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A new method to compute zeros of polynomials

武藤秀夫 (HIDEO MUTO) *　　鈴木智博 (TOMOHIRO SUZUKI)†
鈴木俊夫 (TOSHIO SUZUKI)‡

1 Introduction

Let $f(z) = z^p + a_1 z^{p-1} + \cdots + a_p$, $a_p \neq 0$ be a polynomial with complex coefficients of order p which has distinct s zeros, that is, $f(z) = \prod_{k=1}^{s} (z - z_k)^{n_k}$. The root finding problem for f has a long history and many kinds of methods are known. Among them the Newton's method (or its stable version) is most popularly used because of its rapidness and reliability. We can see thousands of papers in McNamee's on-line bibliography available at http://www.elsevier.nl/local.cam, in which the general subject of polynomial root finding is divided into 29 categories.

Putting $q(z) = \dfrac{f'(z)}{f(z)}$, the r-th derivative $q^{(r)}(z) = \sum_{k=1}^{s} \dfrac{1}{r!} \dfrac{(-1)^r n_k}{(z - z_k)^{r+1}}$ is analysed in Henrici ([3]) as a formula to give an approximation of the nearest zero to z as $r \to \infty$. On the other hand we consider the Fourier coefficient $T(z, \tau, m; f)$ of $q(z)$ using m points on a circle in the complex plane with center z and radius $|\tau|$, satisfying the following representation formula: $T(z, \tau, m; f) = \sum_{k=1}^{s} \dfrac{n_k}{1 - (\frac{z_k - z}{\tau})^m}$, which will be proved in section 2. It is easy to see that $T(z, \tau, m; f)$ as well as $q^{(r)}(z)$ has information on the location of the nearest zero to z.

Our new method is considered to be derived from the error terms of the numerical integral of the following integral:

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{f'(z)}{f(z)} dz, \tag{1.1}$$

where Γ is a circle on the complex plane. It is well known that the integral (1.1) gives the number of zeros of f enclosed in Γ. Taking m points on Γ, we use a modified form of the numerical integral of (1.1). If m is small then it has a large error as a numerical integral. But we can show in this paper that it has a critical information on zeros of f. The main feature of our method is in global convergence which cannot be realized in the simple Newton algorithm.

*Dept. Mathematics and Physics, Yamanashi Univ.
†Dept. Computer Science, Yamanashi Univ.
‡Dept. Mathematics and Physics, Yamanashi Univ. e-mail address: suzuki@grape.kkb.yamanashi.ac.jp

Because of the usage of the modified numerical integral, our theory may look like similar to the papers (eg. Delves & Lynes [2]) in the category of '5. Integral method, esp. Lehmer's' at a glance. But the values of it are used in quite different ways; Their methods use anyway the value of the numerical integral as accurate as possible while our method essentially needs the error of it. So, our new method dose not belong in any one of the 29 categories cited above. Among the 29 categories, only '2. Newton's method' has a relation with ours through the fact that, if the numerical integral in our method is carried out with m=1, our method is essentially reduced to the Newton's method.

In section 2, we study the basic theory, from which algorithms for finding zeros of f are derived. We can prove that our fundamental algorithm is generally convergent. Our algorithm for computing zeros are stated in two ways in section 3 with some remarks. The results of numerical experiments are shown in section 4 with the fact that our method is comparable with the Newton's method in its effectivity. Concluding remarks are stated in section 5.

2 Basic theory

Let f be a polynomial of degree p with zeros z_k, $k = 1, \cdots, s$, of multiplicity n_k. For $\lambda \in \mathbf{C}$, $\tau \in \mathbf{C}$ and a positive integer m, we set $\theta = \theta_m = \frac{2\pi}{m}$ and define $T(\lambda, \tau, m; f)$ and $S(\lambda, \tau, m; f)$ as follows:

$$T(\lambda,\tau,m;f) = \frac{\tau}{m}\sum_{j=0}^{m-1}\frac{f'(\lambda+\tau e^{i\theta j})}{f(\lambda+\tau e^{i\theta j})}e^{i\theta j}, \quad S(\lambda,\tau,m;f) = \frac{\tau^2}{m}\sum_{j=0}^{m-1}\frac{f'(\lambda+\tau e^{i\theta j})}{f(\lambda+\tau e^{i\theta j})}e^{2i\theta j}.$$

Denoting $\nu_k = z_k - \lambda$, we have the following lemma.

