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CONTROLS OF THE OUTPUTS BY MEANS OF INPUTS

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Abstract: Let $\mathbf{f}_j$ be a member of a Hilbert space $\mathcal{H}_j$, S_j be a linear system of $\mathcal{H}_j$ and f_j be the output of $\mathbf{f}_j$ in the system. We assume that the outputs f_j are functions on a same set E. Then we consider the problems :

How to find the sum $f_1 + f_2$, the product $f_1 f_2$, and etc by means of their inputs $\mathbf{f}_j$?

The theory of reproducing kernels will give natural answers in natural situations for these problems.

Surprising enough, for very general nonlinear system S_j, we will be able to discuss the similar problems.

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1. A General Concept

Following Saitoh [1], we shall introduce a general theory for linear transforms in the framework of Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space. Let E be an abstract set and $\mathbf{h}$ be a Hilbert $\mathcal{H}$-valued function on E. Then we shall consider the linear transform

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}}, \mathbf{f} \in \mathcal{H} \tag{1.1}$$

from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ comprising all the complex valued function on E. In order to investigate the linear transform (1.1), we form a positive matrix $K(p,q)$ on E defined by

$$K(p,q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathcal{H}} \text{ on } E \times E. \tag{1.2}$$

Then, we obtain the following :

(I) The range in the linear transform (1.1) by $\mathcal{H}$ is characterized as the reproducing kernel Hilbert space $H_K(E)$ admitting the reproducing kernel $K(p,q)$.

(II) In general, we have the inequality

$$\|f\|_{H_K(E)} \leq \|\mathbf{f}\|_{\mathcal{H}}.$$

Here, for a member f of $H_K(E)$ there exists a uniquely determined $\mathbf{f}^* \in \mathcal{H}$ satisfying

$$f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathcal{H}} \text{ on } E$$

and

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathcal{H}}.$$

(III) In general, we have the inversion formula in (1.1) in the form

$$f \to \mathbf{f}^* \tag{1.3}$$

in (II) by using the reproducing kernel Hilbert space $H_K(E)$. However, this formula is, in general, involved and delicate. We need, case by case, arguments. In this paper, we assume that the inversion formula (1.3) is established.

(IV) Conversely, we assume that an isometrical mapping $\widetilde{L}$ from a reproducing kernel Hilbert space $H_K(E)$ admitting the reproducing kernel $K(p,q)$ on E onto a Hilbert space $\mathcal{H}$. Then we have the representation (1.1) by

$$\mathbf{h}(p) := \widetilde{L}K(.,p).$$

Furthermore, $\{\mathbf{h}(p); p \in E\}$ is complete in $\mathcal{H}$.

Now we shall consider two systems

$$f_j(p) = (\mathbf{f}_j, \mathbf{h}_j(p))_{\mathcal{H}_j}, \quad \mathbf{f}_j \in \mathcal{H}_j \tag{1.4}$$

in the above way by using $\{\mathcal{H}_j, E, \mathbf{h}_j\}_{j=1}^2$. Here, we assume that E is a same set for the two systems in order to have the output functions $f_1(p)$ and $f_2(p)$ on the same set E.

For example, we can consider the operators

$$f_1(p) + f_2(p)$$

and

$$f_1(p)\ f_2(p)$$

in $\mathcal{F}(E)$. Then, we can consider the following problems : How to represent the sum $f_1(p) + f_2(p)$ and the product $f_1(p)\ f_2(p)$ on E in terms of their inputs $\mathbf{f}_1$ and $\mathbf{f}_2$?

In this paper, we shall show that by using the theory of reproducing kernels we can give natural answers for these problems. Furthermore, for some very general nonlinear systems, we shall show that we can consider similar problems and solutions.

After introducing several operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ based on the above idea, we shall give typical and concrete operators, as examples.

2. Sum

By (I), $f_1 \in H_{K_1}(E)$ and $f_2 \in H_{K_2}(E)$, and we note that for the reproducing kernel Hilbert space $H_{K_1+K_2}(E)$ admitting the reproducing kernel

$$K_1(p,q) + K_2(p,q) \quad \text{on } E,$$

$H_{K_1+K_2}(E)$ is composed of all functions

$$f(p) = f_1(p) + f_2(p); \quad f_j \in H_{K_j}(E) \tag{2.1}$$

and its norm $\|f\|_{H_{K_1+K_2}(E)}$ is given by

$$\|f\|^2_{H_{K_1+K_2}(E)} = \min\{\|f_1\|^2_{H_{K_1}(E)} + \|f_2\|^2_{H_{K_2}(E)}\} \tag{2.2}$$

where the minimum is taken over $f_j \in H_{K_j}(E)$ satisfying (2.1) for f. Hence, in general, we have the inequality

$$\|f_1 + f_2\|^2_{H_{K_1+K_2}(E)} \leq \|f_1\|^2_{H_{K_1}(E)} + \|f_2\|^2_{H_{K_2}(E)}. \tag{2.3}$$

