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Riemannゼータ関数の近似関数等式に対する平均値公式

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1 Statement of result.

Let $\zeta(s)$ be the Riemann zeta-function, and define the remainder term $R_1(s)$ in the approximate functional equation for $\zeta(s)$ by

$$R_1(s) = \zeta(s) - \sum_{n\leq\sqrt{\frac{t}{2\pi}}} \frac{1}{n^s} - \chi(s) \sum_{n\leq\sqrt{\frac{t}{2\pi}}} \frac{1}{n^{1-s}}$$

where

$$\chi(s) = 2^s\pi^{s-1}\sin\left(\frac{1}{2}\pi s\right)\Gamma(1-s). \tag{1}$$

The aim of this note is to derive the $2k$-th power moments of the function $|R_1(s)|$ in the critical strip $0\leq\sigma\leq 1$. We can prove the following

Theorem [1]. *For any positive integers k, we have*

$$\int_1^T |R_1(s)|^{2k}\,dt$$
$$= \begin{cases} \dfrac{(2\pi)^{k\sigma}C_k}{1-k\sigma}T^{1-k\sigma} + Y_{k,\sigma}(T) & \text{if } 0\leq\sigma\leq\dfrac{1}{2k} \quad (2) \\ \dfrac{(2\pi)^{k\sigma}C_k}{1-k\sigma}T^{1-k\sigma} + D_{k,\sigma} + Y_{k,\sigma}(T) & \text{if } \dfrac{1}{2k}<\sigma\leq 1 \text{ and } \sigma\neq\dfrac{1}{k} \quad (3) \\ 2\pi C_k\log T + D_{k,\frac{1}{k}} + Y_{k,\frac{1}{k}}(T) & \text{if } \sigma=\dfrac{1}{k} \quad (4) \end{cases}$$

with

$$Y_{k,\sigma}(T) = O(T^{\frac{1}{2}-k\sigma}) \tag{5}$$

where the constant $D_{k,\sigma}$ depends on k and σ, and

$$C_k = \int_0^1 \left(\frac{\cos\left(2\pi\left(y^2-y-\frac{1}{16}\right)\right)}{\cos(2\pi y)}\right)^{2k} dy.$$

Remark. *This theorem includes the fact that*

$$R_1(s) = \begin{cases} \Omega\left(t^{-\frac{\sigma}{2}}\right) & \text{if } 0 \le \sigma < 1, \\ \Omega\left(t^{-\frac{1}{2}}(\log t)^{\frac{1}{2}}\right) & \text{if } \sigma = 1. \end{cases}$$

2 Proof of the formula(2).

To prove our theorem, we start with the weak form of the "Riemann-Siegel formula" for $\zeta(s)$: For $0 \le \sigma \le 1$, we have

$$\chi(1-s)^{\frac{1}{2}} R_1(s) = (-1)^{\left[\sqrt{\frac{t}{2\pi}}\right]-1} \frac{\cos\left(2\pi\left(\delta^2 - \delta - \frac{1}{16}\right)\right)}{\cos(2\pi\delta)} \left(\frac{t}{2\pi}\right)^{-\frac{1}{4}} + O\left(t^{-\frac{3}{4}}\right)$$

where $\delta = \sqrt{\frac{t}{2\pi}} - \left[\sqrt{\frac{t}{2\pi}}\right]$ with $[x]$ being the integer part of x (see [2]). We assume that $T_1 < T_2 \le 2T_1$. From the above, we have, for $0 \le \sigma \le \frac{1}{2k}$,

$$\int_{T_1}^{T_2} |R_1(s)|^{2k} \, dt = I_1(T_1, T_2) + O\left(\sum_{j=1}^{2k} |I_1(T_1, T_2)|^{1-\frac{j}{2k}} |I_2(T_1, T_2)|^{\frac{j}{2k}}\right), \tag{6}$$

where

$$I_1(T_1, T_2) = \int_{T_1}^{T_2} |\chi(s)|^k \left(\frac{t}{2\pi}\right)^{-\frac{k}{2}} \left(\frac{\cos\left(2\pi\left(\delta^2 - \delta - \frac{1}{16}\right)\right)}{\cos(2\pi\delta)}\right)^{2k} dt$$