Lemma 2.1 *If $\tau^m \neq \nu_k^m$ for all k, then we have $T(\lambda,\tau,m;f) = \sum_{k=1}^{s}\frac{n_k}{1-(\frac{\nu_k}{\tau})^m}$ and $S(\lambda,\tau,m;f) = \sum_{k=1}^{s}\frac{n_k}{1-(\frac{\nu_k}{\tau})^m}(z_k-\lambda)$.*

Proof

$$\begin{aligned} T(\lambda,\tau,m;f) &= \frac{\tau}{m}\sum_{j=0}^{m-1}\sum_{k=1}^{s}\frac{n_k e^{i\theta j}}{\lambda+\tau e^{i\theta j}-z_k} \\ &= \frac{\tau}{m}\sum_{k=1}^{s}\sum_{j=0}^{m-1}\frac{n_k e^{i\theta j}}{-\nu_k+\tau e^{i\theta j}} \end{aligned}$$

and

$$\begin{aligned} \frac{\tau}{m}\sum_{j=0}^{m-1}\frac{n_k e^{i\theta j}}{-\nu_k+\tau e^{i\theta j}} &= \frac{n_k}{m}\sum_{j=0}^{m-1}\frac{\sum_{l=0}^{m-1}(\frac{\nu_k}{\tau})^l e^{-i\theta jl}}{(1-\frac{\nu_k}{\tau}e^{-i\theta j})\sum_{l=0}^{m-1}(\frac{\nu_k}{\tau})^l e^{-i\theta jl}} \\ &= \frac{n_k}{m}\sum_{j=0}^{m-1}\frac{\sum_{l=0}^{m-1}(\frac{\nu_k}{\tau})^l e^{-i\theta jl}}{1-(\frac{\nu_k}{\tau})^m} \\ &= \frac{n_k}{1-(\frac{\nu_k}{\tau})^m} \end{aligned}$$

We can derive the second equality by the similar way. ∎

Now we consider zeros of f. Put $T = T(\lambda, \tau, m; f)$. Let z_1 be the nearest zero to λ. Expressing that $T = \frac{n_1}{1-(\frac{\nu_1}{\tau})^m} + \delta$, we can solve $z_1 = \lambda + \tau\left(\frac{T-n_1-\delta}{T-\delta}\right)^{\frac{1}{m}}$. So if the parameters λ, τ and m can be set properly so that δ is small enough, then we have an approximation formula $\lambda + \tau\left(\frac{T-n_1}{T}\right)^{\frac{1}{m}}$ for z_1.

$$\text{Since} \quad T = \sum_{k=1}^{s} \frac{n_k}{1-(\frac{\nu_k}{\tau})^m} = \frac{n_1}{1-(\frac{\nu_1}{\tau})^m}\left\{1 + \sum_{k=2}^{s} \frac{n_k}{n_1}\frac{1-(\frac{\nu_1}{\tau})^m}{1-(\frac{\nu_k}{\tau})^m}\right\}$$
$$= \frac{n_1}{1-(\frac{\nu_1}{\tau})^m}\left\{1 + \sum_{k=2}^{s} \frac{n_k}{n_1}\left(\frac{\nu_1}{\nu_k}\right)^m \frac{1-(\frac{\tau}{\nu_1})^m}{1-(\frac{\tau}{\nu_k})^m}\right\},$$

putting $\Delta = \sum_{k=2}^{s} \frac{n_k}{n_1}\left(\frac{\nu_1}{\nu_k}\right)^m \frac{1-(\frac{\tau}{\nu_1})^m}{1-(\frac{\tau}{\nu_k})^m}$, we have

$$\frac{T-n_1}{T} = \left(\frac{\nu_1}{\tau}\right)^m \left\{1 + \left(\frac{\tau}{\nu_1}\right)^m \Delta\right\}\{1+\Delta\}^{-1}.$$

Choosing a proper branch in the complex plane, the following proposition is derived formally from this equality .

