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ハンケル作用素の積が再びハンケル作用素になる為の条件について

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If $e_n(z) = z^n$ for $|z| = 1$ and $n = 0, \pm 1, \pm 2, \cdots$, then the functions e_n constitute an orthonormal basis for L^2 and the functions e_n, $n = 0, 1, 2, \cdots$ constitute an orthonormal basis for H^2. Let L^∞ be the set of all essentially bounded functions in L^2 and let $H^\infty = H^2 \cap L^\infty$. For a function $\varphi \in L^\infty$, the Toeplitz operator T_φ on H^2 is given by $T_\varphi f = P(\varphi f)$ for $f \in H^2$ where P is the orthogonal projection from L^2 onto H^2 and the Hankel operator H_φ on H^2 is given by $H_\varphi f = J(I - P)(\varphi f)$ for $f \in H^2$ where J is the unitary operator on L^2 defined by $Je_{-n} = e_{n-1}$.

Concerning these operators, the following results are known.

Proposition 1. If $\mathcal{M}$ is a non-zero closed invariant subspace of T_z, then there exists an inner function g uniquely,up to a unimodular constant, such that

$$\mathcal{M} = T_g H^2 \quad \text{and} \quad \mathcal{M}^\perp = H_{\overline{g}}{}^* H^2.$$

Proposition 2. If φ is a non-constant function in L^∞, then $\sigma_p(T_\varphi) \cap \overline{\sigma_p(T_\varphi{}^*)} = \emptyset$ where $\sigma_p(\cdot)$ denotes the point spectrum.

Proposition 3. For any $\psi \in H^\infty$, $H_\varphi T_\psi = H_{\varphi\psi}$ and $T_\psi{}^* H_\varphi = H_{\varphi\psi^*} = H_\varphi T_{\psi^*}$.

Proposition 4. $H_\psi{}^* H_\varphi = T_{\overline{\psi}\varphi} - T_{\overline{\psi}} T_\varphi$.

Proposition 5. The following assertions are equivalent.

(1) $\mathcal{N}_{H_\varphi} \neq \{o\}$.

(2) $[H_\varphi H^2]^{\sim L^2} \neq H^2$.

(3) $\varphi = \overline{g}h$ for some inner function g and $h \in H^\infty$ such that g and h have no common non-constant inner factor.

Now we shall consider the following theorems.

Theorem 1. $H_\varphi H_\psi = O$ if and only if $H_\varphi = O$ or $H_\psi = O$.

Proof. By Proposition 4, we have

$$\begin{aligned} O = H_\varphi H_\psi &= T_{\overline{\varphi^*\psi}} - T_{\overline{\varphi^*}} T_\psi \\ \rightleftharpoons \quad & \varphi^* \in H^\infty \quad \text{or} \quad \psi \in H^\infty \\ \rightleftharpoons \quad & H_\varphi = H_{\varphi^*}{}^* = O \quad \text{or} \quad H_\psi = O. \end{aligned}$$

Theorem 2. The product $H_\varphi H_\psi$ of two non-zero Hankel operators H_φ and H_ψ is also a Hankel operator if and only if

$$\varphi = \overline{q}h \quad \text{and} \quad \psi = \overline{q}k$$

where $q(z) = (z - \overline{\lambda})(1 - \lambda z)^{-1}$ for some complex number λ such as $|\lambda| < 1$ and $h,\ k \in H^\infty$ such that each h and k is non-zero and has no inner factor q. And, in this case,

$$H_\varphi H_\psi = \alpha_q H_{\overline{q}hk}$$

where α_q is the non-zero eigenvalue of $H_{\overline{q}}$.

To prove Theorem 2, we need the following lemmas.

Let $H_\varphi H_\psi = H_u$ for non-zero Hankel operators H_φ and H_ψ. Then we have the following.

Lemma 1. $0 \in \sigma_p(H_\varphi) \cap \sigma_p(H_\psi)$.

Proof. Since

$$\begin{aligned} H_\varphi T_z H_\psi &= T_z{}^* H_\varphi H_\psi = T_z{}^* H_u \\ &= H_u T_z = H_\varphi H_\psi T_z = H_\varphi T_z{}^* H_\psi, \end{aligned}$$

$H_\varphi(T_z - T_z{}^*)H_\psi = O$. If $0 \notin \sigma_p(H_\varphi)$, then $(T_z - T_z{}^*)H_\psi = O$ and $H_\psi = O$ because $0 \notin \sigma_p(T_z - T_z{}^*)$ by Proposition 2. This contradicts the assumption that H_ψ is non-zero. Therefore $0 \in \sigma_p(H_\varphi)$.

If $0 \notin \sigma_p(H_\psi)$, then $H_\psi H^2$ is dense in H^2 by Proposition 5 and $H_\varphi(T_z - T_z{}^*) = O$ and hence $H_\varphi = O$ because $(T_z - T_z{}^*)H^2$ is dense in H^2 by Proposition 2. And this also contradicts the assumption and hence $0 \in \sigma_p(H_\psi)$.