For the positive matrix $K_1 + K_2$ on E, we assume the expression in the form

$$K_1(p,q) + K_2(p,q) = (\mathbf{h}_S(q), \mathbf{h}_S(p))_{\mathcal{H}_S} \quad \text{on } E \times E \tag{2.4}$$

with a Hilbert space $\mathcal{H}_S$-valued function on E and further we assume that

$$\{\mathbf{h}_S(p); p \in E\} \text{ is complete in } \mathcal{H}_S. \tag{2.5}$$

Such a representation is, in general, possible (Saitoh [1], page 36 and see chapter 1, §5). Then, we can consider the linear mapping from $\mathcal{H}_S$ onto $H_{K_1+K_2}(E)$

$$f_S(p) = (\mathbf{f}_S, \mathbf{h}_S(p))_{\mathcal{H}_S}, \ \mathbf{f}_S \in \mathcal{H}_S \tag{2.6}$$

and we obtain the isometrical identity

$$\|f_S\|_{H_{K_1+K_2}(E)} = \|\mathbf{f}_S\|_{\mathcal{H}_S}. \tag{2.7}$$

Hence, for such representations (2.4) with (2.5), we obtain the isometrical mappings among the Hilbert space $\mathcal{H}_S$.

Now, for the sum $f_1(p)+f_2(p)$ there exists a uniquely determined $\mathbf{f}_S \in \mathcal{H}_S$ satisfying

$$f_1(p)+f_2(p) = (\mathbf{f}_S, \mathbf{h}_S(p))_{\mathcal{H}_S} \text{ on } E. \tag{2.8}$$

Then, $\mathbf{f}_S$ will be considered as a sum of $\mathbf{f}_1$ and $\mathbf{f}_2$ through these transforms and so, we shall introduce the notation

$$\mathbf{f}_S = \mathbf{f}_1[+]\mathbf{f}_2. \tag{2.9}$$

This sum for the members $\mathbf{f}_1 \in \mathcal{H}_1$ and $\mathbf{f}_2 \in \mathcal{H}_2$ is introduced through the three transforms induced by $\{\mathcal{H}_j,\ E,\ \mathbf{h}_j\}$ $(j=1,2)$ and $\{\mathcal{H}_s,\ E,\ \mathbf{h}_S\}$.

The operator $\mathbf{f}_1[+]\mathbf{f}_2$ is expressible in terms of $\mathbf{f}_1$ and $\mathbf{f}_2$ by the inversion formula

$$(\mathbf{f}_1, \mathbf{h}_1(p))_{\mathcal{H}_1} + (\mathbf{f}_2, \mathbf{h}_2(p))_{\mathcal{H}_2} \longrightarrow \mathbf{f}_1[+]\mathbf{f}_2 \tag{2.10}$$

in the sense (II) from $H_{K_1+K_2}(E)$ onto $\mathcal{H}_S$. Then, from (II) and (2.5) we have

Theorem 2.1. *We have a triangle inequality*

$$\|\mathbf{f}_1[+]\mathbf{f}_2\|^2_{\mathcal{H}_S} \le \|\mathbf{f}_1\|^2_{\mathcal{H}_1} + \|\mathbf{f}_2\|^2_{\mathcal{H}_2}. \tag{2.11}$$

If $\{\mathbf{h}_j(p);\ p \in E\}$ are complete in $\mathcal{H}_j$ $(j=1,2)$, then $\mathcal{H}_j$ and H_{K_j} are isometrical. By using the isometrical mappings induced by Hilbert space valued function $\mathbf{h}_j$ $(j=1,2)$ and $\mathbf{h}_S$, we can introduce the sum space of $\mathcal{H}_1$ and $\mathcal{H}_2$ in the form

$$\mathcal{H}_1[+]\mathcal{H}_2 \tag{2.12}$$

through the transforms.

For example, if for some positive number γ

$$K_1 \ll \gamma^2 K_2 \text{ on } E \tag{2.13}$$

that is, $\gamma^2 K_2 - K_1$ is a positive matrix on E, we have

$$H_{K_1}(E) \subset H_{K_2}(E) \tag{2.14}$$

and

$$\|f_1\|_{H_{K_2}(E)} \le \gamma \|f_1\|_{H_{K_1}(E)} \text{ for } f_1 \in H_{K_1}(E) \tag{2.15}$$

(Saitoh [1], page 37). Hence, in this case, we need not to introduce a Hilbert space $\mathcal{H}_S$ and the linear mapping (2.6) in Theorem (2.1) and we can use the linear mapping

$$(\mathbf{f}_2, \mathbf{h}_2(p))_{\mathcal{H}_2},\quad \mathbf{f}_2 \in \mathcal{H}_2$$

instead of (2.6) in Theorem 2.1.