Proposition 2.1 $\lambda + \tau\left(\frac{T-n_1}{T}\right)^{\frac{1}{m}} - z_1 = \nu_1\left[\left(1+(\frac{\tau}{\nu_1})^m\Delta\right)^{\frac{1}{m}}\left(\frac{1}{1+\Delta}\right)^{\frac{1}{m}} - 1\right]$ *holds.*

Throughout this paper, let each zero of f be numbered according to the distance from λ in the complex plane, that is, $|\nu_1| \le |\nu_2| \le \cdots \le |\nu_s|$. Here we have the following theorem.

Theorem 2.1 *If $|\tau| < |\nu_1| < |\nu_2|$ then there exists a positive integer N such that, for all $m > N$, the inequality $|\lambda + \tau\left(\frac{T-n_1}{T}\right)^{\frac{1}{m}} - z_1| \le \frac{16p}{m}\frac{|\nu_1|}{1-|\frac{\tau}{\nu_2}|^m}\left|\frac{\nu_1}{\nu_2}\right|^m$ holds.*
If $|\nu_1| < |\tau| < |\nu_2|$ then there exists a positive integer N such that, for all $m > N$, the inequality $|\lambda + \tau\left(\frac{T-n_1}{T}\right)^{\frac{1}{m}} - z_1| \le \frac{16p}{m}\frac{|\nu_1|}{1-|\frac{\tau}{\nu_2}|^m}\left|\frac{\tau^2}{\nu_1\nu_2}\right|^m$ holds, provided that $\left|\frac{\tau^2}{\nu_1\nu_2}\right| < 1$.

Proof It is easy to derive the inequality $|\Delta| \le p|\frac{\nu_1}{\nu_2}|^m \frac{2}{1-|\frac{\tau}{\nu_2}|^m}$, so we can choose an positive integer N such that $|\Delta| < \frac{1}{2}$ for all $m > N$. First, we consider the case $|\tau| < |\nu_1| < |\nu_2|$. Since

$$\left(\frac{1+(\frac{\tau}{\nu_1})^m\Delta}{1+\Delta}\right)^{\frac{1}{m}} - 1 = \frac{1}{(1+\Delta)^{\frac{1}{m}}}\left\{(1+(\frac{\tau}{\nu_1})^m\Delta)^{\frac{1}{m}} - (1+\Delta)^{\frac{1}{m}}\right\}$$
$$= \frac{1}{(1+\Delta)^{\frac{1}{m}}}\left\{\left[1 + \frac{1}{m}(\frac{\tau}{\nu_1})^m\Delta + \frac{1}{2}\frac{1}{m}(\frac{1}{m}-1)(\frac{\tau}{\nu_1})^{2m}\Delta^2 + \cdots\right]\right.$$
$$\left. - \left[1 + \frac{1}{m}\Delta + \frac{1}{2}\frac{1}{m}(\frac{1}{m}-1)\Delta^2 + \cdots\right]\right\}$$
$$= \frac{-1}{(1+\Delta)^{\frac{1}{m}}}\left\{\frac{1}{m}\left(1-(\frac{\tau}{\nu_1})^m\right)\Delta + \frac{1}{2}\frac{1}{m}(\frac{1}{m}-1)\left(1-(\frac{\tau}{\nu_1})^{2m}\right)\Delta^2 + \cdots\right\},$$

we have, for $m > N$,

$$|\lambda + \tau\left(\frac{T-n_1}{T}\right)^{\frac{1}{m}} - z_1| = |\nu_1|\left|\left(1+(\frac{\tau}{\nu_1})^m\Delta\right)^{\frac{1}{m}}\left(\frac{1}{1+\Delta}\right)^{\frac{1}{m}} - 1\right|$$

$$\leq |\nu_1|\frac{1}{(1-|\Delta|)^{\frac{1}{m}}}\left\{\frac{1}{m}2|\Delta|+\frac{1}{m}2|\Delta|^2+\cdots\right\}$$
$$\leq |\nu_1|\frac{1}{(1-|\Delta|)^{1+\frac{1}{m}}}\frac{1}{m}2|\Delta|$$
$$\leq \frac{16p}{m}\frac{|\nu_1|}{1-|\frac{\tau}{\nu_2}|^m}\left|\frac{\nu_1}{\nu_2}\right|^m.$$

