数理解析研究所講究録
1085 巻 1999 年 91-98

Inflection points and singularities on planar rational cubic curve segments

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We obtain the distribution of inflection points and singularities on a parametric rational cubic curve segment with a great help of *Mathematica* (A System of for Doing Mathematics by Computer). The reciprocal numbers of the magnitudes of the end slopes determine the occurrence of inflection points and singularities on the segment. Its use enables us to check whether the segment has inflection points or a singularity (a loop or a cusp) without practical calculation the segment and to get an idea how to place control vertices and how to choose weights for the rational Bézier cubic curve segment to preserve the fair shape.
keywords: inflection points, singularities, rational cubic segments

1 Introduction

Polynomial cubic and rational cubic curves have been widely used in computer-aided design. However, the polynomial cubic curves do not always generate "visually pleasing", "shape preserving" (or simply "fair") interpolants which do not contain *unwanted* or *unplaned* interior inflection points and singularities to a set of planar data points. There is a considerable literature on numerical methods for generating shape preserving interpolations; see for example, Farin(1995), Späth(1995a,1995b), and the references therein. A way to overcome this problem is to consider the rational cubic curve segments $z(t), 0 \leq t \leq 1, u = 1-t$ with a *single* rationality parameter $p > 0$, for example, in Sakai(1996,1997)

$$\boldsymbol{z}(t) = \boldsymbol{a}_0 t + \boldsymbol{b}_0 u + \boldsymbol{c}_0 t^3/(1+pu) + \boldsymbol{d}_0 u^3/(1+pt) \tag{1.1}$$

and

$$\boldsymbol{z}(t) = \boldsymbol{a}_1 t + \boldsymbol{b}_1 u + \boldsymbol{c}_1 t^2 u/(1+ptu) + \boldsymbol{d}_1 t u^2/(1+ptu). \tag{1.2}$$

The object of this paper is to obtain the distribution of inflection points and a singularity (a loop or a cusp) on the planar rational Bézier cubic curve of the nonstandard

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form:

$$\sum_{i=0}^{3} B_i(t)w_i\boldsymbol{p}_i / \sum_{i=0}^{3} B_i(t)w_i, \quad B_i(t) = \binom{3}{i} t^i u^{3-i}. \tag{1.3}$$

The control points $\boldsymbol{p}_i$ belong to R^2 and we assume that the weights w_i are all positive. We may always transform the above nonstandard form to the standard one with the end weights being unity by replacing w_i with $c^i w_i, c = \sqrt[3]{w_0/w_3}$ where the new weights correspond to the new control vertices. The present paper considers the rational Bézier curve segment (1.3) in nonstandard form since (i) little difference is in the analysis by means of *Mathematica* required for rational cubic segments in nonstandard and standard forms, (ii) little difference is in complexity of representation of the obtained results, (iii) rational Bézier in nonstandard form can arise, and (iv) rational segments in nonstandard form are easier to use [Farin(1995)]. Note that it has more flexibility than the cubic curve segments (1.1) and (1.2) since it has more degrees of freedom where the segment (1.2) is a special case of (1.3) with $(w_0, w_1, w_2, w_3) = (1, 1+p/3, 1+p/3, 1)$ and that the distribution of the inflection points and singularities on (1.3) in the present paper extends the one obtained in Sakai(1997). Sections 2-3 describe the distribution on the rational *cubic/cubic* curve of the form:

$$\boldsymbol{z}(t) = \frac{w_0u^3\boldsymbol{z}_0 + u^2t(w_0\boldsymbol{z}'_0 + 3w_1\boldsymbol{z}_0) + ut^2(-w_3\boldsymbol{z}'_1 + 3w_2\boldsymbol{z}_1) + w_3t^3\boldsymbol{z}_1}{w_0u^3 + 3w_1u^2t + 3w_2ut^2 + w_3t^3} \tag{1.4}$$

which satisfies Hermite data: $\boldsymbol{z}^{(k)}(i) = \boldsymbol{z}_i^{(k)}, i = 0, 1, k = 0, 1$. We derive the shape classification of the curve segment (1.4) in terms of coefficients λ and μ of $\Delta\boldsymbol{z}(= \boldsymbol{z}_1 - \boldsymbol{z}_0) = \lambda\boldsymbol{z}'_0 + \mu\boldsymbol{z}'_1$. In Section 4, note that the above segments (1.3) and (1.4) coincide if