3. Product

The product $K_1(p,q)\,K_2(p,q)$ is a positive matrix on E and the reproducing kernel Hilbert space $H_{K_1K_2}(E)$ admitting the reproducing kernel $K_1(p,q)\,K_2(p,q)$ is composed of all functions

$$f(p) = \sum_{n=1}^{\infty} f_{1,n}(p) f_{2,n}(p) \text{ on } E; \tag{3.1}$$

$$f_{j,n}(p) \in H_{K_j}(E) \quad (j = 1, 2)$$

and the norm in $H_{K_1K_2}(E)$ is given by

$$\|f\|^2_{H_{K_1K_2}(E)} = \min \sum_{n=1}^{\infty} \|f_{1,n}\|^2_{H_{K_1}(E)} \|f_{2,n}\|^2_{H_{K_2}(E)} \tag{3.2}$$

where the minimum is taken over all functions satisfying (3.1) for f. In particular, (3.1) converges absolutely on E. Especially we obtain the inequality

$$\|f_1 f_2\|_{H_{K_1K_2}(E)} \leq \|f_1\|_{H_{K_1}(E)} \|f_2\|_{H_{K_2}(E)}. \tag{3.3}$$

As in the sum, we assume that the representation

$$K_1(p,q)K_1(p,q) = (\mathbf{h}_P(q), \mathbf{h}_P(p))_{\mathcal{H}_P} \text{ on } E \times E \tag{3.4}$$

with a Hilbert space $\mathcal{H}_P$-valued function on E, and we assume that

$$\{\mathbf{h}_P(p); p \in E\} \text{ is complete in } \mathcal{H}_P. \tag{3.5}$$

Then we consider the linear mapping

$$f_P(p) = (\mathbf{f}_P, \mathbf{h}_P(p))_{\mathcal{H}_P}, \ \mathbf{f}_P \in \mathcal{H}_P \tag{3.6}$$

and we obtain the isometrical identity

$$\|f_P\|_{H_{K_1K_2}}(E) = \|\mathbf{f}_P\|_{H_P}. \tag{3.7}$$

Hence, for any product $f_1(p)f_2(p)$ there exists a uniquely determined $\mathbf{f}_P \in \mathcal{H}_P$ satisfying

$$f_1(p)f_2(p) = (\mathbf{f}_P, \mathbf{h}_P(p))_{\mathcal{H}_P} \text{ on } E. \tag{3.8}$$

Then, $\mathbf{f}_P$ will be considered as a product of $\mathbf{f}_1$ and $\mathbf{f}_2$ through these transforms and so, we shall introduce the notation

$$\mathbf{f}_P = \mathbf{f}_1[\times]\mathbf{f}_2. \tag{3.9}$$

This product for the members $\mathbf{f}_j \in \mathcal{H}_j$ $(j = 1, 2)$ is introduced through the three transforms induced by $\{\mathcal{H}_j,\ E,\ \mathbf{h}_j\}$ $(j = 1, 2)$ and $\{\mathcal{H}_P,\ E,\ \mathbf{h}_P\}$. The operator $\mathbf{f}_1[\times]\mathbf{f}_2$ is expressible in terms of $\mathbf{f}_1$ and $\mathbf{f}_2$ by the inversion formula

$$(\mathbf{f}_1, \mathbf{h}_1(p))_{\mathcal{H}_1}(\mathbf{f}_2, \mathbf{h}_2(p))_{\mathcal{H}_2} \longrightarrow \mathbf{f}_1[\times]\mathbf{f}_2 \tag{3.10}$$

in the sense (III) from $H_{K_1K_2}(E)$ onto $\mathcal{H}_P$. Then, we obtain

Theorem 3.1. *We have a Schwarz type inequality*

$$\|\mathbf{f}_1[\times]\mathbf{f}_2\|_{\mathcal{H}_P} \leq \|\mathbf{f}_1\|_{\mathcal{H}_1}\|\mathbf{f}_2\|_{\mathcal{H}_2}. \tag{3.11}$$

As in the sum space $\mathcal{H}_1[+]\mathcal{H}_2$ we can introduce the product space

$$\mathcal{H}_1[\times]\mathcal{H}_2 \tag{3.12}$$

through the three transforms under the completeness assumptions of $\mathbf{h}_j$ in $\mathcal{H}_j$ $(j = 1, 2)$.

For example, if for a positive γ

$$K_1K_2 \ll \gamma^2 K_1 \text{ on } E, \tag{3.13}$$

as in the sum, we can consider the linear transform

$$(\mathbf{f}_1, \mathbf{h}_1(p))_{\mathcal{H}_1}, \quad \mathbf{f}_1 \in \mathcal{H}_1$$

instead of (3.6).