As for the case $|\nu_1| < |\tau| < |\nu_2|$, considering that $\Delta = \sum_{k=2}^{s}\frac{n_k}{n_1}\left(\frac{\tau}{\nu_k}\right)^m\frac{(\frac{\nu_1}{\tau})^m-1}{1-(\frac{\tau}{\nu_k})^m}$ and $|(1-(\frac{\tau}{\nu_1})^m\Delta| < 2|\frac{\tau}{\nu_1}|^m|\Delta|$, we can prove the second inequality by the similar way for the first case. ∎

Remark 2.1 Since the measure of the set $\{\lambda \in \mathbf{C}||\nu_1| = |\nu_2|\}$ is equal to zero, the sequence $\{\lambda_m = \lambda + \tau_m\left(\frac{T(\lambda,\tau_m,m;f)-n_1}{T(\lambda,\tau_m,m;f)}\right)^{\frac{1}{m}}$ with $|\tau_m| < |\nu_1|$ for each $m\}$ converges to z_1 for almost all $\lambda \in \mathbf{C}$.

Remark 2.2 Let C be a circle with center λ and radius $|\tau|$, then we have

$$\lim_{m\to\infty} T(\lambda,\tau,m;f) = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz \quad \text{and} \quad \lim_{m\to\infty} S(\lambda,\tau,m;f) = \frac{1}{2\pi i}\oint_C \frac{f'(z)(z-\lambda)}{f(z)}dz,$$

provided that no zero of f is situated on C. We should note that the first one is zero or n_1 corresponding to the cases: $|\tau| < |\nu_1|$ or $|\nu_1| < |\tau| < |\nu_2|$.

Put $A = -\sum_{k\neq 1}\frac{n_k}{1-(\frac{\nu_k}{\tau})^m}$, $B = -\sum_{k\neq 1}\frac{n_k\nu_k}{1-(\frac{\nu_k}{\tau})^m}$ and $C = \frac{1-(\frac{\nu_1}{\tau})^m}{n_1}$. Then we have the following proposition directly from Lemma 2.1.

Proposition 2.2 $z_1 - \left(\lambda + \frac{S}{T}\right) = C\frac{\nu_1 A - B}{1-CA}$ *holds.*

Theorem 2.2 *Assume that a polynomial $f(z)$ satisfies the following conditions (1)-(4): (1) $0 < |z_1 - \lambda|, |\tau| < r$, (2) $\min_{k\neq 1}|z_k-\lambda| \geq R_2 > r$, (3) $R_3 \geq \max_k|z_k-\lambda|$, (4) $(\frac{r}{R_2})^m < \frac{1}{(2p-1)}$. Then, putting $\kappa = \max\{|\tau|, |z_1-\lambda|\}$, we have*

$$|z_1 - (\lambda + \frac{S}{T})| \leq \frac{2p(R_3+\kappa)}{1-(2p-1)(\frac{\kappa}{R_2})^m}\left(\frac{\kappa}{R_2}\right)^m.$$

Proof From the definitions of A, B and C we have

$$|A| \leq \sum_{k\neq 1}\frac{n_k}{1-|\frac{\tau}{z_k-\lambda}|^m}|\frac{\tau}{z_k-\lambda}|^m$$
$$\leq \sum_{k\neq 1}\frac{n_k}{1-(\frac{|\tau|}{R_2})^m}(\frac{|\tau|}{R_2})^m$$
$$= (p-n_1)\frac{1}{1-(\frac{|\tau|}{R_2})^m}(\frac{|\tau|}{R_2})^m,$$
$$|B| \leq \sum_{k\neq 1}\frac{n_k|z_k-\lambda|}{1-|\frac{\tau}{z_k-\lambda}|^m}|\frac{\tau}{z_k-\lambda}|^m$$
$$\leq \sum_{k\neq 1}\frac{n_kR_3}{1-(\frac{|\tau|}{R_2})^m}(\frac{|\tau|}{R_2})^m$$
$$= (p-n_1)\frac{R_3}{1-(\frac{|\tau|}{R_2})^m}(\frac{|\tau|}{R_2})^m,$$
$$|C| \leq \left|\frac{\tau^m-\nu_1^m}{n_1\tau^m}\right| \leq 2\frac{\kappa^m}{|\tau|^m}.$$

