the statement (c). It is obvious that (c) implies (a). □

Remark 1. *Theorem 1 can be applied in the following situation: Let X be a compact subset of a real locally convex Hausdorff vector space F with its dual space F^* and $G = \{u|_X : u \in F^*\}$, where $u|_X$ denotes the restriction of u to X. If X is a compact convex subset of F, then G can be taken as the space of all real-valued continuous affine functions on X.*

3. Bernstein-type operators

Let $B[E]$ denote the normed algebra of all bounded linear operators of E into itself with the identity operator I. Let $X_1, X_2, \cdots, X_r$ be compact Hausdorff spaces and we here consider their product space

$$X = \prod_{i=1}^r X_i = \{x = (x_1, x_2, \cdots, x_r) : x_i \in X_i, i = 1, 2, \cdots, r\}.$$

Let $\Phi = \{(\Phi^{(i)}_{n,k})_{n,k\geq 0} : i = 1, 2, \cdots, r\}$ be a set of infinite lower triangular matrices of continuous functions from X_i into $B[E]$ and let

$\mathcal{T} = \{T_{n,k_1,k_2,\cdots,k_r} : 0 \leq k_i \leq n, i = 1, 2, \cdots, r\}$ be a set of bounded linear operators of $C(X, E)$ into E. Then we define

$$B_n(f)(x) = B_{n,\mathcal{T},\Phi}(f)(x) = \sum_{k_1=0}^{n} \cdots \sum_{k_r=0}^{n} \prod_{i=1}^{r} \Phi_{n,k_i}^{(i)}(x_i)(T_{n,k_1,\cdots,k_r}(f)) \tag{5}$$

for all $f \in C(X, E)$ and all $x \in X$. Notice that each B_n is a bounded linear operator of $C(X, E)$ into itself. We call B_n the n-th Bernstein-type operator with respect to $\mathcal{T}$ and Φ.

If we take

$$X_i = \mathbb{I}_1 = [0, 1] \qquad (i = 1, 2, \cdots, r) \tag{6}$$

and

$$\Phi_{n,k}^{(i)}(t) = \varphi_{n,k}^{(i)}(t) I \qquad (t \in X_i, i = 1, 2, \cdots, r),$$

where

$$\varphi_{n,k}^{(i)} \in C(X_i) \qquad (i = 1, 2, \cdots, r),$$

then (5) becomes

$$B_n(f)(x) = B_{n,\mathcal{T},\Phi}(f)(x) = \sum_{k_1=0}^{n} \cdots \sum_{k_r=0}^{n} \prod_{i=1}^{r} \varphi_{n,k_i}^{(i)}(x_i) T_{n,k_1,\cdots,k_r}(f). \tag{7}$$

Furthermore, in particular, if we take

$$\varphi_{n,k}^{(i)}(t) = \binom{n}{k} t^k (1-t)^{n-k} \qquad (t \in X_i, i = 1, 2, \cdots, r)$$

and define

$$T_{n,k_1,k_2,\cdots,k_r}(f) = f(k_1/n, k_2/n, \cdots, k_r/n) \qquad (f \in C(X, E)), \tag{8}$$

then (7) reduces to (1) in case of $E = \mathbb{R}$.

From now on let $X_i, i = 1, 2, \cdots, r$, be as in (6) and each operator $T_{n,k_1,k_2,\cdots,k_r}$ is defined by (8).

Lemma 3. *Suppose that for all $t \in X_i, i = 1, 2, \cdots, r$,*

$$\sum_{k=0}^{n} \Phi_{n,k}^{(i)}(t) = I, \quad \sum_{k=1}^{n} k\Phi_{n,k}^{(i)}(t) = ntI \tag{9}$$

and

$$\sum_{k=2}^{n} k(k-1)\Phi_{n,k}^{(i)}(t) = n(n-1)t^2 I. \tag{10}$$

Then we have

$$B_n(1_X \otimes a) = 1_X \otimes a, \quad B_n(e_j \otimes a) = e_j \otimes a$$

and

$$B_n(e_j^2 \otimes a) = e_j^2 \otimes a + \frac{1}{n}(e_j - e_j^2) \otimes a$$

for all $a \in E, n \geq 1$ *and* $j = 1, 2, \cdots, r$. *Here,* e_j *denotes the* j*-th coordinate function on* X *defined by*

