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Representations of finite groups and Hilbert modular forms for real quadratic fields

東京理大理工 浜畑芳紀 (Yoshinori Hamahata)

1. Introduction.

In this paper, we would like to report our results about the representation of finite groups on the space of Hilbert modular cusp forms.

In his paper [3], Hecke considered the representation π of $SL_2(\mathbb{F}_p)$ on the space of elliptic cusp forms of weight 2 for $\Gamma(p)$, and he determined how tr π decomposes into irreducible characters. Above all, he showed that the difference of the multiplicities of certain two irreducible characters yields the Dirichlet expression for $h(\mathbb{Q}(\sqrt{-p}))$, the class number of $\mathbb{Q}(\sqrt{-p})$. This result was generalized to cusp forms of several variables, i.e., Hilbert cusp forms by H. Yoshida-H. Saito and W. Meyer-R. Sczech, and Siegel cusp forms of degree 2 by K. Hashimoto.

Using his trace formula, Eichler [1] obtained another expression for the difference of the multiplicities above. This expression can be rewritten as the Dirichlet expression for $h(\mathbb{Q}(\sqrt{-p}))$. This Eichler's result was generalized to Hilbert cusp forms for real quadratic fields by H. Saito [6], and for totally real cubic fields by the author [2].

The purpose of this paper is to report that Saito's result can be generalized to the case where the level of the Hilbert modular group is the product of the distinct prime ideals lying over odd primes. The plan of this paper is as follows. In section 2 we review the definiton of Hilbert cusp forms and then recall the results of Hecke, Eichler, Yoshida-Saito and Meyer-Sczech. In section 3, we recall Saito's result on Hilbert cusp forms for real quadratic fields. In section 4, our result is stated. In section 5, we give a result on which we cannot talk at RIMS. This result is a generalization of that of Meyer-Sczech.

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Notation. Let $\mathbb{R}, \mathbb{Q}$ be the field of real, and rational numbers, respectively, and $\mathbb{F}_q$ the finite field of q-elements. For a number field K, let $h(K)$ denote the class number of K. Put $\mathbf{e}[\bullet] = \exp(2\pi i \bullet)$. By $\#(S)$, we mean the cardinality of the set S.

2. Hilbert modular forms.

In this section we first review the definition of Hilbert cusp forms. Next we recall the results of Hecke, Eichler, Yoshida-Saito and Meyer-Sczech.

Let K be a totally real number field of degree n, and $\mathfrak{o}_K$ the ring of integers of K. There exist n different embeddings of K into $\mathbb{R}$. Denote them by $K \hookrightarrow \mathbb{R}, \quad x \mapsto x^{(i)} \ (x \in K)$. Let $\mathfrak{H}$ be the upper half plane of all complex numbers with positive imaginary part. The group $SL_2(\mathfrak{o}_K)$ acts on $\mathfrak{H}^n$, the n-th fold product of $\mathfrak{H}$, as follows: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathfrak{o}_K)$ and $z = (z_1, \cdots, z_n) \in \mathfrak{H}^n$ we have

$$\gamma \cdot z = \left(\frac{a^{(1)} z_1 + b^{(1)}}{c^{(1)} z_1 + d^{(1)}}, \cdots, \frac{a^{(n)} z_n + b^{(n)}}{c^{(n)} z_n + d^{(n)}} \right).$$

Let $\mathfrak{n}$ be an integral ideal of K, and set

$$\Gamma(\mathfrak{n}) = \{\gamma \in SL_2(\mathfrak{o}_K) \mid \gamma \equiv 1_2 \pmod{\mathfrak{n}}\}.$$

Then $\Gamma(\mathfrak{n})$ also acts on $\mathfrak{H}^n$. Let k be an even positive integer. For any element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathfrak{o}_K)$, put $j_k(\gamma, z) = \prod_{i=1}^n (c^{(i)} z_i + d^{(i)})^{-k}$. We now define Hilbert modular cusp forms.