And

$$|z_1 - (\lambda + \frac{S}{T})| \leq |C|\frac{|A\nu_1 - B|}{1 - |C||A|}$$
$$\leq \frac{|\nu_1| + R_3}{1 - |C|(p-n_1)\frac{1}{1-(\frac{|\tau|}{R_2})^m}(\frac{|\tau|}{R_2})^m}|C|(p-n_1)\frac{1}{1-(\frac{|\tau|}{R_2})^m}(\frac{|\tau|}{R_2})^m$$
$$\leq \frac{2p(|\nu_1| + R_3)}{1-(2p-1)(\frac{\kappa}{R_2})^m}(\frac{\kappa}{R_2})^m. \blacksquare$$

Remark 2.3 The second case of Theorem 2.1 is not easy to check, so $\lambda + \frac{S}{T}$ can be used as a complementary value for that case.

3 Algorithms for finding zeros of f

In this section we propose algorithms to compute zeros of f, based on the theory developed in the preceding section. We assume hereafter that the numerical computation in our algorithm is carried out under double precision arithmetic of 16 decimal places. We set a value ε such that if $|f(z)| < \varepsilon$ then z is considered to be a zero of f numerically.

For almost all fixed $\lambda \in \mathbf{C}$, we can get z_1 by the following algorithm as is stated in Remark 2.1.

Algorithm 1: Fundamental algorithm for computing z_1.

(step 1) Check that $|f(\lambda)| > \varepsilon$. Put $R_1 = \min(p\left|\frac{f(\lambda)}{f'(\lambda)}\right|, |f(\lambda)|^{1/p})$.

(step 2) Set initial parameters: $m = 2$, $\tau_{max} = R_1$, $\tau_{min} = 0$, $\tau = R_1/p$

(step 3) Compute $T = T(\lambda, \tau, m; f)$.

(step 4) If $|T| \leq 10^{-5}$ then $\tau_{min} = \tau$, $\tau = \frac{\tau_{max}+\tau_{min}}{2}$ and go to (step 3).

If $|T| \geq .99$ then $\tau_{max} = \tau$, $\tau = \frac{\tau_{max}+\tau_{min}}{2}$ and go to (step 3).

(step 5) Put $n_1 = 1$. Get m values of $\left(\frac{T-n_1}{T}\right)^{\frac{1}{m}}$ as $\{x_1, \cdots, x_m\}$. Choose x from $\{x_1, \cdots, x_m\}$ such that $|f(\lambda + \tau x)| = \min_k |f(\lambda + \tau x_k)|$. Put $z = \lambda + \tau x$.

(step 6) If $|f(z)| \geq \varepsilon$ then $m = m \times 2$ and go to (step 3).

(step 7) z is a zero of f. End.

Remark 3.1 R_1 is set as a value such that $|z_1 - \lambda| < R_1$ (cf. [3], p454 Theorem 6.4e).

Remark 3.2 For fixed λ, m and f, $T(\tau) = T(\lambda, \tau, m; f)$ is a continuous function of τ. When $|\tau|$ varies from 0 to over $|\nu_1|$, $|T(\tau)|$ dose from 0 to over n_1 continuously as is seen from Remark 2.2. So (step 4) can be passed after at most several times returns to (step 3). Practically $|\tau|$ can be any value as far as it can pass the criterion in (step 4).

Remark 3.3 If z_1 may be of multiplicity greater than 1, for example, of multiplicity 3, then it would be better that n_1 of (step 5) is set to range over three cases: $n_1 = 1, 2, 3$ and that x is chosen from $3m$ values of $\left(\frac{T-n_1}{T}\right)^{\frac{1}{m}}$, $n_1 = 1, 2, 3$.

Remark 3.4 Since the less $\left|\frac{\nu_1}{\nu_2}\right|$ is the more effective this algorithm is, we would be able to get the more practical algorithm by replacing λ with a better approximation. So we propose the following more practical algorithm, though the convergence of which is not proved yet.

Algorithm 2: Applied algorithm which converge to a zero of f.

(step 1) Take $\lambda \in \mathbf{C}$. Check that $|f(\lambda)| > \varepsilon$. Let $R_1 = \min(p\left|\frac{f(\lambda)}{f'(\lambda)}\right|, |f(\lambda)|^{1/p})$, $m = 5$, $L = 1$, $\tau_{max} = R_1$, $\tau_{min} = 0$ and $\tau = R_1/p$.

