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ON THE VANISHING OF IWASAWA INVARIANTS OF ABSOLUTELY ABELIAN p-EXTENSIONS

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ABSTRACT. Let p be any odd prime. We determine all absolutely abelian p-extension fields such that Iwasawa λ_p, μ_p and ν_p-invariants of the cyclotomic $\mathbb{Z}_p$-extension are zero, in terms of congruent conditions, p-th power residues, and genus fields.

1. INTRODUCTION

Let p be a prime and $\mathbb{Z}_p$ the ring of p-adic integers. Let k be a finite extension of the rational number field $\mathbb{Q}$, k_∞ a $\mathbb{Z}_p$-extension of k, k_n the n-th layer of k_∞/k, and A_n the p-Sylow subgroup of the ideal class group of k_n. Iwasawa proved the well-known theorem about the order $\#A_n$ of A_n that there exist integers $\lambda = \lambda(k_\infty/k) \geq 0, \mu = \mu(k_\infty/k) \geq 0, \nu = \nu(k_\infty/k)$, and $n_0 \geq 0$ such that

$$\#A_n = p^{\lambda n + \mu p^n + \nu}$$

for all $n \geq n_0$. These integers $\lambda = \lambda(k_\infty/k)$, $\mu = \mu(k_\infty/k)$ and $\nu = \nu(k_\infty/k)$ are called *Iwasawa invariants* of k_∞/k for p. If k_∞ is the cyclotomic $\mathbb{Z}_p$-extension of k, we write $\lambda_p(k)$, $\mu_p(k)$ and $\nu_p(k)$ for the above invariants, respectively.

In [7], Greenberg conjectured that if k is a totally real, $\lambda_p(k) = \mu_p(k) = 0$. We call this conjecture *Greenberg conjecture.* For Iwasawa λ_p, μ_p-invariants of abelian p-extension fields of $\mathbb{Q}$, there are results by Greenberg ([7], V), Iwasawa([9]), Fukuda, Komatsu, Ozaki and Taya([6]), Fukuda([4]), and the author([12]), etc. On the other hand, Ferrero and Washington have shown that $\mu_p(k) = 0$ for any abelian extension field k of $\mathbb{Q}$.

In this paper, we will consider a stronger condition than Greenberg conjecture that $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$ and determine all absolutely abelian p-extensions k, i.e. k is an abelian extension of the rational number field $\mathbb{Q}$, with $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$ for an odd prime p, using the results of G. Cornell and M. Rosen([1]).

2. MAIN THEOREM

Throughout this section, we fix an odd prime p. For an absolutely abelian p-extension field k, let f_k be its conductor, i.e. f_k is the minimum positive integer with $k \subseteq \mathbb{Q}(\zeta_{f_k})$. Then, it follows easily that $f_k = p^a p_1 \cdots p_t$, where a is a non-negative integer and $p_1, \cdots, p_t$ are distinct primes which are congruent to 1 modulo p. We denote k_G by the genus field of k. So k_G is the maximal unramified abelian extension of k such that $k_G/\mathbb{Q}$ is an abelian extension. In general, if $k/\mathbb{Q}$ is an abelian extension of odd degree, then it has shown by Leopoldt that

$$[k_G : k] = \frac{e_1 e_2 \cdots e_t}{[k : \mathbb{Q}]},$$

where $e_1, \cdots e_t$ are ramification indices of primes which ramify in $k/\mathbb{Q}$. Hence in our case, k_G is also an abelian p-extension of $\mathbb{Q}$. For instance we denote by $(\frac{\cdot}{\cdot})_p$ the p-th power residue symbol, i.e., for integers x, y, $(\frac{x}{y})_p = 1$ if and only if x is the p-th power modulo y.

Our main theorem gives a necessary and sufficient condition for $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$ in terms of p-th power residue symbol, congruent conditions and genus fields:

Theorem 1. *Let k be an abelian p-extension of $\mathbb{Q}$, and $f_k = p^a p_1 \cdots p_t$ the prime decomposition of its conductor, where primes $p_1, \cdots, p_t$ are distinct. If*

$$\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0, \tag{1}$$

then $t \leq 2$. Conversely, in each case of $t = 0$ or 1 or 2, the followings are a necessary and sufficient condition of (1):

***In case of* $\mathbf{t = 0}$:** (1) *holds.*

***In case of* $\mathbf{t = 1}$:** (1) *is equivalent to $k_1 = k_{1,G}$ and,*

$$\left(\frac{p}{p_1}\right)_p \neq 1 \, or \; p_1 \not\equiv 1 \pmod{p^2}. \tag{2}$$