$$\boldsymbol{z}_0 = \boldsymbol{p}_0, \boldsymbol{z}'_0 = (3w_1/w_0)(\boldsymbol{p}_1 - \boldsymbol{p}_0), \boldsymbol{z}'_1 = (3w_2/w_3)(\boldsymbol{p}_3 - \boldsymbol{p}_2), \boldsymbol{z}_1 = \boldsymbol{p}_3. \tag{1.5}$$

to obtain the distribution of inflection points and singularities (a loop and a cusp) on the nonstandard planar rational Bézier cubic curve (1.3) which gives us an idea how to place the control vertices and how to choose the weights for the fair rational Bézier cubic curve segment.

2 Inflection points and singularities on rational cubic curve segments (1.4)

In this paper, we assume that the tangent vectors $\boldsymbol{z}'_i, i = 0, 1$ are not parallel, i.e., $\boldsymbol{z}'_0 \times \boldsymbol{z}'_1 \neq 0$ where given two vectors $\boldsymbol{A} = (A_1, A_2), \boldsymbol{B} = (B_1, B_2)$, we write $\boldsymbol{A} \times \boldsymbol{B} = A_1B_2 - A_2B_1$. Note that if $\boldsymbol{z}'_i, i = 1, 2$ are not parallel, then $\Delta\boldsymbol{z}(= \boldsymbol{z}_1 - \boldsymbol{z}_0)$ can be represented as $\Delta\boldsymbol{z} = \lambda\boldsymbol{z}'_0 + \mu\boldsymbol{z}'_1$ where (λ, μ) are easily determined from the given data, i.e., $(\lambda, \mu) = (\Delta\boldsymbol{z} \times \boldsymbol{z}'_1, -\Delta\boldsymbol{z} \times \boldsymbol{z}'_0)/(\boldsymbol{z}'_0 \times \boldsymbol{z}'_1)$. The coefficient are to be considered to be " reciprocal numbers of the magnitudes of the end slopes". Use of these λ and μ gives simpler shape classification than traditional use of of the magnitudes of $1/\lambda$ and $1/\mu$ in (Su & Liu,1989). We state the main Theorem 2.1 and Fig. 1 concerning the distribution of interior inflection points and singularities on the parametric rational cubic segment (1.4). In order to display the occurrences of inflection points and singularities depending on these parameters, we introduce an auxiliary plane with the coordinates λ and μ. In Figure 1, the plane is divided into several regions by the $\lambda-$axis, the $\mu-$axis, the straight lines $\lambda = w_0/(3w_1)$ and $\mu = w_3/(3w_2)$, A(the segment of the hyperbola): $w_0\mu^2 = \lambda(3w_2\mu - w_3)$ limited by the second quadrant and B(the segment of the hyperbola): $w_3\lambda^2 = \mu(3w_1\lambda - w_0)$ limited by the fourth quadrant and the curve C is $(u(\sigma), v(\sigma)), 0 < \sigma < \infty$:

$$\begin{aligned} &(i) \qquad u(\sigma) = \frac{w_0(-w_0\sigma^4 + 3w_2\sigma^2 + 2w_3\sigma)}{3\{2w_0w_2\sigma^3 + (3w_1w_2 + w_0w_3)\sigma^2 + 2w_1w_3\sigma\}} \\ &(ii) \qquad v(\sigma) = \frac{w_3(2w_0\sigma^3 + 3w_1\sigma^2 - w_3)}{3\{2w_0w_2\sigma^3 + (3w_1w_2 + w_0w_3)\sigma^2 + 2w_1w_3\sigma\}}. \end{aligned} \tag{2.1}$$

Mathematica helps us check that the curve C:$(u(\sigma), v(\sigma)), 0 < \sigma < \infty$ is a branch of $k(\lambda, \mu) = 0$ limited by $\lambda < w_0/(3w_1), \mu < w_3/(3w_2)$:

$$\begin{aligned} k(\lambda,\mu) = {} & 4w_3^2w_0(3w_2\mu - w_3)\lambda^3 + 4w_0^2w_3(3w_1\lambda - w_0)\mu^3 - 3(w_0w_3\lambda\mu)^2 \\ & +\{(3w_1\lambda - w_0)(3w_2\mu - w_3)\}^2 - 6w_0w_3(3w_1\lambda - w_0)(3w_2\mu - w_3)\lambda\mu. \end{aligned} \tag{2.2}$$

$k(\lambda, \mu) = 0$ has two straight lines $\lambda = w_0/(3w_1)$ and $\mu = w_3/(3w_2)$ as its asymptotic lines.