In particular, in the setting in Section 1, we obtain

Corollary 3.1. *If $K^2 \ll \gamma^2 K$ on E for a positive constant γ and $\{\mathbf{h}(p); p \in E\}$ is complete in $\mathcal{H}$, then $\mathcal{H}$ is a commutative ring with the product* $\mathrm{f}[\times]\mathrm{g}$ *through the same three transforms $\{\mathcal{H},\ E,\ \mathbf{h}\}$. Furthermore if $\gamma = 1$, $\mathcal{H}$ is a Banach ring with the product.*

4. Differential

In the setting in Section 1, for simplicity we shall assume that E is an interval on the real line and the reproducing kernel Hilbert space $H_K(E)$ is composed of C^1-class functions on E. This smoothness reflects to the smoothness of the reproducing kernel $K(p,q)$ that

$$K_{1,1}(p,q) := \frac{\partial^2 K(p,q)}{\partial p \partial q} \text{ belongs to } C^1(E \times E) \tag{4.1}$$

and $\mathbf{h}(p)$ is differentiable on E in the space $\mathcal{H}$. Furthermore, we have

$$f'(p) = \left(\mathbf{f}, \frac{\partial \mathbf{h}(p)}{\partial p}\right) \text{ on } E \tag{4.2}$$

(Saitoh [1], pages 40-41). For the positive matrix (4.1) we have the expression

$$K_{1,1}(p,q) = \left(\frac{\partial \mathbf{h}(q)}{\partial q}, \frac{\partial \mathbf{h}(p)}{\partial p}\right) \quad \text{on } E \times E. \tag{4.3}$$

Here, we assume that

$$\{\frac{\partial \mathbf{h}(p)}{\partial p}; p \in E\} \text{ is complete in } \mathcal{H}. \tag{4.4}$$

Then, the derived function $f'(p)$ belongs to the reproducing kernel Hilbert space $H_{K_{1,1}(E)}$ admitting the reproducing kernel $K_{1,1}(p,q)$ and we have the isometrical identity

$$\|f'\|_{K_{1,1}} = \|\mathbf{f}\|_{\mathcal{H}}. \tag{4.5}$$

However, for the positive matrix $K_{1,1}(p,q)$ if we use a representation

$$K_{1,1}(p,q) = (\mathbf{h}_D(q), \mathbf{h}_D(p))_{\mathcal{H}_D} \quad \text{on } E \times E \tag{4.6}$$

for a Hilbert space $\mathcal{H}_D$-valued function such that

$$\{\mathbf{h}_D(p); p \in E\} \text{ is complete in } \mathcal{H}_D,$$

there exists a unique vector $\mathbf{f}_D \in \mathcal{H}_D$ satisfying

$$f'(p) = (\mathbf{f}_D, \mathbf{h}_D(p))_{\mathcal{H}_D} \quad \text{on } E. \tag{4.7}$$

Then, $\mathbf{f}_D$ will be considered as a derived vector through the transforms induced from $\{\mathcal{H}, E, \mathbf{h}_D\}$ and $\{\mathcal{H}_D, E, \mathbf{h}\}$. So, we shall write

$$\mathbf{f}_D = D\mathbf{f} \tag{4.8}$$

and the operator $D\mathbf{f}$ is expressible in terms of $\mathbf{f}$ by the inversion formula

$$\frac{\partial}{\partial p}(\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}} \longrightarrow D\mathbf{f} \tag{4.9}$$

in the sense (III) from $H_{K_{1,1}(E)}$ onto $\mathcal{H}_D$. Then we have

Theorem 4.1. *We have the inequality*

$$\|D\mathbf{f}\|_{\mathcal{H}_D} \leq \|\mathbf{f}\|_{\mathcal{H}}. \tag{4.10}$$

As in the sum and the product spaces, we can introduce the derived Hilbert space

$$D\mathcal{H} \tag{4.11}$$

through the transforms induced by $\{\mathcal{H}, E, \mathbf{h}\}$ and $\{\mathcal{H}_D, E, \mathbf{h}_D\}$ if $\{\mathbf{h}(p); p \in \mathcal{H}\}$ is complete in $\mathcal{H}$.

5. Integral

In the setting in Section 4, we can consider an integral of a Hilbert space $\mathcal{H}$ as in the derived space $D\mathcal{H}$.