Definition 2.1. A holomorphic function f on $\mathfrak{H}^n$ is called *Hilbert cusp form* of weight k for $\Gamma(\mathfrak{n})$ if it satisfies

i) $f(\gamma z) j_k(\gamma, z) = f(z)$ for any $\gamma \in \Gamma(\mathfrak{p})$,

ii) f is holomorphic at each cusp of $\Gamma(\mathfrak{n})$, and its Fourier expansion at each cusp has no the constant term.

Let $S_k(\Gamma(\mathfrak{n}))$ be the set of Hilbert cusp forms with weight k for $\Gamma(\mathfrak{n})$. Put $f|_k[\gamma] = f(\gamma z) j_k(\gamma, z)$ for $\gamma \in SL_2(\mathfrak{o}_K)$. Then $SL_2(\mathfrak{o}_K)$ acts on $S_k(\Gamma(\mathfrak{n}))$ by $(\gamma, f) \mapsto f|_k[\gamma]$. Since $\Gamma(\mathfrak{n})$ acts on it trivially, $SL_2(\mathfrak{o}_K)/\Gamma(\mathfrak{n})$ acts on it. Let π be the representation associated to this action. We are interested in the representation π.

In the rest of this section, we assume that $\mathfrak{n}$ is a prime ideal $\mathfrak{p}$ lying over an odd prime. Let q be a power of an odd prime. Then, there are two pairs of irreducible characters of $SL_2(\mathbb{F}_q)$ whose values are conjugate mutually. We give a list of values at $\epsilon = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\epsilon' = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$ (η is a nonsquare element of $\mathbb{F}_q^*$) of such pairs (ψ^+, ψ^-) and (ψ'^+, ψ'^-) as follows:

	ϵ	ϵ'
ψ^+	$\frac{1+\sqrt{q^*}}{2}$	$\frac{1-\sqrt{q^*}}{2}$
ψ^-	$\frac{1-\sqrt{q^*}}{2}$	$\frac{1+\sqrt{q^*}}{2}$
ψ'^+	$\frac{-1+\sqrt{q^*}}{2}$	$\frac{-1-\sqrt{q^*}}{2}$
ψ'^-	$\frac{-1-\sqrt{q^*}}{2}$	$\frac{-1+\sqrt{q^*}}{2}$

Here we put $q^* = q(-1)^{(q-1)/2}$. Note that each pair has the same values on other conjugacy classes. The characters ψ^+ and ψ^- are of degree $(q+1)/2$, and ψ'^+ and ψ'^- are of degree $(q-1)/2$. If $q \equiv 1 \pmod 4$, then ψ'^+ and ψ'^- do not appear. If $q \equiv 3 \pmod 4$, then ψ^+ and ψ^- do not appear. Let $m(\bullet)$ be the multiplicity of $\bullet$ in tr π. Now Let q be the norm of $\mathfrak{n} = \mathfrak{p}$. Hecke proved the following result.

Theorem 2.2(Hecke [3]). *If $n = 1$ and $k = 2$, then*

$$m(\psi^+) - m(\psi^-) + m(\psi'^+) - m(\psi'^-) = \begin{cases} 0 & (q \equiv 1 \bmod 4), \\ h(\mathbb{Q}(\sqrt{-q})) & (q \equiv 3 \bmod 4), \end{cases}$$

Eichler got the following result.

Theorem 2.3(Eichler [1]). *If $n = 1$ and $k = 2$, then*

$$m(\psi^+) - m(\psi^-) + m(\psi'^+) - m(\psi'^-) = \frac{1}{\sqrt{p^*}} \sum_{i=1}^{p-1} \left(\frac{i}{p}\right) \nu(i),$$

where $\left(\frac{\ }{p}\right)$ is the quadratic residue symbol mod p ($\mathfrak{p} = (p)$), and $\nu(i) = -\mathbf{e}[i/p]/(1 - \mathbf{e}[i/p])$.

Using the Selberg trace formula, H. Yoshida and H. Saito generalized Theorem 2.2 to Hilbert cusp forms independently:

Theorem 2.4(Yoshida and Saito, cf. [6]). *If $k \geq 4$, then we have*

$$|m(\psi^+) - m(\psi^-) + m(\psi'^+) - m(\psi'^-)| = 2^{n-1} \sum_{K_j} \frac{h(K_j)}{h(K)},$$

where K_j runs over totally imaginary quadratic extensions of K with the relative discriminant $\mathfrak{n}$.