Theorem 1 *Assume that* $\Delta z = \lambda z_0' + \mu z_1'$ *with* $z_0' \times z_1' \neq 0$. *Then, Figure 1 gives the distribution of inflections and singularity on the curve of the form (1.4) with respect to* (λ, μ) *where (i)* $N_i, 0 \leq i \leq 2$ *represent the regions for which the curve has i-inflection points and no singularity, (ii)* C *(or* L *limited by* A, B, C*) means the region for the curve to have a cusp (or a loop) and no inflection point. The region* N_0 *contains the boundaries* A *and* B*; and* N_1 *contains the two straight lines:* $\lambda = w_0/(3w_1), \mu < w_3/(3w_2)$ *and* $\lambda < w_0/(3w_1), \mu = w_3/(3w_2)$.

The implicit form (2.2) is more useful when determining on which side of the curve the point (λ, μ) lies, while the parametric form (2.1) is more useful for displaying the curve. When $(w_0, w_1, w_2, w_3) = (1, 1, 1, 1)$ (i.e., the polynomial cubic case), A, B are $\mu^2 = \lambda(3\mu - 1), \lambda^2 = \mu(3\lambda - 1)$, and C reduces to a branch of the hyperbola: $(\lambda - 1/3)(\mu - 1/3) = 1/36$ limited by $\lambda, \mu < 1/3$. In another paper, *Mathematica* will determine a subregion ($\in N_{0)}$) in the first qudrant for the parametric cubic segment to be a spiral of monotone curvature having several advantages of containg nether inflection points, singularities nor curvature extrema; see Figure 1. Here we give the result without its proof: the T-cubic spline is a spiral if and only if $(\lambda - 1/2)(\mu - 1/2) \leq 0, (\lambda - 1/3)(\mu - 1/3) \geq 1/36$. The spiral is useful as a transition curve between straight line segment and circular arc segment, and between circular arc segments of different radii and is also used in data fitting.

Theorem 1 says that the rational "*cubic/cubic*" curve has the same behavior (the number of inflection points, loops and cusps) mentioned for the cubic polynomial (Wang,1981).

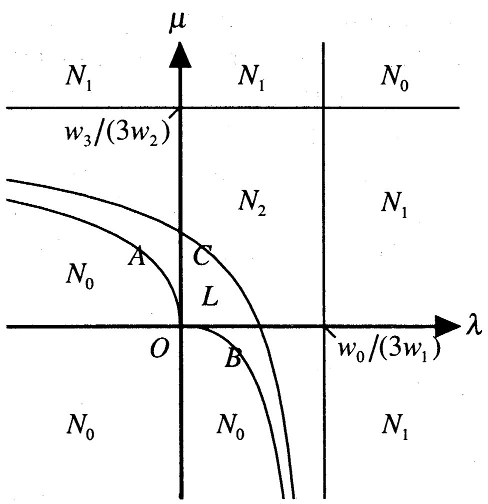

Fig. 1. Distribution of inflections and singularity.

3 Proof of Theorem 1

Inflection points: Let $\varphi(t)$ be the denominator of (1.4), i.e., $\varphi(t) = w_0u^3+3w_1u^2t+3w_2ut^2+w_3t^3$. Use $\Delta \boldsymbol{z} = \lambda \boldsymbol{z}_0' + \mu \boldsymbol{z}_1'$ to obtain

$$\begin{aligned} \varphi(t)^2\boldsymbol{z}'(t) &= a(t)\boldsymbol{z}_0' + b(t)\boldsymbol{z}_1' \\ \varphi(t)^3\boldsymbol{z}''(t) &= \{a'(t)\varphi(t) - 2a(t)\varphi'(t)\}\boldsymbol{z}_0' + \{b'(t)\varphi(t) - 2b(t)\varphi'(t)\}\boldsymbol{z}_1' \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} a(t) &= u(w_0^2u^3 - 3w_0w_2t^2u - 2w_0w_3t^3) + 3\lambda tu(2w_0w_2u^2 + 3w_1w_2tu + w_0w_3tu + 2w_1w_3t^2) \\ b(t) &= t(w_3^3t^3 - 3w_1w_3tu^2 - 2w_0w_3u^3) + 3\mu tu(2w_0w_2u^2 + 3w_1w_2tu + w_0w_3tu + 2w_1w_3t^2) \end{aligned} \tag{3.2}$$