We assume that

$$K^{1,1}(p,q) := \int_{p_0}^{p}\int_{q_0}^{q} K(\tilde{p},\tilde{q})d\tilde{p}d\tilde{q} \text{ on } E \times E \tag{5.1}$$

exists, and $K^{1,1}(p,q)$ is expressible in the form

$$K^{1,1}(p,q) := (\mathbf{h}_I(q), \mathbf{h}_I(p))_{\mathcal{H}_I} \text{ on } E \times E \tag{5.2}$$

for a Hilbert space $\mathcal{H}_I$-valued function $\mathbf{h}_I$ on E such that

$$\{\mathbf{h}_I(p); p \in E\} \text{ is complete in } \mathcal{H}_I.$$

Then, as in the derived vector we can introduce the integrated vector

$$\int_{p_0}^{p} \mathbf{f} \tag{5.3}$$

and the space

$$\int_{p_0}^{p} \mathcal{H}. \tag{5.4}$$

The vector (5.3) is expressible in terms of $\mathbf{f}$ by the inversion foumula

$$\int_{p_0}^{p} (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}} dp \longrightarrow \int_{p_0}^{p} \mathbf{f} \tag{5.5}$$

in the sense (III) from $H_{K^{1,1}}(E)$ onto $\mathcal{H}_I$. Then we have

Theorem 5.1. *We have the inequality*

$$\| \int_{p_0}^{p} \mathbf{f} \|_{\mathcal{H}_I} \leq \|f\|_{\mathcal{H}}. \tag{5.6}$$

6. Integral of Hilbert spaces

In the setting in Section 2, we shall consider systems with a continuous parameter t on an index set T as follows :

$$\{\mathcal{H}_t,\ E,\ \mathbf{h}_t\},\ t \in T. \tag{6.1}$$

We assume that the associated reproducing kernel

$$K_t(p,q) = (\mathbf{h}_t(q), \mathbf{h}_t(p))_{\mathcal{H}_t} \text{ on } E \times E. \tag{6.2}$$

is integrable on T with respect to a σ-finite positive measure $d\sigma$ on T and

$$K_T(p,q) = \int_T K_t(p,q)d\sigma(t) \text{ on } E \times E. \tag{6.3}$$

As a generalization of the sum of reproducing kernels, note that the reproducing kernel Hilbert space $H_{K_T}(E)$ is composed of all functions $f(p)$ which are expressible in the form

$$f(p) = \int_T f(p,t)d\sigma(t), \quad f(p,t) \in H_{K_t}(E) \tag{6.4}$$

and the norm $\|f\|_{H_{K_T}(E)}$ is given by

$$\|f\|^2_{H_{K_T}(E)} = \min \int_T \|f(p,t)\|^2_{H_{K_T}(E)} d\sigma(t) \tag{6.5}$$

where the minimum is taken over all the expressions (6.4) for f.

We shall assume the expression

$$K_T(p,q) = (\mathbf{h}_T(q), \mathbf{h}_T(p))_{\mathcal{H}_T} \text{ on } E \times E \tag{6.6}$$

by a Hilbert space $\mathcal{H}_T$-valued function $\mathbf{h}_T$ on E and $\{\mathbf{h}_T(p); p \in E\}$ is complete in $\mathcal{H}_T$. Then, we can introduce the integral of $\mathbf{f}_t$

$$\int_T \mathbf{f}_t \tag{6.7}$$

and

$$\int_T \mathcal{H}_t \tag{6.8}$$

similarly. The integral (6.7) is expressible in terms of $\mathbf{f}_t$ by the inversion formula

$$\int_T (\mathbf{f}_t, \mathbf{h}_t(p))_{\mathcal{H}_t} d\sigma(t) \longrightarrow \int_T \mathbf{f}_t \tag{6.9}$$

from $H_{K_T}(E)$ onto $\mathcal{H}_T$. Then we have

Theorem 6.1. *We have the inequality*

$$\| \int_T \mathbf{f}_t \|^2_{\mathcal{H}_T} \leq \int_T \|\mathbf{f}_t\|^2_{\mathcal{H}_T} d\sigma(t). \tag{6.10}$$

As shown in Appendix 1 in Saitoh [1], for very general nonlinear transforms of a reproducing kernel Hilbert space, their ranges belong to naturally determined reproducing kernel Hilbert spaces and norm inequalities hold in the nonlinear transforms. Therefore, for very general nonlinear transforms we can obtain similar results as in the linear transforms. In order to reduce this paper in a reasonable size, we shall present their exact formulations and applications in another papers. In the last part, we shall give a concrete example as a prototype result.

Our background idea comes from the idea of "convolution" and for our concrete applications we can give various solution of integral equations. In fact, the product $\mathbf{f}_1[\times]\mathbf{f}_2$ will be regarded as a convolution of $\mathbf{f}_1$ and $\mathbf{f}_2$. As a first step paper in our new concept, in the sequel we shall present typical concrete examples in the general and abstract operators. It seems that to examine concrete operators in various concrete transforms is to rich our mathematics.