$$e_j(x) = x_j \qquad (x = (x_1, x_2, \cdots, x_r) \in X).$$

Proof. Let $x \in X$. Then we have

$$B_n(1_X \otimes a)(x) = \sum_{k_1=0}^{n} \cdots \sum_{k_r=0}^{n} \prod_{i=1}^{r} \Phi_{n,k_i}^{(i)}(x_i)(a) = I(a) = a,$$

$$B_n(e_j \otimes a)(x) = \sum_{k_j=1}^{n} \Phi_{n,k_j}^{(j)}(x_j)\left(\frac{k_j}{n}a\right) = \frac{1}{n}(nx_j I)(a) = x_j a$$

and

$$B_n(e_j^2 \otimes a)(x) = \sum_{k_j=1}^{n} \Phi_{n,k_j}^{(j)}(x_j)\left(\frac{k_j^2}{n^2}a\right)$$

$$= \frac{1}{n^2}\left\{\sum_{k_j=1}^{n} k_j \Phi_{n,k_j}^{(j)}(x_j)(a) + \sum_{k_j=2}^{n} k_j(k_j-1)\Phi_{n,k_j}^{(j)}(x_j)(a)\right\}$$

$$= \frac{1}{n^2}\left\{(nx_j I)(a) + (n(n-1)x_j^2 I)(a)\right\} = x_j^2 a + \frac{1}{n}(x_j - x_j^2)a,$$

which implies desired result. □

Theorem 2. *Suppose that for every* $t \in X_i, i = 1, 2, \cdots, r$, *each operator* $\Phi_{n,k}^{(i)}(t)$ *is positive, and (9) and (10) are fulfilled. Then we have* $\lim_{n\to\infty} \|B_n(f) - f\| = 0$ *for all* $f \in C(X, E)$.

Proof. We take $G = \{e_1, e_2, \cdots, e_r\}$, which clearly separates the points of X (cf. Remark 1). Since each B_n is positive, by Lemma 1, it is quasi-positive and $\|B_n\| = \|B_n(1_X \otimes e)\|$. Therefore, the desired result follows from Theorem 1 and Lemma 3. □

Lemma 4. *Let $\{(\Psi^{(i)}_{n,k})_{n,k\geq 0} : i = 1, 2, \cdots, r\}$ be a set of infinite matrices of continuous mappings from X_i into $B[E]$ such that for all $t \in X_i, i = 1, 2, \cdots, r$,*

$$\Psi^{(i)}_{n,k+m}(t) = t^m \Psi^{(i)}_{n,k}(t) \qquad (n, k = 0, 1, 2, \cdots, m = 1, 2) \tag{11}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \Psi^{(i)}_{n-k,k}(t) = I \qquad (n = 0, 1, 2, \cdots). \tag{12}$$

Then we have

$$\sum_{k=1}^{n} k \binom{n}{k} \Psi^{(i)}_{n-k,k}(t) = ntI \tag{13}$$

and

$$\sum_{k=2}^{n} k(k-1) \binom{n}{k} \Psi^{(i)}_{n-k,k}(t) = n(n-1)t^2 I \tag{14}$$

for all $t \in X_i, i = 1, 2, \cdots, r$.

Proof. Since

$$k\binom{n}{k} = n\binom{n-1}{k-1} \qquad (1 \leq k \leq n)$$

and

$$k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2} \qquad (2 \leq k \leq n),$$

it follow from (11) and (12) that

$$\sum_{k=1}^{n} k \binom{n}{k} \Psi^{(i)}_{n-k,k}(t) = n \sum_{k=1}^{n} \binom{n-1}{k-1} \Psi^{(i)}_{n-k,k}(t)$$

$$= n \sum_{j=0}^{n-1} \binom{n-1}{j} \Psi^{(i)}_{n-j-1,j+1}(t) = nt \sum_{j=0}^{n-1} \binom{n-1}{j} \Psi^{(i)}_{n-1-j,j}(t) = ntI$$

and

$$\sum_{k=2}^{n} k(k-1) \binom{n}{k} \Psi^{(i)}_{n-k,k}(t) = n(n-1) \sum_{k=2}^{n} \binom{n-2}{k-2} \Psi^{(i)}_{n-k,k}(t)$$

$$= n(n-1)\sum_{j=0}^{n-2}\binom{n-2}{j}\Psi^{(i)}_{n-j-2,j+2}(t)$$

$$= n(n-1)t^2\sum_{j=0}^{n-2}\binom{n-2}{j}\Psi^{(i)}_{n-2-j,j}(t) = n(n-1)t^2 I.$$