***In case of* $\mathbf{t = 2}$:** (1) *is equivalent to $k_1 = k_{1,G}$, and for $(i, j) = (1, 2)$ or $(2, 1)$,*

$$\left(\frac{p}{p_i}\right)_p \neq 1, \left(\frac{p_i}{p_j}\right)_p \neq 1, p_j \not\equiv 1 \pmod{p^2}, \tag{3}$$

and, there exist $x, y, z \in \mathbb{F}_p$ such that

$$\left(\frac{p_j p^x}{p_i}\right)_p = 1, \left(\frac{p p_i^y}{p_j}\right)_p = 1, p_i p_j^z \equiv 1 \pmod{p^2}, \, and \; xyz \neq -1 in \; \mathbb{F}_p \tag{4}$$

In case of $t = 2$, the conditions in Theorem 1 are complicated. So we will give an example. We consider the case $p = 3, p_1 = 7$ and $p_2 = 19$. We denote k_7(resp. k_{19}) by the subfield of $\mathbb{Q}(\zeta_7)$ (resp.$\mathbb{Q}(\zeta_{19})$) with degree 3 over $\mathbb{Q}$. As for the condition $k_1 = k_{1,G}$, there exists a field F such that $k_7 \subsetneq F \subsetneq k_7 k_{19} \mathbb{Q}_1$ and $F \neq k_7 k_{19}, k_7 \mathbb{Q}_1$, where $\mathbb{Q}_1$ is the first layer of cyclotomic $\mathbb{Z}_3$-extension of $\mathbb{Q}$. Then $k_7 k_{19} \mathbb{Q}_1 / F$ is a nontrivial unramified extension and $k_7 k_{19} \mathbb{Q}_1$ is abelian, hence $F \subsetneq k_7 k_{19} \mathbb{Q}_1 \subseteq F_G$. But, for $F_1 = k_7 k_{19} \mathbb{Q}_1$, it follows easily that $F_1 = F_{1,G}$. If we restrict the case p is unramified in k, i.e. $a = 0$, then the statement $k_1 = k_{1,G}$ can be simplified to $k = k_G$ because $k_1 = k\mathbb{Q}_1$. This restriction is not so strong: In general, for an absolutely abelian p-extension field k, there exists an absolutely abelian extension field k' such that p is unramified in k' and $k_\infty = k'_\infty$. Note that if k is the maximal subfield of $\mathbb{Q}(\zeta_m)$ ($m = p^a p_1 \cdots p_t$ as above) which is abelian p-extension of $\mathbb{Q}$, then $k = k_G$.

We continue to examine the above example. If we put $(i, j) = (1, 2)$, then $p_j = 19 \equiv 1 \pmod{3^2}$, so the condition (3) is not satisfied. But if we put $(i, j) = (2, 1)$, then we can verify that $p_i = 19$ and $p_j = 7$ satisfy the conditions (3) and (4). Hence, for example, if K is the maximal subfield of $\mathbb{Q}(\zeta_{7\cdot 19})$ which is 3-extension of $\mathbb{Q}$, then K satisfies the conditions of Theorem 1. Therefore we get

$$\lambda_p(K) = \mu_p(K) = \nu_p(K) = 0.$$

As for Greenberg conjecture, we can also get the following: In general, it is known that if $L \subseteq M$ then $\lambda_p(L) \leq \lambda_p(M)$ and $\mu_p(L) \leq \mu_p(M)$ for number fields L, M. Hence for any subfield k of $\mathbb{Q}(\zeta_{7\cdot 19})$ which is 3-extension of $\mathbb{Q}$, i.e. $k \subseteq K$, then $\lambda_p(k) = \mu_p(k) = 0$. This consideration is generalized as follows:

Corollary 2. *Let $m = p^a p_1 \cdots p_t$ satisfy the condition (2) or (3), (4). Then for any subfield k of $\mathbb{Q}(\zeta_m)$ which is p-extension of $\mathbb{Q}$, Greenberg conjecture for k and p is valid.*

3. The results of G. Cornell and M. Rosen

In this section, we review briefly part of [1]. Let $K/\mathbb{Q}$ be an abelian p-extension, p a prime. In the 1950's, A. Fröhlich determined all such fields with class number prime to p (cf. [2]). In [1],G. Cornell and M. Rosen reconsidered this problem in the case where p is an odd prime, and reduced the problem to the case when $\mathrm{Gal}(K/\mathbb{Q})$ is an elementary abelian p-group, i.e. $\mathrm{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^m$ for some integer m .