7. Examples of Operators

7.1. We shall consider two linear transforms

$$f_j(p) = \int_T F_j(t)\overline{h(t,p)}\rho_j(t)dm(t), \quad p \in E \tag{7.1.1}$$

where ρ_j are positive continuous functions on T,

$$\int_T |h(t,p)|^2 \rho_j(t)dm(t) < \infty \text{ on } p \in E \tag{7.1.2}$$

and

$$\int_T |F_j(t)|^2 \rho_j(t)dm(t) < \infty. \tag{7.1.3}$$

We assume that $\{h(t,p); p \in E\}$ is complete in the spaces satisfying (7.1.3). Then, we consider the associated reproducing kernel on E

$$K_j(p,q) = \int_T h(t,p)\overline{h(t,p)}\rho_j(t)dm(t)$$

and, for example we consider the expression

$$K_1(p,q) + K_2(p,q) = \int_T h(t,p)\overline{h(t,p)}(\rho_1(t) + \rho_2(t))dm(t). \tag{7.1.4}$$

So, we can consider the linear transform

$$f(p) = \int_T F(t)\overline{h(t,p)}(\rho_1(t) + \rho_2(t))dm(t) \tag{7.1.5}$$

for functions F satisfying

$$\int_T |F(t)|^2(\rho_1(t) + \rho_2(t))dm(t) < \infty.$$

Hence, through the three transforms (7.1.1) and (7.1.5) we have the sum

$$(F_1[+]F_2)(t) = \frac{F_1(t)\rho_1(t) + F_2(t)\rho_2(t)}{\rho_1(t) + \rho_2(t)}. \tag{7.1.6}$$

7.2. We shall consider two linear transforms

$$f_j(x) = \int_{-\infty}^{\infty} e^{ixt} F_j(t)\rho_j(t)dt \tag{7.2.1}$$

for positive $L_1(-\infty,\infty)$ integrable functions ρ_j and for funtions F_j satisfying

$$\int_{-\infty}^{\infty} |F_j(t)|^2 \rho_j(t)dt < \infty. \tag{7.2.2}$$

We consider the associated reproducing kernels

$$K_j(x,y) = \int_{-\infty}^{\infty} e^{ixt} e^{-iyt} \rho_j(t)dt$$

and, for example we have the expression

$$K_1(x,y)K_2(x,y) = \int_{-\infty}^{\infty} e^{ixt} e^{-iyt} (\rho_1 * \rho_2)(t)dt \tag{7.2.3}$$

and the induced linear transform

$$f(x) = \int_{-\infty}^{\infty} F(t) e^{ixt} (\rho_1 * \rho_2)(t)dt \tag{7.2.4}$$

for functions F satisfying

$$\int_{-\infty}^{\infty} |F(t)|^2 (\rho_1 * \rho_2)(t)dt < \infty. \tag{7.2.5}$$

Meanwhile, we have the expression from (7.2.1)

$$f_1(x)f_2(x) = \int_{-\infty}^{\infty} e^{ixt} (F_1\rho_1) * (F_2\rho_2)(t)dt. \tag{7.2.6}$$

We thus have the product $F_1[\times]F_2$ through the transforms (7.2.1) $(j = 1, 2)$ and (7.2.4)

$$(F_1[\times]F_2)(t) = \frac{(F_1\rho_1)(F_2\rho_2)(t)}{(\rho_1 * \rho_2)(t)}. \tag{7.2.7}$$

In particular, we obtain the inequality

$$\int_{-\infty}^{\infty} \frac{|\int_{-\infty}^{\infty} F_1(\xi)\rho_1(\xi)F_2(t-\xi)\rho_2(t-\xi)d\xi|^2}{\int_{-\infty}^{\infty} \rho_1(\xi)\rho_2(t-\xi)d\xi} dt$$

$$\leq \int_{-\infty}^{\infty} |F_1(t)|^2 \rho_1(t)dt \int_{-\infty}^{\infty} |F_2(t)|^2 \rho_2(t)dt. \tag{7.2.8}$$

7.3. Let $K_j(z, \bar{u})$ be two reproducing kernels on $\{|z| < r_j\}$ defined by the expansions

$$K_j(z, \bar{u}) = \sum_{n=0}^{\infty} C_n^{(j)} z^n \bar{u}^n, \quad (C_n^{(j)} > 0). \tag{7.3.1}$$

Then, the reproducing kernel Hilbert spaces H_{K_j} are composed of all analytic funtions $f_j(z)$ defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_n^{(j)} z^n \text{ on } \{|z| < r_j\} \tag{7.3.2}$$

with finite norms

$$\|f_j\|_{H_{K_j}}^2 = \sum_{n=0}^{\infty} \frac{|a_n^{(j)}|^2}{C_n^{(j)}}, \tag{7.3.3}$$

as we see easily, respectively. We have the expressions

$$K_1(z, \bar{u}) K_2(z, \bar{u}) = \sum_{n=0}^{\infty} \left(\sum_{\nu+\mu=n} C_\nu^{(1)} C_\mu^{(2)} \right) z^n \bar{u}^n, \tag{7.3.4}$$

and

$$f_1(z) f_2(z) = \sum_{n=0}^{\infty} \left(\sum_{\nu+\mu=n} a_\nu^{(1)} a_\mu^{(2)} \right) z^n. \tag{7.3.5}$$