Inflection points on (1.4) are determined by $\boldsymbol{z}'(t) \times \boldsymbol{z}''(t) = 0, 0 < t < 1$ or $a'(t)b(t) - a(t)b'(t) = 0, 0 < t < 1$. *Mathematica* helps us check that substitution of $t = 1/(1+\sigma), 0 < \sigma < \infty$ equivalently rewrites the above determining equation $a'(t)b(t)-a(t)b'(t) = 0$ of degree *six* as a product of two *cubic* polynomials:

$$\{w_0^2(3w_2\mu - w_3)\sigma^3 + 3w_0^2w_3\mu\sigma^2 + 3w_0w_3^2\lambda\sigma + w_3^2(3w_1\lambda - w_0)\}\varphi(\sigma) = 0. \tag{3.3}$$

Since $w_i > 0, 0 \le i \le 3$, from above we obtain a *cubic* equation:

$$w_0^2(3w_2\mu - w_3)\sigma^3 + 3w_0^2w_3\mu\sigma^2 + 3w_0w_3^2\lambda\sigma + w_3^2(3w_1\lambda - w_0) = 0. \tag{3.4}$$

The number of the inflection points being equal to the number of the *simple* positive roots of the *cubic* equation (3.4), easily we have

(a) $\lambda \ge w_0/(3w_1), \mu \ge w_3/(3w_2)$: $(\lambda, \mu) \in N_0$.

(b) $\{\lambda - w_0/(3w_1)\}\{\mu - w_3/(3w_2)\} < 0$ or $\lambda = w_0/(3w_1), \mu < w_3/(3w_2)$ or $\lambda < w_0/(3w_1), \mu = w_3/(3w_2)$: $(\lambda, \mu) \in N_1$.

(c) $\lambda < w_0/(3w_1), \mu < w_3/(3w_2)$: Descartes' Rule of Signs implies that the number of the positive roots of (3.4) is either zero or two, counting any double root twice. Remark that (λ, μ) is on the boundary between these cases if a double root occurs. At a positive double root σ, the cubic (3.4) and its first derivative must vanish, which gives two equations that are linear in λ and μ:

$$\begin{aligned} &3w_3^2(w_0\sigma + w_1)\lambda + 3w_0^2(w_2\sigma^3 + w_3\sigma^2)\mu = w_0w_3(w_0\sigma^3 + w_3) \\ &w_0w_3^2\lambda + w_0^2(3w_2\sigma^2 + 2w_3\sigma)\mu = w_0^2w_3\sigma^2 \end{aligned} \tag{3.5}$$

Thus it is straightforward to identify the required boundary: $(\lambda, \mu) = (u(\sigma), v(\sigma))$. Taking into account of the signs of the coefficients of (3.4), $(\lambda, \mu) \in N_0$ for $\lambda = u(\sigma), \mu \leq v(\sigma)$ and $(\lambda, \mu) \in N_2$ for $\lambda = u(\sigma), \mu > v(\sigma)$, respectively. Hence we have

Lemma 2 *If $(\lambda, \mu) \in N_i, 0 \leq i \leq 2$, the curve (1.4) has i-inflection points where $N_0 = \{(\lambda, \mu) | \ \lambda \geq w_0/(3w_1), \mu \geq w_3/(3w_2)$ or $k(\lambda, \mu) \geq 0, \lambda \leq w_0/(3w_1), \mu \leq w_3/(3w_2)\}$, $N_1 = \{(\lambda, \mu) | \ (\lambda - w_0/(3w_1))(\mu - w_3/(3w_2)) \leq 0$ or $\lambda = w_0/(3w_1), \mu < w_3/(3w_2)$ or $\lambda < w_0/(3w_1), \mu = w_3/(3w_2)\}$ and $N_2 = \{(\lambda, \mu) | \ k(\lambda, \mu) < 0, \lambda < w_0/(3w_1), \mu < w_3/(3w_2)\}$.*