If $0 \in \sigma_p(H_\varphi)$, then, by Proposition 5, $\varphi = \overline{g}h$ for some inner function g and $h \in H^\infty$ and $H_{\varphi g} = H_h = O$. And we have the following.

Lemma 2. For an inner function g, the following assertions are equivalent.

$$(1)\ H_{\varphi g} = O, \quad (2)\ H_{\psi g} = O \quad \text{and} \quad (3)\ H_{ug} = O.$$

Proof. Since, by Proposition 3

$$\begin{aligned} H_{\varphi g} H_\psi &= T_{g^*}{}^* H_\varphi H_\psi = T_{g^*}{}^* H_u = H_u T_g \\ &= H_{ug} = H_\varphi H_\psi T_g = H_\varphi H_{\psi g} \end{aligned}$$

and since H_φ and H_ψ are non-zero by the assumption, the assertion follows from Theorem 1.

Lemma 3. $\dim[H_u{}^* H^2]^{\sim L^2} = 1$.

Proof. Since $\mathcal{N}_{H_u} \neq \{o\}$ by Lemma 1,we have, by Proposition 1,

$$[H_u{}^* H^2]^{\sim L^2} = H_{\overline{q}}{}^* H^2 \quad \text{and} \quad \mathcal{N}_{H_u} = T_q H^2$$

for some inner function q. If $\dim[H_u{}^* H^2]^{\sim L^2} \geq 2$, then

$$\dim[T_q H^2]^\perp = \dim[H_u{}^* H^2]^{\sim L^2} \geq 2$$

and there exists a closed invariant subspace $\mathcal{M}$ of T_z such as $T_q H^2 \subset \mathcal{M} \subset H^2$. Since $\mathcal{M} = T_{q_1} H^2$ for some non-constant inner function q_1 by Proposition 1, $q = q_1 q_2$ for some non-constant inner function q_2. Since, by Proposition 3,

$$\begin{aligned} O = H_u T_q &= H_u T_{q_1} T_{q_2} = T_{q_1{}^*}{}^* H_u T_{q_2} \\ &= T_{q_1{}^*}{}^* H_\varphi H_\psi T_{q_2} = H_\varphi T_{q_1} H_\psi T_{q_2} = H_{\varphi q_1} H_{\psi q_2}, \end{aligned}$$

$H_{\varphi q_1} = O$ or $H_{\psi q_2} = O$ by Theorem 1 and, by Lemma 2, $H_{uq_1} = O$ or $H_{uq_2} = O$. If $H_{uq_1} = O$, then, by Proposition 3, $T_{q_1} H^2 \subseteq \mathcal{N}_{H_u} = T_q H^2 = T_{q_1} T_{q_2} H^2$ and

$H^2 \subseteq T_{q_2}H^2$ because T_{q_1} is an isometry and this contradicts that q_2 is a non-constant inner function. Hence $H_{uq_1} \neq O$. By the same reason, $H_{uq_2} \neq O$. These contradict the above result that $H_{uq_1} = O$ or $H_{uq_2} = O$. Therefore $\dim[H_u{}^*H^2]^{\sim L^2} \leq 1$. By Theorem 1, $H_u \neq O$ because H_φ and H_ψ are non-zero by the assumption and $\mathcal{N}_{H_u} \neq H^2$ and hence $\dim[H_u{}^*H^2]^{\sim L^2} = \dim[\mathcal{N}_{H_u}]^\perp \geq 1$. Therefore $\dim[H_u{}^*H^2]^{\sim L^2} = 1$.

Proof of Theorem 2. $\underline{(\rightarrow)}$; By Lemma 3 and its proof, we have

$$\dim \mathcal{N}_{T_q{}^*} = \dim[H_u{}^*H^2]^{\sim L^2} = 1$$

and $\mathcal{N}_{T_q{}^*}$ is an eigenspace of $T_z{}^*$ and hence, for some $\lambda \in \mathbb{C}$ such as $|\lambda| < 1$, $q(z) = (z - \overline{\lambda})(1 - \lambda z)^{-1}$. Since $\mathcal{N}_{H_u} = T_qH^2$, $H_{uq} = H_uT_q = O$ by Proposition 3 and, by Lemma 2, $H_{\varphi q} = O$ and $H_{\psi q} = O$ and hence $\varphi q = h$ and $\psi q = k$ for some $h,\ k \in H^\infty$. Therefore $\varphi = \overline{q}h$ and $\psi = \overline{q}k$ because q is inner. Since $H_\varphi \neq O$ and $H_\psi \neq O$ by the assumption, each h and k is non-zero and has no inner factor q.