In the case $n = 2$, Meyer and Sczech removed the absolute value of $|m(\psi^+) - m(\psi^-) + m(\psi'^+) - m(\psi'^-)|$:

Theorem 2.5(Meyer and Sczech [5]) *If $n = 2$, then we have*

$$m(\psi^+) - m(\psi^-) + m(\psi'^+) - m(\psi'^-) = -2 \sum_{K_j} \frac{h(K_j)}{h(K)},$$

where K_j runs over totally imaginary quadratic extensions of K with the relative discriminant $\mathfrak{p}$.

3. The result of Saito.

In this section we first review Hilbert modular surfaces in order to explain Saito's result, and then recall his result, which is an analogue of Eichler's formula (Theorem 2.3). In the rest of this paper, we assume $n = 2$.

Let the notation be as above. Since $\Gamma(\mathfrak{n})$ acts on $\mathfrak{H}^2$, we have the quotient space $\mathfrak{H}^2/\Gamma(\mathfrak{n})$. One can compactify it by adding all cusps of $\Gamma(\mathfrak{n})$. We denote by $\overline{\mathfrak{H}^2/\Gamma(\mathfrak{n})}$ the resulting surface. The surface $\overline{\mathfrak{H}^2/\Gamma(\mathfrak{n})}$ has two kinds of singularities, i.e., quotient singularities and cusp singularities. Let $X(\mathfrak{n})$ be the desingularization of $\overline{\mathfrak{H}^2/\Gamma(\mathfrak{n})}$. It is called the Hilbert modular surface obtained from $\Gamma(\mathfrak{n})$. If we assume that $\overline{\mathfrak{H}^2/\Gamma(\mathfrak{n})}$ has no quotient singularities and that $h(K) = 1$, then the resolution of singularities can be described by a complex Σ obtained from the pair $(\mathfrak{o}_K, U(\mathfrak{n}))$. Here $U(\mathfrak{n})$ denotes the group of units of K congruent to 1 modulo $\mathfrak{n}$. Let γ be any element of $SL_2(\mathfrak{o}_K)$. Since $\Gamma(\mathfrak{n})$ is a normal subgroup, γ induces f_γ, the automorphism of $\mathfrak{H}^2/\Gamma(\mathfrak{n})$ defined by $(z_1, z_2) \mapsto (\gamma^{(1)} z_1, \gamma^{(2)} z_2)$. Here $\gamma^{(i)}$ denotes the matrix defined by exchanging the components of γ for the images of them by the i-th embedding of K. The automorphism f_γ can be extended to that of $\overline{\mathfrak{H}^2/\Gamma(\mathfrak{n})}$, and moreover that of $X(\mathfrak{n})$, which is also denoted by f_γ.

We now recall the result of Saito [3]. Let K be a real quadratic field, and $\mathfrak{n}$ an integral ideal of K such that $\mathfrak{n}$ is generated by a totally positive element μ, prime to $6 \cdot d_K$ (d_K is the discriminant of K). Let U be the unit group of K, and $U(\mathfrak{n})$ the group of units congruent to 1 modulo $\mathfrak{n}$. Let $[U : U(\mathfrak{n})] = t$. There exists an element $w \in \mathfrak{o}_K$ such that $\mathfrak{o}_K = \mathbb{Z} + \mathbb{Z}w$ and $0 < w' < 1 < w$. Here w' denotes the conjugate of w. We have the continued fraction

$$w = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r - \cfrac{1}{w}}}}.$$

Then we define positive integers p_k and q_k by

$$\frac{p_k}{q_k} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_{k-1} - \cfrac{1}{b_k}}}}$$

for a positive integer k $(1 \leq k \leq r)$. For any element $\alpha \in \mathfrak{o}_K$, H. Saito defines
(*)
$\nu(\alpha) :=$