We suppose that p is an odd prime and $\mathrm{Gal}(K/\mathbb{Q})$ is an abelian p-group. Then the genus field K_G of K is also abelian p-extension. If p does not divide the class number h_K of K, then K does not have any non-trivial unramified abelian p-extension by class field theory, hence $K_G = K$. In the following we will assume $K_G = K$. Further, we consider the central p-class field K_C of K, i.e. K_C is the maximal p-extension of K such that K_C/K is abelian and unramified, $K_C/\mathbb{Q}$ is Galois and $\mathrm{Gal}(K_C/K)$ is in the center of $\mathrm{Gal}(K_C/\mathbb{Q})$. Since a p-group must have a lower central series that terminates in the identity, one sees that $p \not| h_K$ if and only if $K_C = K$. So we are interested in which case $K_C = K$. This can be reduced the case when $\mathrm{Gal}(K/\mathbb{Q})$ is an elementary abelian p-group by the following result:

Lemma 3 ([1] **Theorem 1).** *Let $K/\mathbb{Q}$ be an abelian p-extension with $K_G = K$. Let k be the maximal intermediate extension between $\mathbb{Q}$ and K such that* $\mathrm{Gal}(k/\mathbb{Q})$ *is an elementary abelian p-group. Then p-rank of* $\mathrm{Gal}(K_C/K)$ *is equal to the p-rank of* $\mathrm{Gal}(k_C/k)$.

In the case $\mathrm{Gal}(K/\mathbb{Q})$ is an elementary abelian p-group, by the results of Furuta and Tate, we have the following lemma:

Lemma 4 ([1] **Section 1).** *Let K be an absolutely abelian p-extension such that* $\mathrm{Gal}(K/\mathbb{Q})$ *is an elementary abelian p-group and $K_G = K$. Then, we have*

$$\mathrm{Gal}(K_C/K) \simeq \mathrm{Coker}(\oplus_{i=1}^{n} \wedge^2 (G_i) \longrightarrow \wedge^2(G)),$$

where G_i's are the decomposition groups of primes ramified in $K/\mathbb{Q}$ and $G =$ $\mathrm{Gal}(K/\mathbb{Q})$.

We will assume $\mathrm{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^m$. Let $p_1, \cdots p_t$ be the primes ramified in K and h_K the class number of K. From genus theory, it follows that if h_K is not divisible by p, then $t = m$. Also it follows that if $m \geq 4$ then p divides h_K by Lemma 4. So, we assume $t = m$ and $m = 2$ or 3. (In case of $t = m = 1$, $p \not| h_K$. cf. [8].)

Lemma 5 (**[1] Proposition 2**). *Suppose $m = 2$ and $p_i \neq p$ for $i = 1, 2$. Then $p | h_K$ if and only if $(\frac{p_1}{p_2})_p = 1$ and $(\frac{p_2}{p_1})_p = 1$.*

Next, we consider the case where one of the ramified primes is p. Suppose $m = 2$ and p and p_1 are the only primes ramified in K. Then we can get easily $K = k(p_1)\mathbb{Q}_1$ and $p_1 \equiv 1 \pmod p$, where $k(p_1)$ is the unique subfield of $\mathbb{Q}(\zeta_{p_1})$ which is cyclic over $\mathbb{Q}$ of degree p, ζ_{p_1} is a primitive p_1-th root of unity, and $\mathbb{Q}_1$ is the first layer of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.

Lemma 6 (**[1] Proposition 3**). *Suppose $m = 2$ and p and p_1 are the only primes ramified in K. Then $p | h_K$ if and only if $(\frac{p}{p_1})_p = 1$ and $p_1 \equiv 1 \pmod{p^2}$.*

Suppose $t = m = 3$ and p_1, p_2 and p_3 all the primes ramified in K. We put D_{p_i} the decomposition field of $p_i (i = 1, 2, 3)$ in K. In [1], the following simple result is given:

Lemma 7 (**[1] Theorem 2**). *Suppose $t = m = 3$. Following statements* (a) *and* (b) *are equivalent*:

(a) *h_K is not divisible by p,*
(b) *$[D_{p_1} : \mathbb{Q}] = [D_{p_2} : \mathbb{Q}] = [D_{p_3} : \mathbb{Q}] = p$ and $D_{p_1} D_{p_2} D_{p_3} = K$.*

In the next section, we shall prove Theorem 1,using these results.