(step 2) Compute $T = T(\lambda, \tau, m; f)$.

(step 3) If $|T| \leq 10^{-5}$ then $\tau_{min} = \tau$, $\tau = \frac{\tau_{max}+\tau_{min}}{2}$ and go to (step 2).

If $|T| \geq .99$ then $\tau_{max} = \tau$, $\tau = \frac{\tau_{max}+\tau_{min}}{2}$ and go to (step 2).

(step 4) Get the $3m$ values of $\left(\frac{T-n_1}{T}\right)^{\frac{1}{m}}$ as $\{x_1^{n_1}, \cdots, x_m^{n_1}\}$, $n_1 = 1, 2, 3$. Choose x from $\{x_1^{n_1}, \cdots, x_m^{n_1}\}$ such that $|f(\lambda + \tau x)| = \min |f(\lambda + \tau x_k^{n_1})|$.

If $|f(\lambda + \tau x)| > |f(\lambda)|$ then let $m = 2m$ and go to (step 2) else let $\lambda = \lambda + \tau x$.

If $|f(\lambda)| < \varepsilon$ then stop

else let $R_1 = \min(p\left|\frac{f(\lambda)}{f'(\lambda)}\right|, |f(\lambda)|^{1/p})$, $\tau_{max} = R_1$, $\tau_{min} = 0$ and $\tau = R_1/p$.

$$
\begin{aligned}
&Let \quad m = 5 + M, \quad if \quad \tau > 10^{-2} \\
or \quad &let \quad m = 3 + M, \quad if \quad 10^{-2} \geq \tau > 10^{-9} \\
&or \quad let \quad m = 1, \quad if \quad 10^{-9} \geq \tau.
\end{aligned}
$$

(step 5) Go to (step 2).

Remark 3.5 We should remark that τ may be decided easily, satisfying $|\tau| < |\nu_1|$, as far as $|T|$ is not too small. In fact for polynomials of lower order, we can set τ more roughly, say $|\tau| = |f(\lambda)|$. The criterion to determine the parameters m and τ in (step 3) is suitable for the polynomials of order less than 50. As is seen from Theorem 2.1, we had better to take m larger than this case for polynomials of higher order.

Remark 3.6 We assume here that the multiplicities of zeros are rarely greater than 3. So we set $n_1 = 1, 2, 3$. If the multiplicity or the number of close zeros is guessed to be t_0 then it will be more effective to set $n_1 = 1, 2, \cdots, t_0$. If x is chosen from $\{x_1^1, \cdots, x_m^1\}$ then it does not need to compute for $n_1 \geq 2$ in the following iterations.

Remark 3.7 The case of $m = 1$ is essentially coincide with the Newton's method.

Remark 3.8 When τ and m are fixed, this algorithm is not generally convergent as is proved in McMullen [4] that there is no generally convergent purely iterative algorithm, rational over $\mathbf{C}$, for finding roots of polynomials of degree ≥ 4. There are examples which does not converge to a zero of f. In fact Professor Sugiura of Nagoya Univ. gave examples of triples $(f(z), m, \tau)$ each of which guarantees the existence of a region D such that if the initial λ is taken in D then the iteration of this algorithm for fixed m and τ gives a sequence that approaches to two different values periodically.

Remark 3.9 When a cluster of very close zeros apart from other zeros exists, we can easily see from Lemma 2.1 that $\lambda + \frac{S}{T}$ yields a certain weighted mean of zeros contained in the cluster by taking a small circle with center λ and radius $|\tau|$ which enclose the cluster and exclude all the other zeros. Then the value of T gives the number of zeros contained in it. Moreover considering the weighted mean like a multiple zero z_1, the estimate in Theorem 2.2 can be applied to this case. That is, a set of very close zeros is treated as a multiple zero in the formula of $\lambda + \frac{S}{T}$.

4 Results of numerical experiments

Numerical experiments were carried out for many examples. Algorithm 1 was testified to work well; That is, Algorithm 1 is generally convergent and the errors of test computations are theoretically reasonable values, except machine errors. So we show only a few results which characterize how Algorithm 1 works.