$$\sum_i \frac{\mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{p_i - q_i w'}{w - w'}\right] \cdot \mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{-p_{i-1} + q_{i-1} w'}{w - w'}\right]}{\left(1 - \mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{p_i - q_i w'}{w - w'}\right]\right)\left(1 - \mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{-p_{i-1} + q_{i-1} w'}{w - w'}\right]\right)}$$
$$+ \sum_j \frac{\mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{p_j - q_j w'}{w - w'}\right]}{\left(1 - \mathbf{e}\left[-\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{p_j - q_j w'}{w - w'}\right]\right)} \left\{-1 + \frac{b_j}{1 - \mathbf{e}\left[-\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{p_j - q_j w'}{w - w'}\right]}\right\},$$

where i runs over such indices as $1 \leq i \leq rt$ and neither $\mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{p_i - q_i w'}{w - w'}\right]$ nor $\mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{-p_{i-1} + q_{i-1} w'}{w - w'}\right]$ equal 1, and j runs over such indices as $1 \leq j \leq rt$ and $\mathbf{e}\left[\operatorname{tr}\left(\frac{\alpha}{\mu}\right) \cdot \frac{-p_{j-1} + q_{j-1} w'}{w - w'}\right] = 1$. Notice that each integer

$-b_j$ is the selfintersection number of some irreducible curve arising from the cusp resolution of α/μ. Using the holomorphic Lefschetz formula of Atiyah-Singer, H. Saito proved the following:

Theorem 3.1(Saito [6]). *Let $\mathfrak{n}$ be a prime ideal $\mathfrak{p}$ lying over an odd prime. Then on $S_2(\Gamma(\mathfrak{p}))$ we have*

$$m(\psi^+)-m(\psi^-)+m(\psi'^+)-m(\psi'^-) = \frac{1}{\sqrt{q^*}}\cdot\frac{2}{[U:U(\mathfrak{p})]}\sum_{\alpha\in(\mathfrak{o}_K/\mathfrak{p})^\times}\left(\frac{\alpha}{\mathfrak{p}}\right)\nu(\alpha),$$

where $q = N(\mathfrak{p})$ and $\left(\frac{\ }{\mathfrak{p}}\right)$ denotes the quadratic residue symbol mod $\mathfrak{p}$.

4. The main result.

Let $\mathfrak{n} = \mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_t$ be a product of distinct prime ideals $\mathfrak{p}_i$. Put $q_i = N(\mathfrak{p}_i)$, $q_i^* = q_i(-1)^{(q_i-1)/2}$. We assume that each q_i is a power of an odd prime. Let $\psi_i^\pm$, $\psi_i'^\pm$ be irreducible characters of $SL_2(\mathbf{F}_{q_i})$ as in the table of section 2.

Theorem 4.1. *We assume that $h(K) = 1$, that $\mathfrak{n}$ is generated by a totally positive element μ, and that $\mathfrak{n}$ is prime to $6\cdot d_K$. Then*

$$\sum_{e_1,\cdots,e_t\in\{\pm1\}}\sum_{\varphi_1\in\{\psi_1,\psi_1'\}}\cdots\sum_{\varphi_t\in\{\psi_t,\psi_t'\}} e_1\cdots e_t\cdot m(\varphi_1^{e_1}\cdots\varphi_t^{e_t})$$
$$=\frac{1}{\sqrt{q_1^*\cdots q_t^*}}\cdot\frac{2^t}{[U:U(\mathfrak{n})]}\sum_{\alpha\in(\mathfrak{o}_K/\mathfrak{n})^\times}\left(\frac{\alpha}{\mathfrak{n}}\right)\nu(\alpha),$$

where $\nu(\alpha)$ is (*).

This result is a generalization of Theorem 3.1.