4. Proof of Theorem 1

Notations are as in previous section.

Firstly, we suppose $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$. Clearly, this condition is equivalent to $A(k_n) = 0$ for any sufficiently large n. Then, k_n satisfies $k_n = k_{n,G}$ and $k_1 = k_{1,G}$, because all ramified primes are totally ramified in k_n/k_1. Since k_n is also an abelian p-extension of $\mathbb{Q}$, we can apply the results of Cornell-Rosen:

Let L be the maximal subfield of k_n such that $\mathrm{Gal}(L/\mathbb{Q})$ is an elementary abelian extension of $\mathbb{Q}$. Since $k_n = k_{n,G}$, $\mathrm{Gal}(k_n/\mathbb{Q})$ is the direct sum of the inertia groups of primes ramified in $k_n/\mathbb{Q}$, hence it follows that $L = k(p_1) \cdots k(p_t)\mathbb{Q}_1$. By Lemma 3, $A(k_n) = 0$ is equivalent to $p \not| h_L$. Note that if $t \geq 3$ then we always have $p | h_L$ as in the previous section. Hence we may examine in each case of $t = 0$ or 1 or 2.

If $t = 0$ then $L = \mathbb{Q}_1$, hence it is well known that $A(L) = A(\mathbb{Q}_1) = 0$ (cf. [8]).

If $t = 1$ then $L = k(p_1)\mathbb{Q}_1$. By lemma 6, we get the statement in Theorem 1.

In the following we assume that $t = 2$. In this case, $L = k(p_1)k(p_2)\mathbb{Q}_1$. Let $G_p, G_{p_i} (i = 1, 2)$ be the decomposition groups for p, p_i in $\mathrm{Gal}(L/\mathbb{Q})$ and let D_p, D_{p_i} be the fixed field of G_p, G_{p_i}, respectively. We note that $D_p \subset k(p_1)k(p_2), D_{p_1} \subset k(p_2)\mathbb{Q}_1$ and $D_{p_2} \subset k(p_1)\mathbb{Q}_1$.

Now, from our assumption $p \not| h_L$, we have $[D_p : \mathbb{Q}] = [D_{p_1} : \mathbb{Q}] = [D_{p_2} : \mathbb{Q}] = p$ and $D_p D_{p_1} D_{p_2} = L$ by Lemma 7. Here, we assume that either $(\frac{p}{p_1})_p = 1$ or $(\frac{p_1}{p_2})_p = 1$ or

$p_2 \equiv 1 \pmod{p^2}$ holds, and either $(\frac{p}{p_2})_p = 1$ or $(\frac{p_2}{p_1})_p = 1$ or $p_1 \equiv 1 \pmod{p^2}$. This is equivalent to

$$D_p = k(p_i) \text{ or } D_{p_i} = k(p_j) \text{ or } D_{p_j} = \mathbb{Q}_1 \text{ for } (i,j) = (1,2) \text{ and } (2,1), \tag{5}$$

because $[D_p : \mathbb{Q}] = [D_{p_1} : \mathbb{Q}] = [D_{p_2} : \mathbb{Q}] = p$.

If $D_p = k(p_1)$, then $D_{p_2} \neq k(p_1)$ because $D_p D_{p_1} D_{p_2} = L$. Hence by (5) (put $(i,j) = (2,1)$), we have $D_{p_1} = \mathbb{Q}_1$. Then $D_{p_2} \subseteq k(p_1)\mathbb{Q}_1 = D_p D_{p_1}$, which contradicts $D_p D_{p_1} D_{p_2} = L$. In the same way, if $D_p = k(p_2)$, then $D_{p_1} \neq k(p_2)$ and we have $D_{p_2} = \mathbb{Q}_1$ by (5), which contradicts. Thus, it follows that the assumption (5) cause contradiction. Therefore, for $(i,j) = (1,2)$ or $(2,1)$, $(\frac{p}{p_i})_p \neq 1, (\frac{p_i}{p_j})_p \neq 1$, and $p_j \not\equiv 1 \pmod{p^2}$.