and

$$I_2(T_1, T_2) = \int_{T_1}^{T_2} t^{-\frac{3k}{2}} |\chi(s)|^k \, dt.$$

By using the asymptotic formula of (1), we have

$$|\chi(s)|^k = \left(\frac{t}{2\pi}\right)^{k\left(\frac{1}{2}-\sigma\right)} + G_{k,\sigma}(t) \qquad (t > 0) \tag{7}$$

where

$$G_{k,\sigma} = O\left(t^{k\left(\frac{1}{2}-\sigma\right)-1}\right). \tag{8}$$

It is easily seen that

$$I_2(T_1, T_2) = O\left(T_1^{1-k-k\sigma}\right). \tag{9}$$

From (7) and (8), we get

$$I_1(T_1, T_2) = I_{1,1}(T_1, T_2) + O\left(T_1^{-1} |I_{1,1}(T_1, T_2)|\right), \tag{10}$$

where

$$I_{1,1}(T_1,T_2)=\int_{T_1}^{T_2}\left(\frac{t}{2\pi}\right)^{-k\sigma}\left(\frac{\cos\left(2\pi\left(\delta^2-\delta-\frac{1}{16}\right)\right)}{\cos(2\pi\delta)}\right)^{2k}dt.$$

Let N_1 be the smallest integer such that $\sqrt{\frac{T_1}{2\pi}}\le N_1$ and N_2 the largest integer such that $N_2\le\sqrt{\frac{T_2}{2\pi}}$. We obtain, for $0\le\sigma\le\frac{1}{2k}$,

$$\begin{aligned}
&I_{1,1}(T_1,T_2)\\
&=4\pi\int_0^1\left(\frac{\cos\left(2\pi\left(y^2-y-\frac{1}{16}\right)\right)}{\cos(2\pi y)}\right)^{2k}\sum_{n=N_1}^{N_2-1}(y+n)^{1-2k\sigma}dy+O\left(T_1^{\frac{1}{2}-k\sigma}\right)\\
&=\frac{2\pi}{1-k\sigma}\int_0^1\left(\frac{\cos\left(2\pi\left(y^2-y-\frac{1}{16}\right)\right)}{\cos(2\pi y)}\right)^{2k}\left\{\left(y+\sqrt{\frac{T_2}{2\pi}}\right)^{2-2k\sigma}-\left(y+\sqrt{\frac{T_1}{2\pi}}\right)^{2-2k\sigma}\right\}dy\\
&\quad+O\left(T_1^{\frac{1}{2}-k\sigma}\right)\\
&=\frac{(2\pi)^{k\sigma}C_k}{1-k\sigma}\left(T_2^{1-k\sigma}-T_1^{1-k\sigma}\right)+O\left(T_1^{\frac{1}{2}-k\sigma}\right).
\end{aligned}\tag{11}$$

Substituting (9), (10) and (11) into (6), we obtain the formula (2).

3 Proof of the formulas (3) and (4).

From Lemma 1 in [1], we have, for any positive integers k and $\frac{1}{2k}<\sigma\le 1$,

$$\int_1^T|R_1(s)|^{2k}dt=J_1(1,T)+J_2(1,T),\tag{12}$$

where

$$J_1(T_1,T_2)=\int_{T_1}^{T_2}|\chi(s)|^k\left|\chi\left(\frac{1}{2k}+it\right)\right|^{-k}\left|R_1\left(\frac{1}{2k}+it\right)\right|^{2k}dt$$

and

$$J_2(T_1,T_2)=\sum_{j=1}^{k}\binom{k}{j}\int_{T_1}^{T_2}|\chi(s)|^k\left|\chi\left(\frac{1}{2k}+it\right)\right|^{-(k-j)}\left|R_1\left(\frac{1}{2k}+it\right)\right|^{2(k-j)}F_k(t)^j dt$$

with

$$F_k(t)=O\left(t^{-\frac{3}{4}}\left|\chi\left(\frac{1}{2k}+it\right)\right|^{-\frac{1}{2}}\left|R_1\left(\frac{1}{2k}+it\right)\right|+t^{-\frac{3}{2}}\right).$$