Therefore, The equalities (13) and (14) hold. □

Theorem 3. *Let* $(\Psi^{(i)}_{n,k})_{n,k\geq 0}, i = 1,2,\cdots,r$, *be as in Lemma 4 with the additional assumption that all the operators* $\Psi^{(i)}_{n,k}(t)$ *are positive for each* $t \in X_i, i = 1,2,\cdots,r$, *and define*

$$\Phi^{(i)}_{n,k} = \begin{cases} \binom{n}{k}\Psi^{(i)}_{n-k,k} & (0 \leq k \leq n) \\ 0 & (k \geq n). \end{cases}$$

Then we have $\lim_{n\to\infty}\|B_n(f) - f\| = 0$ *for all* $f \in C(X,E)$.

Proof. This follows from Lemma 4 and Theorem 2. □

Let $\{\{\varphi^{(i)}_k\}_{k\geq 0} : i = 1,2,\cdots,r\}$ be a set of sequences of continuous mappings from X_i into $B[E]$, and we define

$$\Delta^n\varphi^{(i)}_k(t) = \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\varphi^{(i)}_{n+k-j}(t) \qquad (n,k = 0,1,2,\cdots). \quad (15)$$

Suppose that for all $t \in X_i, i = 1,2,\cdots,r$,

$$\varphi^{(i)}_{k+m}(t) = t^m\varphi^{(i)}_k(t) \qquad (k = 0,1,2,\cdots,m = 1,2) \quad (16)$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\Delta^{n-k}\varphi^{(i)}_k(t) = I \qquad (n = 0,1,2,\cdots). \quad (17)$$

Corollary 1. *Assume that all the operator* $\Delta^n\varphi^{(i)}_k(t)$ *given by (15) are positive for each* $t \in X_i, i = 1,2,\cdots r$, *and define*

$$\Phi^{(i)}_{n,k} = \begin{cases} \binom{n}{k}\Delta^{n-k}\varphi^{(i)}_k & (0 \leq k \leq n) \\ 0 & (k \geq n) \end{cases}$$

Then we have $\lim_{n\to\infty}\|B_n(f) - f\| = 0$ *for all* $f \in C(X,E)$.

Indeed, setting

$$\Psi_{n,k}^{(i)} = \Delta^n \varphi_k^{(i)} \qquad (n, k = 0, 1, 2, \cdots, i = 1, 2, \cdots r),$$

the conditions (16) and (17) imply the equalities (11) and (12), respectively. Thus, by Theorem 3, we have the claim of the corollary.

In particular, we take

$$\varphi_k^{(i)}(t) = t^k I \qquad (t \in X_i, i = 1, 2, \cdots, r, k = 0, 1, 2, \cdots).$$

Then we have

$$\Delta^n \varphi_k^{(i)}(t) = (1-t)^n t^k I \qquad (n, k = 0, 1, 2, \cdots, t \in X_i, i = 1, 2, \cdots, r),$$

and the conditions (16) and (17) are also satisfied. Furthermore, we get again the Bernstein operators given by (1).

Remark 2. *Suppose that E is a Banach space. Let $r = 1$ and let $\Phi_{n,k}^{(1)}$ be as in Corollary 1. Then $B_n(f)$ becomes the Φ-Bernstein approximation of f of order n due to Tucker [13]. Also, conversely if we have* $\lim_{n\to\infty} \|B_n(f) - f\| = 0$ *for every $f \in C(X_1, E)$, then*

$$\varphi_k^{(1)}(t) = t^k I \qquad (t \in X_1, k = 0, 1, 2, \cdots)$$

([13; Corollary]).

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