Without loss of generality, we may assume $(i,j) = (1,2)$. Since $(\frac{p}{p_1})_p \neq 1$, p is inert in $k(p_1)$. Hence $\sigma = (\frac{k(p_1)/\mathbb{Q}}{p}) \neq 1$, where $(\frac{k(p_1)/\mathbb{Q}}{p})$ is the Artin symbol, and σ generates $\mathrm{Gal}(k(p_1)/\mathbb{Q})$: $<\sigma> = \mathrm{Gal}(k(p_1)/\mathbb{Q})$. We often regard $<\sigma> = \mathrm{Gal}(k(p_1)k(p_2)/k(p_2))$ or $\mathrm{Gal}(L/k(p_2)\mathbb{Q}_1)$ in the natural way. Similarly, we put $\tau = (\frac{k(p_2)/\mathbb{Q}}{p_1})$ and $\eta = (\frac{\mathbb{Q}_1/\mathbb{Q}}{p_2})$, then $<\tau> = \mathrm{Gal}(k(p_2)/\mathbb{Q})$ and $<\eta> = \mathrm{Gal}(\mathbb{Q}_1/\mathbb{Q})$.

Since $(\frac{p}{p_1})_p \neq 1$, there exists $x \in \mathbb{F}_p$ such that $(\frac{p_2 p^x}{p_1})_p = 1$. Then

$$\left(\frac{p_2 p^x}{p_1}\right)_p = 1 \Leftrightarrow \left(\frac{k(p_1)/\mathbb{Q}}{p_2 p^x}\right) = \left(\frac{k(p_1)/\mathbb{Q}}{p_2}\right)\left(\frac{k(p_1)/\mathbb{Q}}{p}\right)^x = 1.$$

Therefore $(\frac{k(p_1)/\mathbb{Q}}{p_2}) = \sigma^{-x}$. Similarly, we obtain $y, z \in \mathbb{F}_p$ such that $(\frac{p p_1^y}{p_2})_p = 1$ and $p_1 p_2^z \equiv 1 \pmod{p^2}$, and hence $(\frac{k(p_2)/\mathbb{Q}}{p}) = \tau^{-y}$ and $(\frac{\mathbb{Q}_1/\mathbb{Q}}{p_1}) = \eta^{-z}$.

Since $(\frac{k(p_1)k(p_2)/\mathbb{Q}}{p}) = (\frac{k(p_1)/\mathbb{Q}}{p})(\frac{k(p_2)/\mathbb{Q}}{p}) = \sigma\tau^{-y}$, D_p is the fix field of $<\sigma\tau^{-y}>$ in $k(p_1)k(p_2)$. Therefore, when we consider G_p in $\mathrm{Gal}(L/\mathbb{Q})$,

$$G_p = <\eta, \sigma\tau^{-y}> .$$

And similarly,

$$G_{p_1} = <\sigma, \tau\eta^{-z}>,$$

and

$$G_{p_2} = <\tau, \eta\sigma^{-x}>,$$

in $\mathrm{Gal}(L/\mathbb{Q})$.

By a direct computation, we have,

$$G_p \cap G_{p_1} = <\sigma\tau^{-y}\eta^{yz}> .$$

Hence,

$$\begin{aligned} G_p \cap G_{p_1} \cap G_{p_2} &= <\sigma\tau^{-y}\eta^{yz}> \cap <\tau, \eta\sigma^{-x}> \\ &= \begin{cases} \{1\} & \text{, if } xyz \neq -1, \\ <\sigma\tau^{-y}\eta^{yz}> & \text{, if } xyz = -1. \end{cases} \end{aligned}$$

But , our assumption $D_p D_{p_1} D_{p_2} = L$ implies $G_p \cap G_{p_1} \cap G_{p_2} = \{1\}$. Hence $xyz \neq -1$.

Conversely, we assume k satisfies the conditions of Theorem 1 in case of $t=2$. Since $k_1 = k_{1,G}$, it follows easily that $L = k(p_1)k(p_2)\mathbb{Q}_1$ is the maximal intermediate extension between $\mathbb{Q}$ and $k_n (n \geq 1)$ such that $\mathrm{Gal}(L/\mathbb{Q})$ is an elementary abelian p-group. Without loss of generality, we may assume $(i,j) = (1,2)$. Since $\mathrm{Gal}(k(p_1)k(p_2)/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^2$ and p is unramified in $k(p_1)k(p_2)$, p must decompose in $k(p_1)k(p_2)$. But the condition $(\frac{p}{p_1})_p \neq 1$ implies p is inert in $k(p_1) \subset k(p_1)k(p_2)$, hence we obtain $[D_p : \mathbb{Q}] = p$. Similarly, $(\frac{p_1}{p_2})_p \neq 1$ and $p_2 \not\equiv 1 \pmod{p^2}$ imply $[D_{p_1} : \mathbb{Q}] = [D_{p_2} : \mathbb{Q}] = p$. Therefore, as in the above computation of G_p, G_{p_i}, we have $D_p D_{p_1} D_{p_2} = L$, by $xyz \neq -1$. □

5. Remarks

The condition of Theorem 1 in [12] means $xyz = 0$ which is a special case of $xyz \neq -1$. Hence, our Corollary 2 contains some known results and there exist infinitely many fields satisfying the conditions of Theorem 1 (cf. [12]).