Hence, we have the sum $\mathbf{a}^{(1)}[+]\mathbf{a}^{(2)}$ $\left(\mathbf{a}^{(j)} = (a_0^{(j)}, a_1^{(j)}, \cdots)\right)$ satisfying (7.3.3) for $j = 1, 2$

$$\{ \sum_{\nu+\mu=n} a_\nu^{(1)} a_\mu^{(2)} \}_{n=0}^{\infty} \tag{7.3.6}$$

through the two transforms (7.3.2) satisfying (7.3.3) and the transform

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{7.3.7}$$

satisfying

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{\sum_{\nu+\mu=n} C_\nu^{(1)} C_\mu^{(2)}} < \infty. \tag{7.3.8}$$

In particular, we have the inequality

$$\sum_{n=0}^{\infty} \frac{|\sum_{\nu+\mu=n} a_\nu^{(1)} a_\mu^{(2)}|^2}{\sum_{\nu+\mu=n} C_\nu^{(1)} C_\mu^{(2)}} < \left(\sum_{n=0}^{\infty} \frac{|a_n^{(1)}|^2}{C_n^{(1)}}\right)\left(\sum_{n=0}^{\infty} \frac{|a_n^{(2)}|^2}{C_n^{(2)}}\right). \tag{7.3.9}$$

7.4. Recall that

$$K(x,y) = \frac{1}{2} e^{-|x-y|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ixt} e^{-iyt}}{t^2+1} dt \tag{7.4.1}$$

is the reproducing kernel for the Sobolov space H_K comprising all absolutely continuous function $f(x)$ on $(-\infty, \infty)$ with finite norms

$$\|f\|_{H_K}^2 = \int_{-\infty}^{\infty} \left(|f(x)|^2 + |f'(x)|^2\right) dx. \tag{7.4.2}$$

Then,

$$\begin{aligned} K(x,y)^2 &= \frac{1}{4} e^{-2|x-y|} \\ &= \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i2xt} e^{-i2yt}}{t^2+1} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ixt} e^{-iyt}}{t^2+4} dt. \end{aligned} \tag{7.4.3}$$

Hence,

$$K(x,y)^2 \ll K(x,y) \text{ on } (-\infty, \infty). \tag{7.4.4}$$

We consider the linear transform induced from (7.4.1)

$$f_j(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F_j(t) e^{ixt}}{t^2+1} dt \tag{7.4.5}$$

for functions F_j satisfying

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|F_j(t)|^2}{t^2+1} dt < \infty. \tag{7.4.6}$$

Then, we have

$$f_1(x)f_2(x) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{1}{t^2+1}\left(\left(\frac{F_1(t))}{t^2+1}\right) * \left(\frac{F_2(t))}{t^2+1}\right)\right)(t)(t^2+1)e^{ixt}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(t)e^{ixt}}{t^2+1}dt. \tag{7.4.7}$$

Hence, through the same three transforms (7.4.5) we have the product

$$(F_1[\times]F_2)(t) = \frac{1}{2\pi}\{\left(\frac{F_1(t))}{t^2+1}\right) * \left(\frac{F_2(t))}{t^2+1}\right)\}(t)(t^2+1). \tag{7.4.8}$$

By Corollary 3.1, the space (7.4.6) is a Banach ring under the product (7.4.8).

7.5. For any separable Hilbert space H and its complete orthonormal system $\{\mathbf{f}_n\}_{n=1}^{\infty}$, we shall consider the linear transform from H onto l^2

$$a_n = (\mathbf{f}, \mathbf{f}_n)_H. \tag{7.5.1}$$

Then, the reproducing kernel for l^2 is Kronecker's δ_{nm}, and of course

$$\delta_{nm}^2 \ll \delta_{nm}. \tag{7.5.2}$$

Hence for

$$\mathbf{f} = \sum_{n=1}^{\infty} a_n\mathbf{f}_n \in H$$

and

$$\mathbf{g} = \sum_{n=1}^{\infty} b_n\mathbf{f}_n \in H$$

through the three transforms (7.5.1), we have the product

$$\mathbf{f}[\times]\mathbf{g} = \sum_{n=1}^{\infty} a_n b_n \mathbf{f}_n. \tag{7.5.3}$$

Under this product, H is a Banach ring.