The polynomial for the experiment to test Algorithm 1 is $f(z) = z^3 - 3z + 3$ whose zeros are $z_1 = 1.051\cdots + 0.565\cdots i$, $z_2 = 1.051\cdots - 0.565\cdots i$ and $z_3 = -2.103\cdots + 0i$. We denote the theoretical error bound stated in Theorem 2.1 by E, i.e. $E = \frac{16}{m}\frac{|\nu_1|}{1-|\frac{\tau}{\nu_2}|^m}\left|\frac{\nu_1}{\nu_2}\right|^m$. Let $z = \lambda + \tau\left(\frac{T-1}{T}\right)^{\frac{1}{m}}$.

Table 1-1 by Algorithm 1 $\quad f(z) = z^3 - 3z + 3$

The case $|z_1 - \lambda| \ll |z_2 - \lambda|$. $\quad \lambda = 1.051 + .566i \quad \frac{|z_1-\lambda|}{|z_2-\lambda|} = 1.045 \times 10^{-3}$

τ	m	$\lvert z_1 - z\rvert$	E	$\lvert f(z)\rvert$
0.001	2	7.09×10^{-10}	1.03×10^{-8}	2.57×10^{-9}
0.001	3	8.34×10^{-13}	7.19×10^{-12}	3.02×10^{-12}
0.001	4	1.42×10^{-15}	5.63×10^{-15}	3.13×10^{-15}

Table 1-1 shows that if λ is a good approximation of zero then Algorithm 1 converges even for small m.

Table 1-2 by Algorithm 1 $\quad f(z) = z^3 - 3z + 3$

The case $|z_1 - \lambda| \sim |z_2 - \lambda|$. $\quad \lambda = 2.0 + 0.001i \quad \frac{|z_1-\lambda|}{|z_2-\lambda|} = 0.9991$

τ	m	$\lvert z_1 - z\rvert$	E	$\lvert f(z)\rvert$
1.1016	512	3.89×10^{-3}	2.97×10^{-2}	1.41×10^{-2}
1.1016	1024	5.57×10^{-4}	7.23×10^{-3}	2.02×10^{-3}
1.1016	2048	7.97×10^{-5}	1.30×10^{-3}	2.89×10^{-4}
1.1016	4096	5.95×10^{-6}	9.64×10^{-5}	2.16×10^{-5}

We can seee in Table 1-2 that if $\frac{|z_1-\lambda|}{|z_2-\lambda|} \sim 1$ then, though it needs a large amount of computation, it converges very slowly.

As for the examples for Algorithm 2, we tried some polynomials of order higher than 100 with satisfactory results, added to many ones of lower order. Here we show some results which illustrate how the convergent process of Algorithm 2 is. In these experiments roughly determined τ such that

$|\tau| = O(|f(\lambda)|)$ is used. Table 2-1 shows that Algorithm 2 converges even if $|\nu_1| = |\nu_2|$. Table 2-2 is the result for $f(z) = z^{20} + 1$ which has simple zeros distributed on the unit circle. Table 2-3 is the result for a polynomial which has double zeros. We set the range of n_1 $\{1, 2, 3\}$ in these experiments. It is easily seen from our theorems that if $|z_1 - \lambda| < |\tau|$ then $\lambda + \frac{S}{T}$ approximates z_1 better than $\lambda + \tau(\frac{T-n_1}{T})^{\frac{1}{m}}$. So we added to the result of Algorithm 2 the value $|f(\lambda + \frac{S}{T})|$ in the following tables to compare two values.

Table 2-1 by Algorithm 2 $f(z) = z^3 - 3z + 3$

iteration k	m_k	τ_k	n_1	λ_k	$\lvert f(\lambda_k)\rvert$	$\lvert f(\lambda_{k-1} + \frac{S}{T})\rvert$
0	/	/	/	2.0	5.0	
1	5	0.1	2	$1.0876\cdots - 0.6629\cdots i$	4.164×10^{-1}	1.363×10^{0}
2	3	0.01	1	$1.0519\cdots - 0.5652\cdots i$	7.343×10^{-5}	2.161×10^{-3}
3	4	7.34×10^{-5}	1	$1.0519\cdots - 0.5652\cdots i$	2.003×10^{-15}	4.089×10^{-16}
4	1	1.0×10^{-9}	1	$1.0519\cdots - 0.5652\cdots i$	4.089×10^{-16}	3.624×10^{-9}
$\lambda = 1.\underline{051901701367768} - 0.\underline{5652358516771708}i$, $\lvert z_1 - \lambda\rvert = 5.99 \times 10^{-16}$						