Singularities: A loop occurs if $z(\alpha) = z(\beta)$ for $0 < \alpha < \beta < 1$. Since z_0' and z_1' are independent, letting the coefficients of the two vectors in $\{z(\alpha) - z(\beta)\}$ be zero gives

$$\begin{aligned} &\lambda\left[\beta^2\{w_3\beta + 3w_2(1-\beta)\}\varphi(\alpha) - \alpha^2\{w_3\alpha + 3w_2(1-\alpha)\}\varphi(\beta)\right] \\ &= w_0\{(1-\alpha)^2\alpha\varphi(\beta) - (1-\beta)^2\beta\varphi(\alpha)\} \\ &\mu\left[\beta^2\{w_3\beta + 3w_2(1-\beta)\}\varphi(\alpha) - \alpha^2\{w_3\alpha + 3w_2(1-\alpha)\}\varphi(\beta)\right] \\ &= w_3\{(1-\beta)\beta^2\varphi(\alpha) - (1-\alpha)\alpha^2\varphi(\beta)\}. \end{aligned} \tag{3.6}$$

Note $\alpha \neq \beta$ to obtain from the above (3.6)

$$\begin{aligned} \lambda/w_0 &= \{-w_0(1-\alpha)^2(1-\beta)^2 + w_3\alpha\beta(\alpha+\beta-2\alpha\beta) + 3w_2\alpha\beta(1-\alpha)(1-\beta)\}/D \\ \mu/w_3 &= \{w_0(1-\alpha)(1-\beta)(\alpha+\beta-2\alpha\beta) - w_3\alpha^2\beta^2 + 3w_1\alpha\beta(1-\alpha)(1-\beta)\}/D \end{aligned} \tag{3.7}$$

with

$$\begin{aligned} D = {} & w_0w_3\{\beta^2(1-\alpha)^2 + \alpha\beta(1-\alpha)(1-\beta) + \alpha^2(1-\beta)^2\} + 3w_1w_3\alpha\beta(\alpha+\beta-2\alpha\beta) \\ &+ 3w_0w_2(1-\alpha)(1-\beta)(\alpha+\beta-2\alpha\beta) + 9w_1w_2(1-\alpha)(1-\beta)\alpha\beta. \end{aligned} \tag{3.8}$$

where

$$0 < \alpha < \beta < 1. \tag{3.9}$$

Hence, we consider the image of (λ, μ) by (3.7)-(3.8) under (3.9) to get the necessary and sufficient conditions for the existence of the loop on (1.4). First the image of the boundary of the region determined by inequalities (3.9) is given by:

$$\begin{array}{ll} (i) & \alpha = 0, 0 < \beta < 1 \Rightarrow w_0\mu^2 = \lambda(3w_2\mu - w_3) \\ (ii) & 0 < \alpha < 1, \beta = 1 \Rightarrow w_3\lambda^2 = \mu(3w_1\lambda - w_0) \\ (iii) & 0 < \alpha = \beta < 1 \quad \text{or} \quad \alpha = \beta = 1/(1+\sigma), 0 < \sigma < \infty \Rightarrow (\lambda, \mu) = (u(\sigma), v(\sigma)). \end{array} \tag{3.10}$$

Next, *Mathematica* helps us check that the Jacobian matrix of (λ, μ) with respect to (α, β) is nonsingular for $(\alpha, \beta) = (1/(1+c), 1/(1+d)), 0 < d < c$ as follows;

$$\begin{aligned} &\{3w_0w_2cd(c+d) + (w_0w_3 + 9w_1w_2)cd + 3w_1w_3(c+d) + w_0w_3(c^2+d^2)\}^3 \begin{vmatrix} \lambda_\alpha & \lambda_\beta \\ \mu_\alpha & \mu_\beta \end{vmatrix} \\ &= (d-c)\{w_0w_3(1+c)(1+d)\}^2(w_0c^3 + 3w_1c^2 + 3w_2c + w_3) \times \\ &(w_0d^3 + 3w_1d^2 + 3w_2d + w_3) \quad (< 0). \end{aligned} \tag{3.11}$$

Note that a cusp on a curve can be regarded as the limit of a loop to obtain

Lemma 3 *If* $(\lambda, \mu) \in L$ *or* C, *then a loop or a cusp occurs on the curve segment (1.4) where* $L = \{(\lambda, \mu) |\ k(\lambda, \mu) > 0, \lambda < w_0/(3w_1), \mu < w_3/(3w_2), w_3\lambda^2 > \mu(3w_1\lambda - w_0),\ w_0\mu^2 > \lambda(3w_2\mu - w_3)\ \}$.