$\underline{(\leftarrow)}$; Conversely, if $\varphi = \overline{q}h$ and $\psi = \overline{q}k$ where $h,\ k \in H^\infty$ and $q(z) = (z - \overline{\lambda})(1 - \lambda z)^{-1}$ for some $\lambda \in \mathbb{C}$ such as $|\lambda| < 1$, then $H_{\overline{q}}$ is a partial isometry by Proposition 4 and

$$H_{\overline{q}}H^2 = H_{\overline{q}}H_{\overline{q}}{}^*H^2 = (I - T_{q^*}T_{q^*}{}^*)H^2 = \mathcal{N}_{T_{q^*}{}^*}.$$

Since $\mathcal{N}_{T_{q^*}{}^*} = \{\mathbb{C}(1 - \overline{\lambda}z)^{-1}\}$ because $q^*(z) = (z - \lambda)(1 - \overline{\lambda}z)^{-1}$, $H_{\overline{q}}(1 - \overline{\lambda}z)^{-1} = \alpha_q(1 - \overline{\lambda}z)^{-1}$ for some $\alpha_q \in \mathbb{C}$. Hence, for any $f \in H^2$, we have, by Proposition 3,

$$\begin{aligned} H_\varphi H_\psi f &= H_{\overline{q}h}H_{\overline{q}k}f = T_{h^*}{}^*H_{\overline{q}}H_{\overline{q}}T_kf \\ &= T_{h^*}{}^*H_{\overline{q}}\{\mu(1 - \overline{\lambda}z)^{-1}\} \quad \text{for some } \mu \in \mathbb{C} \\ &\quad \text{(because } H_{\overline{q}}T_kf \in H_{\overline{q}}H^2 = \{\mathbb{C}(1 - \overline{\lambda}z)^{-1}\}) \\ &= T_{h^*}{}^*\{\mu\alpha_q(1 - \overline{\lambda}z)^{-1}\} = \alpha_qT_{h^*}{}^*H_{\overline{q}}T_kf \\ &= \alpha_qH_{\overline{q}hk}f \end{aligned}$$

and $H_\varphi H_\psi = \alpha_qH_{\overline{q}hk} = H_{\alpha_q\overline{q}hk}$. Therefore $H_\varphi H_\psi$ is a Hankel operator and $\alpha_q \neq 0$ by Theorem 1.

Corollary. Every non-zero idempotent Hankel operator is of the form $\frac{1}{\alpha_q}H_{\overline{q}}$ where $q(z) = (z - \overline{\lambda})(1 - \lambda z)^{-1}$ for some $\lambda \in \mathbb{C}$ such as $|\lambda| < 1$ and α_q is the non-zero eigenvalue of $H_{\overline{q}}$.

Proof. If ${H_\varphi}^2 = H_\varphi$, then, by Theorem 2, $\varphi = \overline{q}h$ where $h \in H^\infty$ and $q(z) = (z - \overline{\lambda})(1 - \lambda z)^{-1}$ for some complex number λ such as $|\lambda| < 1$ and

$$H_{\overline{q}h} = H_\varphi = {H_\varphi}^2 = \alpha_q H_{\overline{q}h^2} = H_{\alpha_q \overline{q}h^2}$$

where α_q is the non-zero eigenvalue of $H_{\overline{q}}$ and hence $H_{\overline{q}h(1-\alpha_q h)} = H_{\overline{q}h} - H_{\alpha_q\overline{q}h^2} = O$. Therefore, by using Theorem 2 again, we have

$$H_{\overline{q}h} H_{\overline{q}(1-\alpha_q h)} = \alpha_q H_{\overline{q}h(1-\alpha_q h)} = O$$

and $H_{\overline{q}} - \alpha_q H_{\overline{q}h} = H_{\overline{q}(1-\alpha_q h)} = O$ by Theorem 1 because $H_{\overline{q}h} = H_\varphi \neq O$ by the assumption and hence $H_\varphi = H_{\overline{q}h} = \frac{1}{\alpha_q} H_{\overline{q}}$. Conversely $\left(\frac{1}{\alpha_q} H_{\overline{q}}\right)^2 = \frac{1}{\alpha_q} H_{\overline{q}}$ by Theorem 2.

Remark. The concrete value of α_q in Theorem 2 is given, by the direct calculation, as follows : Since $H_{\overline{q}}(1-\overline{\lambda}z)^{-1} = \alpha_q(1-\overline{\lambda}z)^{-1}$ because $H_{\overline{q}}H^2 = \{\mathbb{C}(1-\overline{\lambda}z)^{-1}\}$ and since

$$\begin{aligned}
(\overline{z}-\lambda)\quad & (1-\overline{\lambda}\overline{z})^{-1}(1-\overline{\lambda}z)^{-1} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \overline{\lambda}^{m+n}\overline{z}^{m+1}z^n - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda\overline{\lambda}^{m+n}\overline{z}^m z^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \overline{\lambda}^{2n+k}\overline{z}^{k+1} - \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \lambda\overline{\lambda}^{2n+k}\overline{z}^k \\
&\quad + (\text{analytic part}),
\end{aligned}$$

$$\begin{aligned}
H_{\overline{q}}&(1-\overline{\lambda}z)^{-1} \\
&= J(I-P)(\overline{z}-\lambda)\quad (1-\overline{\lambda}\overline{z})^{-1}(1-\overline{\lambda}z)^{-1} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \overline{\lambda}^{2n+k} z^k - \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \lambda\overline{\lambda}^{2n+k} z^{k-1} \\
&= (1-|\lambda|^2)(1-\overline{\lambda}^2)^{-1}(1-\overline{\lambda}z)^{-1}.
\end{aligned}$$