Table 2-2 by Algorithm 2 $f(z) = z^{20} + 1$

iteration k	m_k	τ_k	n_1	λ_k	$\lvert f(\lambda_k)\rvert$	$\lvert f(\lambda_{k-1} + \frac{S}{T})\rvert$
0	/	/	/	$0.4 + 0.5i$	$1.000080\cdots$	
1	5	1.0×10^{-1}	3	$0.7466\cdots + 0.1402\cdots i$	9.97×10^{-1}	1.01×10^{0}
2	3	1.0×10^{-2}	3	$0.5237\cdots + 0.5338\cdots i$	9.97×10^{-1}	9.72×10^{-1}
3	4	1.0×10^{-2}	1	$0.7214\cdots + 0.7165\cdots i$	4.02×10^{-1}	9.27×10^{-1}
4	3	0.1×10^{-2}	1	$0.7071\cdots + 0.7071\cdots i$	2.68×10^{-5}	2.12×10^{-3}
5	4	0.1×10^{-5}	1	$0.7071\cdots + 0.7071\cdots i$	1.52×10^{-12}	$o(1. \times 10^{-16})$
6	1	0.1×10^{-9}	1	$0.7071\cdots + 0.7071\cdots i$	2.94×10^{-16}	2.00×10^{-8}
$\lambda_6 = 0.\underline{7071067811865476} + 0.\underline{7071067811865476}i$, $\lvert z_1 - \lambda_6\rvert = 1. \times 10^{-16}$						

λ_6 in Table 2-2 is eqaul to the exact zero to the full digits. Note that $\lambda_2 + \frac{S}{T}$ in Table 2-1 and $\lambda_4 + \frac{S}{T}$ in Table 2-2 can be considered to be the numerical solutions.

Remark 4.1 Since $\lambda_{k-1} + \frac{S}{T}$ can be computed cheaply by adding one process to Algorithm 2, we can expect that such a device combining two algorithms will give another more efficient algorithm.

Table 2-3 by Algorithm 2 $f(z) = (z-2)^2(z+1)^2(z-0.5)(z-0.501)(z-0.503)$

iteration k	m_k	τ_k	n_1	λ_k	$\lvert f(\lambda_k)\rvert$	$\lvert f(\lambda_{k-1} + \frac{S}{T})\rvert$
0	/	/	/	$-5 + 3i$	$356755.7\cdots$	
1	5	$1. \times 10^{-1}$	3	$-0.8138\cdots + 0.2347\cdots i$	1.71×10^{0}	2.90×10^{0}
2	3	$1. \times 10^{-2}$	2	$-0.9990\cdots + 0.0015\cdots i$	9.64×10^{-5}	2.27×10^{-2}
3	4	9.64×10^{-5}	2	$\lambda = -1.0000\cdots - 0.0000\cdots$	4.12×10^{-16}	5.93×10^{-16}
$\lambda_3 = -1.\underline{000000000003040} - 1.91 \times 10^{-13}i$, $\lvert z_1 - \lambda_3\rvert = 3.05 \times 10^{-12}$						

Table 2-3 shows that in the case of zeros of multiplicity greater than 1, our method has the advantage of rapidity and of accuracy over Newton's method.

5 Concluding remarks

Remark 5.1 The feature of our algorithm 1 and 2 is in the usage of the error of the numerical integral of (1.1). Our root finding method is a new one which we cannnot find in preceding papers cited in McNamee's biblography.

Remark 5.2 Our fundamental algorithm is proved to be globally convergent for almost all initial values, while our applied algorithm is not proved, yet. But we have the conjecture that Algorithm 2 is globally convergent from the theoretical situation and the results of the numerical experiments.

Remark 5.3 Comparing with the Newton's method, Algorithm 2 is comparable in rapidity and is equal or better in accuracy, especially for zeros of multiplicity greater than 1.

Remark 5.4 Connecting Algorithm 2 and the value $\lambda + \frac{S}{T}$, the more rapid and reliable algorithm will be realized. Comparing the two kinds of estimates in two theorems, we can expect to find the method to distinguish the multiple zeros from the close ones.

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