Lemmas 2–3 give the desired Theorem 1 on the distribution of inflection points and singularities on the planar rational cubic curves of the form (1.4) where note that the inflection points, cusps or loops do not occur simultaneously.

4 Shape classification of rational cubic Bézier curve

As in (Meek & Walton,1990), we want to know the shape classification of the rational cubic curve (1.3) in terms of one of the control vertices. Based on Theorem 1, we consider the distribution of inflection points and singularities on the rational cubic Bézier curve (1.3) or the shape of the curve segment resulting from placing p_1 in various regions of the plane, with $\boldsymbol{p}_0, \boldsymbol{p}_2, \boldsymbol{p}_3$ and w_0, w_1, w_2, w_3 fixed. From (1.3), we equivalently rewrire$\Delta \boldsymbol{z} = \lambda \boldsymbol{z}_0' + \mu \boldsymbol{z}_1'$ as

$$\boldsymbol{p}_3 - \boldsymbol{p}_0 = (3w_1/w_0)\lambda(\boldsymbol{p}_1 - \boldsymbol{p}_0) + (3w_2/w_3)\mu(\boldsymbol{p}_3 - \boldsymbol{p}_2) \tag{4.1}$$

from which follows

$$\boldsymbol{p}_1 - \boldsymbol{p}_2 = u(\boldsymbol{p}_0 - \boldsymbol{p}_2) + v(\boldsymbol{p}_3 - \boldsymbol{p}_2), \qquad u = 1 - \frac{w_0}{3w_1\lambda}, v = \frac{w_0}{3w_1\lambda}(1 - \frac{3w_2\mu}{w_3}), \tag{4.2}$$

Theorem 1 gives Figure 2 (the shape classification of the rational cubic Bézier curve for placement of $\boldsymbol{p}_1$ with $\boldsymbol{p}_0, \boldsymbol{p}_2, \boldsymbol{p}_3$ fixed) where for A, B, C in Theorem 1, (u, v) can be given respectively:

$$A: u = 1 - v + \sqrt{-3w_2^2 v/(w_1 w_3)}, v < 0, B: v = 1 - u - w_0 w_2/(3w_1^2 u), u < 0 \tag{4.3}$$

$$C: u = \frac{w_0 t(w_1 t^2 + 2w_2 t + w_3)}{w_1(w_0 t^3 - 3w_2 t - 2w_3)}, v = -\frac{w_3(w_0 t^2 + 2w_1 t + w_2)}{w_1 t(w_0 t^3 - 3w_2 t - 2w_3)}, \qquad t > 0$$

where $u+v > 1$ for A, B. Since C approaches to the straight line $\boldsymbol{p}_0\boldsymbol{p}_2$ as $w_i, i = 1, 2 \to \infty$ with $w_i, i = 0, 3$ fixed, N_2 disappears. As for the cubic polynomial curve in (Wang,1981; Meek & Walton,1990), an S-shaped control polyline always results in a rational cubic curve with one inflection point and vice versa, regardless of the weights. However, if the polyline forms a loop, the resulting rational curve segment can have a cusp, two inflection points, a loop, or none of those. Note that appropriate values of weights $w_i, 0 \le i \le 3$ would make the desired region N_0 giving "none of those" larger even with fixed control vertices; see Figure 2 and that larger values of $w_0/w_1, w_3/w_2$ would make the resulting curve (1.3) (or (1.4)) be an unacceptable "flat" curve segment. For a choice of the parameters w_i, refer to (Farin,1995[pp.256-258])