If $K = k(p_1)k(p_2)$ satisfies the conditions of Theorem 1, then $\lambda_p(k) = \mu_p(k) = 0$ for any field $k \subseteq K$ with $[k : \mathbb{Q}] = p$. This is a result of Fukuda [4]. He has shown this result using a technic of capitulation of ideal class group. The case $xyz = -1$ is a difficult case. But we can get some results:

Proposition 8. *Notations are as in section 3. Assume that* $(\frac{p}{p_1})_p \neq 1, (\frac{p_1}{p_2})_p \neq 1$, *and* $p_2 \not\equiv 1 \pmod{p^2}$. *Then* $\lambda_p(k) = \mu_p(k) = 0$ *for the decomposition field* k *of* p *in* $k(p_1)k(p_2)$.

Proof. We apply a result of [6]:

Lemma 9 ([6] **Corollary 3.6).** *Let* k *be a cyclic extension of* $\mathbb{Q}$ *of degree* p. *Then the following conditions are equivalent*:

(a) $\lambda_p(k) = \mu_p(k) = 0$,
(b) *For any prime ideal* w *of* k_∞ *which is prime to* p *and ramified in* $k_\infty/\mathbb{Q}_\infty$, *the order of the ideal class of* w *is prime to* p.

If $xyz \neq -1$ then we have $\lambda_p(k) = \mu_p(k) = 0$ by Corollary 2. So we only consider the case $xyz = -1$. In this case we have $k \neq k(p_i)$ $(i = 1, 2)$. It follows easily that $A(k)$, the p-part of the ideal class group of k, is cyclic of order p, and it is generated by products of primes of k above p. On the other hand, for $i = 1, 2$, the prime $\mathfrak{p}_i$ of k above p_i generates $A(k)$, and is inert in k_∞/k. Since the primes of k above p is principal for some k_n by the natural mapping $A(k) \to A(k_n)$ (cf. [7]), $\mathfrak{p}_i$ is principal in k_∞.

Since the primes ramified in $k_\infty/\mathbb{Q}_\infty$ are $\mathfrak{p}_1$ and $\mathfrak{p}_2$, which is principal in k_∞, we can apply Lemma 9 and obtain $\lambda_p(k) = \mu_p(k) = 0$. □

Recently, Fukuda verified Greenberg conjecture for various cubic cyclic fields k with $f_k = p_1 p_2$ and $p = 3$. He gives an example, which is the case $p_1 = 7$ and $p_2 = 223$. Note that there exist two such fields, and these p_1 and p_2 do not satisfy condition (3) in Theorem 1. He verified $\lambda_3 = \mu_3 = 0$ for one of such fields by using his result concerning with the unit group of k (cf. [5]).

When $t \geq 3$, i.e. at least 3 primes are ramified in $k/\mathbb{Q}$, there are a few results for Greenberg conjecture. In this case, the p-rank of $A(k)$ is greater than 2. Greenberg([7]) gave the following example, but the proof are omitted in his paper: $p = 3$ and k is an cubic cyclic field with conductor $7 \cdot 13 \cdot 19$ and 3 is inert in $k/\mathbb{Q}$. He mentioned that by "delicate" arguments one can show $\lambda_3(k) = \mu_3(k) = 0$. The author had a chance to contact Prof. Greenberg, and asked him about this example. He kindly taught the author the "delicate" arguments, which is a system to examine relations of the ideal class group of intermediate fields of $k\mathbb{Q}_1$. Applying his idea, we can show the following result:

Theorem 10 ([13]). *Let p be any odd prime. For any integer $0 \leq m \leq p-1$, there exist infinitely many cyclic extension fields k of $\mathbb{Q}$ with $[k : \mathbb{Q}] = p$ such that p-rank$A(k) = m$ and $\lambda_p(k) = \mu_p(k) = 0$.*

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