The sketch of Proof. By Chinese remainder theorem, we can find an element η_i of $\mathfrak{o}_K$ so that $\left(\frac{\eta_i}{\mathfrak{p}_i}\right) = -1$ and $\left(\frac{\eta_i}{\mathfrak{p}_j}\right) = 1$ $(j \neq i)$ for each i. Then we have the following:

Lemma 4.2.

$$\sum_{\epsilon_1\in\{1,\eta_1\}}\cdots\sum_{\epsilon_t\in\{1,\eta_t\}}\left(\frac{\epsilon_1\cdots\epsilon_t}{\mathfrak{n}}\right)(\mathrm{tr}\ \pi)\left(\begin{pmatrix}1 & \epsilon_1\cdots\epsilon_t\\ 0 & 1\end{pmatrix}\right)$$
$$=\sqrt{q_1^*\cdots q_t^*}\sum_{e_1,\cdots,e_t\in\{\pm1\}}\sum_{\varphi_1\in\{\psi_1,\psi_1'\}}\cdots\sum_{\varphi_t\in\{\psi_t,\psi_t'\}} e_1\cdots e_t\cdot m(\varphi_1^{e_1}\cdots\varphi_t^{e_t}).$$

We write $f_{\epsilon_1\cdots\epsilon_t}$ for the automorphism of $X(\mathfrak{n})$ induced by $(z_1, z_2) \mapsto (z_1 + \epsilon_1 \cdots \epsilon_t, z_2 + \epsilon'_1 \cdots \epsilon'_t)$. Denote by Ω^2 the sheaf of germs of holomorphic 2-forms on $X(\mathfrak{n})$. Since $S_2(\Gamma(\mathfrak{n})) = H^0(X(\mathfrak{n}), \Omega^2)$, we have

$$\operatorname{tr} \pi(\begin{pmatrix} 1 & \epsilon_1 \cdots \epsilon_t \\ 0 & 1 \end{pmatrix}) = \operatorname{tr}(f_{\epsilon_1\cdots\epsilon_t} | H^0(X(\mathfrak{n}), \Omega^2).$$

Put

$$\tau(\epsilon_1 \cdots \epsilon_t) := \sum_{i=0}^{2} (-1)^i \operatorname{tr}(f_{\epsilon_1\cdots\epsilon_t} | H^i(X(\mathfrak{n}), \Omega^2).$$

Since $H^1(X(\mathfrak{n}), \Omega^2) = 0$ and $H^2(X(\mathfrak{n}), \Omega^2) = \mathbb{C}$, the left hand side of the equation in Lemma 4.2 is written as

$$(4.3) \qquad \sum_{\epsilon_1 \in \{1, \eta_1\}} \cdots \sum_{\epsilon_t \in \{1, \eta_t\}} \left(\frac{\epsilon_1 \cdots \epsilon_t}{\mathfrak{n}} \right) \tau(\epsilon_1 \cdots \epsilon_t).$$

We then apply the holomorphic Lefschetz formula to each $\tau(\epsilon_1 \cdots \epsilon_t)$, and rewrite the last equation.

5. Supplement.

In this section, we give a generalization of the result of Meyer-Sczech. Let $\mathfrak{n}$ be an integral ideal of K as in the previous section. We assume $\mathfrak{n}$ to be prime to $6 \cdot d_K$. Then we have the following:

Theorem 5.1.

$$\sum_{e_1, \cdots, e_t \in \{\pm 1\}} \sum_{\varphi_1 \in \{\psi_1, \psi'_1\}} \cdots \sum_{\varphi_t \in \{\psi_t, \psi'_t\}} e_1 \cdots e_t \cdot m(\varphi_1^{e_1} \cdots \varphi_t^{e_t})$$
$$= -2^t \sum_{L/K} \frac{h(L)}{h(K)},$$

where the sum in the right hand side is over all totally imaginary quadratic extensions L of K with relative discriminant $\mathfrak{n}$.

The sketch of Proof. We rewrite (4.3) as the sum of values at 1 of L-series of certain cusps, and furthermore rewrite it as the sum of values at 1 of Hecke's L-series associated with real characters of the group of all ideals of K prime to $\mathfrak{n}$. Finally we apply the analytic class number formula to the resulting expression.

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Yoshinori HAMAHATA
Department of Mathematics
Faculty of Science and Technology
Science University of Tokyo
Noda, Chiba, 278-8510, JAPAN