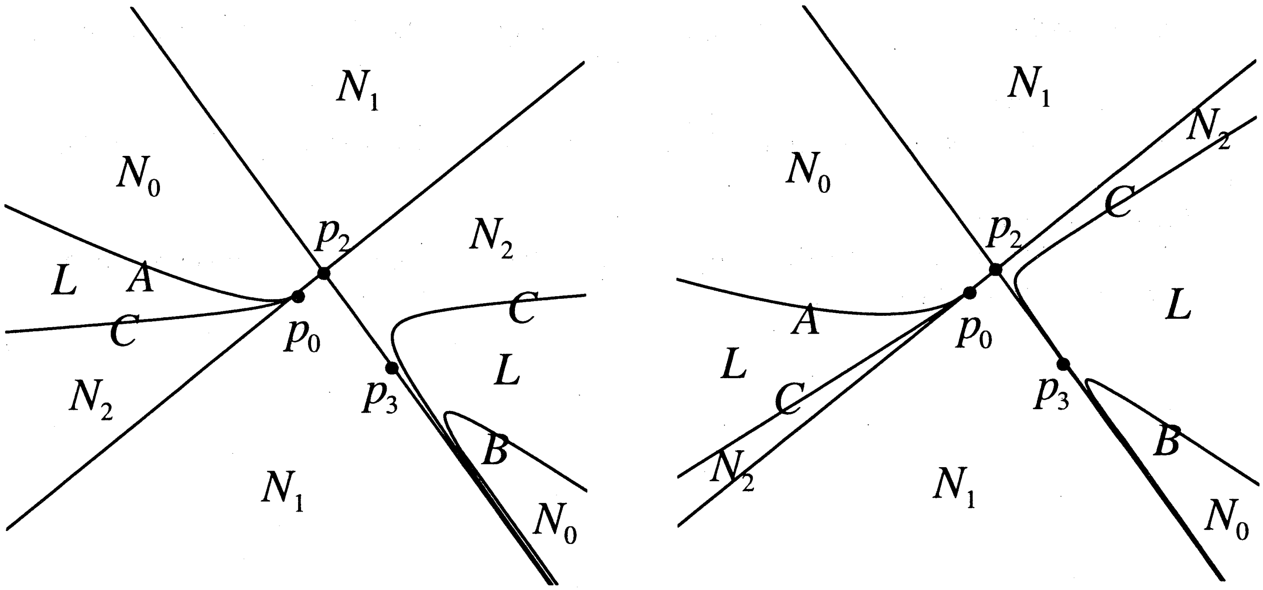

Fig. 2. Shape classification with $(w_0, w_1, w_2, w_3) = (1, 4/3, 1, 1)$ **and** $(1, 16/3, 4, 1)$

In order to obtain the shape classification of the rational cubic Bézier curve for placement of from placing p_3 in various regions of the plane, with $\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2$ and w_0, w_1, w_2, w_3 fixed, we only have to rewrite (4.1) as

$$\boldsymbol{p}_3 - \boldsymbol{p}_2 = u(\boldsymbol{p}_0 - \boldsymbol{p}_2) + v(\boldsymbol{p}_1 - \boldsymbol{p}_2), \quad u = \frac{w_3(w_0 - 3w_1\lambda)}{w_0(w_3 - 3w_2\mu)}, v = \frac{3w_1 w_3 \lambda}{w_0(w_3 - 3w_2\mu)} \tag{4.4}$$

Acknowledgment
The author is grateful to Prof. Meek of Department of Computer Science, University of

Manitoba, Canada R3T 2N2 for his valuable comments and suggestions.

References

Wang, C.Y. (1981), Shape classification of the parametric cubic curve and parametric B-spline curve. Comput. Aided-Des.13,199-206.

Farin, G. (1995), Curves and Surfaces for Computer Aided Geometric Design-Practical Guide. Academic Press, New York.

Meek, D.S. & Walton, D.J. (1990), Shape determination of planar uniform cubic B-spline segments. Comput. Aided-Des. 22, 434-441.

Sakai, M. & Usmani, R.A. (1995), On fair parametric rational cubic curves. BIT 36, 349-367.

Sakai, M (1997), Inflections and singularity on parametric rational cubic curves. Numer. Math.76, 403-417.

Sakai, M. Inflection points and singularities on planar rational cubic curve segments. (to appear in Computer Aided Geometric Design).

Sakai, M. Planar Hermite spiral interpolation (submitted).

Stone, M.C. & DeRose, T.D. (1989), Characterizing cubic Bézier curves. ACM Transaction on Graphics 8, 147-163.

Späth, H. (1995), One Dimensional Spline Interpolation Algorithms. AK Peters Wellesley, Masasachusetts.

Späth, H. (1995), Two Dimensional Spline Interpolation Algorithms. AK Peters Wellesley, Masasachusetts.

B. -Q. Su. & D. -Y. Liu. (1989), Computational Geometry-Curve and Surface Modeling. Academic Press, New York.