Applying (2),(7),(8) and integration by parts, we have, for $k \neq 2$,

$$\int_{T_1}^{T_2} \left|\chi\left(\frac{1}{2k}+it\right)\right|^{-k} \left|R_1\left(\frac{1}{2k}+it\right)\right|^{2k} dt$$

$$= \frac{4\pi}{2-k} C_k \left(\frac{t}{2\pi}\right)^{1-\frac{k}{2}} + \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\frac{k}{2}} Y_{k,\frac{1}{2k}}(t) \Big|_{T_1}^{T_2}$$

$$+ \int_{T_1}^{T_2} H_k(t) \left|R_1\left(\frac{1}{2k}+it\right)\right|^{2k} dt + \frac{k-1}{2}(2\pi)^{\frac{k-1}{2}} \int_{T_1}^{T_2} t^{-\frac{1+k}{2}} Y_{k,\frac{1}{2k}}(t) dt$$

and for $k = 2$,

$$\int_{T_1}^{T_2} \left|\chi\left(\frac{1}{4}+it\right)\right|^{-2} \left|R_1\left(\frac{1}{4}+it\right)\right|^{4} dt$$

$$= 2\pi C_2 \log t + \sqrt{2\pi} t^{-\frac{1}{2}} Y_{2,\frac{1}{4}}(t) \Big|_{T_1}^{T_2} + \int_{T_1}^{T_2} H_2(t) \left|R_1\left(\frac{1}{4}+it\right)\right|^{4} dt$$

$$+ \sqrt{\frac{\pi}{2}} \int_{T_1}^{T_2} t^{-\frac{3}{2}} Y_{2,\frac{1}{4}}(t) dt.$$

From (5),(7) and (8), we have

$$\int_{1}^{T} \left|\chi\left(\frac{1}{2k}+it\right)\right|^{-k} \left|R_1\left(\frac{1}{2k}+it\right)\right|^{2k} dt$$

$$= A_k + B_k(T) + \begin{cases} \frac{4\pi}{2-k} C_k \left(\frac{T}{2\pi}\right)^{1-\frac{k}{2}} & \text{if } k \geq 1 \text{ and } k \neq 2 \\ 2\pi C_2 \log T & \text{if } k = 2 \end{cases}$$

with

$$B_k(T) = O\left(T^{\frac{1-k}{2}}\right),$$

where the constant A_k depends on k. Hence, integrating by parts, we obtain, for $\frac{1}{2k} < \sigma \leq 1$ and $\sigma \neq \frac{1}{k}$,

$$J_1(T_1, T_2) = O\left(T_1^{1-k\sigma}\right)$$

and

$$J_1(1,T) = \frac{(2\pi)^{k\sigma} C_k}{1-k\sigma} T^{1-k\sigma} + L_{k,\sigma} + O\left(T^{\frac{1}{2}-k\sigma}\right) \tag{13}$$

with a certain constant $L_{k,\sigma}$. Similarly in case $\sigma = \frac{1}{k}$, we have

$$J_1(T_1, T_2) = O(1)$$

and

$$J_1(1,T) = 2\pi C_k \log T + L_{k,\frac{1}{k}} + O\left(T^{-\frac{1}{2}}\right). \tag{14}$$

By using Hölder's inequality and the above, we have, for $\frac{1}{2k} < \sigma \leq 1$,

$$J_2(1,T) = J_2(1,\infty) + O\left(T^{\frac{1}{2}-k\sigma}\right). \tag{15}$$

Substituting (13), (14) and (15) into (12), we obtain the formulas (3) and (4).

References

[1] I. Kiuchi and N. Yanagisawa, On the mean value formulas for the approximate functional equation of the Riemann zeta-function, preprint.

[2] C.L.Siegel, Über Riemanns Nachlass zur analytischen Zahlentheorie, Quellen und Studien zur Geschichte der Math. Astr. und Physik, Abt. B, Studien 2(1932), 45-80.