7.6. For $q > \frac{1}{2}$,

$$K_q(z,\bar{u}) = \frac{\Gamma(2q)}{(z+\bar{u})^{2q}}$$

$$= \int_{-\infty}^{\infty} e^{-zt}e^{-\bar{u}t}t^{2q-1}dt \tag{7.6.1}$$

is the reproducing kernel for the Bergman-Selbert space $H_{K_q}(R^+)$ on the complex half plane $R^+ = \{Re\ z > 0\}$ comprising all analytic functions $f(z)$ on R^+ with finite norms

$$\|f\|^2_{H_{K_q}} = \frac{1}{\pi\Gamma(2q-1)} \int\int_{R^+} |f(z)|^2 [2Re\ z]^{2q-2} dx\, dy. \tag{7.6.2}$$

Note that

$$K_1(z, \bar{u}) = \int_0^\infty e^{-zt} e^{-\bar{u}t} t dt \tag{7.6.3}$$

and

$$\begin{aligned} \frac{\partial^2 K_1(z,\bar{u})}{\partial z \partial \bar{u}} &= \frac{6}{(z+\bar{u})^4} \\ &= \int_0^\infty e^{-zt} e^{-\bar{u}t} t^3 dt \\ &= K_2(z, \bar{u}). \end{aligned} \tag{7.6.4}$$

In the transform induced from (7.6.3)

$$f(z) = \int_0^\infty e^{-zt} F(t) t dt \tag{7.6.5}$$

for functions F satisfying

$$\int_0^\infty |F(t)|^2 t\ dt < \infty \tag{7.6.6}$$

we have

$$f'(z) = \int_0^\infty e^{-zt} F(t)(-t^2) dt. \tag{7.6.7}$$

By using the transform induced from (7.6.4)

$$f'(z) = \int_0^\infty e^{-zt} G(t)(t^3) dt \tag{7.6.8}$$

for a function G satisfying

$$\int_0^\infty |G(t)|^2 t^3 dt < \infty, \tag{7.6.9}$$

we have the derived vector of F

$$DF = -\frac{F(t)}{t} \tag{7.6.10}$$

through the transforms (7.6.5) and (7.6.8).

Furthermore, by integrating (7.6.4) from $+\infty$ to z and by using the corresponding transforms to (7.6.8) and (7.6.5) we have the integrated vector of F as follows :

$$\int F = -tF(t). \tag{7.6.11}$$

7.7. Note that for example, for the nonlinear transform

$$d_1 f'' + d_2 f'f + d_3 f^3 \quad (d_j : \text{ constants}) \tag{7.7.1}$$

for $f \in H_{K_1}$ in 7.6, it has a specially simple structure and it belongs to the space H_{K_3}. Furthermore we have the inequality

$$\frac{120}{127}\|d_1 f'' + d_2 f'f + d_3 f^3\|^2_{H_{K_3}}$$

$$\leq \|f\|^2_{H_{K_1}} \left(\frac{1}{120}|d_1|^2 + \frac{1}{6}|d_2|^2\|f\|^2_{H_{K_1}} + |d_3|^2\|f\|^4_{H_{K_1}}\right) \tag{7.7.2}$$

(Saitoh [1], Appendix 2).

In the transform (7.6.5), we have

$$f''(z) = \int_0^\infty e^{-zt}F(t)t^3 dt,$$

$$f'(z)f(z) = \int_0^\infty e^{-zt}\left((F(t)(-t^2)) * (F(t)(t))\right)(t)dt$$

and

$$f(z)^3 = \int_0^\infty e^{-zt}\left(F(t)t\right)^{*3}(t)dt.$$

Here, $(\quad)^{*3}$ means the three times convolution product. Hence we have the expression

$$d_1 f''(z) + d_2 f'(z)f(z) + d_3 f(z)^3 = \int_0^\infty e^{-zt}\{d_1 t^{-2}F(t) + d_2 t^{-5}$$

$$\left((F(t)(-t^2)) * (F(t)t)\right)(t) + d_3 t^{-5}\left(F(t)t\right)^{*3}(t)\}t^5 dt. \tag{7.7.3}$$

Hence, by using the transform

$$f(z) = \int_0^\infty e^{-zt}G(t)t^5 dt \tag{7.7.4}$$

for functions G satisfying

$$\int_0^\infty |G(t)|^2 t^5 dt < \infty \tag{7.7.5}$$

induced from the expression (7.6.1) for $q = 3$, we have

$$\begin{aligned} G(t) &= d_1 t^{-2} F(t) + d_2 t^{-5} \left((F(t)(-t^2)) \right. \\ &\quad \left. * (F(t)t) \right)(t) + d_3 t^{-5} \left(F(t)t \right)^{*3}(t). \end{aligned} \tag{7.7.6}$$

Note that the nonlinear transform (7.7.1) is transformed to the form (7.7.6) containing convolutions.

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Reference

[1] S. Saitoh, *Integral Transforms, Reproducing Kernels and their Applications*, Pitman Research Notes in Mathematics Series, **369**, Addison Wesley Longman, UK, 1998.