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Analogue of Poisson Distribution in Monotone Fock Space

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1 Introduction

The monotone Fock space was introduced by the author to construct a new example of noncommutative "de Moivre Laplace theorem" [1] and of noncommutative "Brownian motion" [2] in quantum probability theory (see [3] [4] for general reference to quantum probability). We note that the essentially same structure was introduced by Lu [5], independently from the author.

In this note, we investigate the "Poisson type" limit distribution for Bernoulli random variables $x_i = q_i + a\delta_i^\circ$ with $q_i = \delta_i^+ + \delta_i^-$ on the monotone Fock space Φ. We determine the probability measure ν of the limit distribution of the operator

$$\frac{q_1 + q_2 + \cdots + q_n}{\sqrt{n}} + c(\delta_1^\circ + \delta_2^\circ + \cdots + \delta_n^\circ) \qquad (n \to \infty)$$

in the vacuum state. Here δ_i^+, δ_i^-, δ_i° are the creation, annihilation, and conservation operators on the discrete monotone Fock space.

2 Monotone Fock Space

Let us give the precise definition of the monotone Fock space and related operators. The discrete monotone Fock space Φ is the Hilbert space direct sum $\Phi = \oplus_{r=0}^{\infty}\mathcal{H}_r$ of the r-particle spaces

$$\mathcal{H}_r = l^2({}_T\mathrm{M}_r) \qquad (r = 0, 1, 2, \cdots).$$

Here, $\mathcal{H}_r$ is the complex l^2-space on the set

$${}_T\mathrm{M}_r \;=\; \{\sigma |\; \sigma = (i_r > i_{r-1} > \cdots > i_2 > i_1);\; i_1, i_2, \cdots, i_r \in T\}$$

of all r-tuples $\sigma = (i_r > i_{r-1} > \cdots > i_2 > i_1)$ from the natural numbers $T = \mathrm{N} = \{1, 2, 3, \cdots\}$. Here $\sigma = (i_r > i_{r-1} > \cdots > i_2 > i_1)$ means the r-tuple $\sigma = (i_r, i_{r-1}, \cdots, i_2, i_1)$ with the property that the components are listed in the increasing order to the left, for example $\sigma = (5 > 3 > 2)$. Note that ${}_T\mathrm{M}_0 = \{\Lambda\}$ with the null sequence Λ, and hence $\mathcal{H}_0 \cong \mathrm{C}$.

Each r-particle space $\mathcal{H}_r$ has the natural complete orthonormal basis $\{e_\sigma\}_{\sigma \in {}_T\mathrm{M}_r}$ labelled with $\sigma \in {}_T\mathrm{M}_r$, where e_σ is defined by

$$e_\sigma(\tau) = \begin{cases} 1 & (\tau = \sigma), \\ 0 & (\tau \neq \sigma). \end{cases}$$

The unit vector e_Λ corresponding to the null sequence Λ is called the *vacuum vector* and denoted by Ω.

The discrete monotone Fock space has the three natural classes of operators δ^+, δ°, δ^-. The *creation operator* δ_i^+ $(i \in T)$ is defined by

$$\delta_i^+ e_{(j_r > \cdots > j_1)} = \begin{cases} e_{(i > j_r > \cdots > j_1)} & (\text{if } i > j_r), \\ 0 & (\text{otherwise}). \end{cases}$$

The *annihilation operator* δ_i^- $(i \in T)$ is defined by

$$\delta_i^- e_{(j_r > \cdots > j_1)} = \begin{cases} e_{(j_{r-1} > \cdots > j_1)} & (\text{if } r \geq 1 \text{ and } i = j_r), \\ 0 & (\text{otherwise}). \end{cases}$$

The *conservation operator* δ_i° $(i \in T)$ is defined by

$$\delta_i^\circ e_{(j_r > \cdots > j_1)} = \begin{cases} e_{(j_r > \cdots > j_1)} & (\text{if } r \geq 1 \text{ and } i = j_r), \\ 0 & (\text{otherwise}). \end{cases}$$

These operators $\delta_i^+, \delta_i^\circ, \delta_i^-$ are bounded operators, and δ_i^+ and δ_i^- are mutually adjoint: $(\delta_i^-)^* = \delta_i^+$.

3 Bernoulli Variables

Let us consider the operators x_i on Φ which can be interpreted as the Bernoulli random variables in the Posson limit theorem (= law of small numbers) of classical probability theory.

Let x_i $(i \in T)$ be an operator on Φ defined by

$$x_i = \delta_i^+ + \delta_i^- + a\delta_i^\circ.$$

The probability distribution of x_i under the vacuum state $\phi(\cdot) = \langle \Omega | \cdot \Omega \rangle$ is the two point distribution given by

$$p \cdot \varepsilon_{x_+} + q \cdot \varepsilon_{x_-}$$

with $p = \frac{1}{2} - \frac{a}{2\sqrt{4+a^2}}$, $q = \frac{1}{2} + \frac{a}{2\sqrt{4+a^2}}$ and $x_\pm = \frac{a}{2} \pm \sqrt{1 + \frac{a^2}{4}}$. Here ε_x denotes the Dirac measure at a point x. This is verified through the calculation of the moment generating function $f(s) = \sum_{p=0}^{\infty} m_p s^p$ for x_i, where m_p is the p-th moment $\phi(x_i^p)$ of x_i. The direct calculation shows

$$f(s) = \frac{1 - as}{1 - as - s^2}$$

and hence we get the above probability measure.

Furthermore we can show that the operators $\{x_i\}$ are independent under the vacuum state ϕ, in the sense of Kümmerer. So the operators $\{x_i\}$ can be viewed as quantum Bernoulli random variables.

4 Moments and Diagrams

We want to know the limit distribution ν of the operators

$$\frac{q_1 + q_2 + \cdots + q_n}{\sqrt{n}} + c(\delta_1^\circ + \delta_2^\circ + \cdots + \delta_n^\circ)$$

at $n \to \infty$ under the vacuum state ϕ, where q_i is given by $q_i = \delta_i^+ + \delta_i^-$. The scaling of this type is motivated by the Fock space interpretation of the classical Poisson process [4]. The limit distribution, if there exists, can be viewd as an analogue of Poisson distribution in the case of monotone Fock space.

To obtain the limit distribution ν, we adopt the moment method. Put

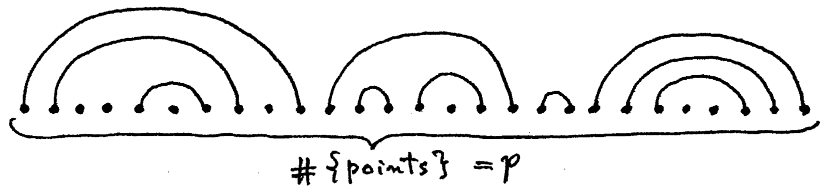
$$\begin{aligned} X_n &= x_1 + x_2 + \cdots + x_n \\ &= q_1 + q_2 + \cdots + q_n + (c\sqrt{n})(\delta_1^\circ + \delta_2^\circ + \cdots + \delta_n^\circ), \end{aligned}$$

where we put $a = c\sqrt{n}$ with some constant c. Let us take the limit of the p-th moments of $\frac{X_n}{\sqrt{n}}$:

$$m_p = \lim_{n\to\infty} \langle \left(\frac{X_n}{\sqrt{n}} \right)^p \rangle.$$

Here $\langle \cdot \rangle$ denotes the vacuum expectation $\phi(\cdot)$.

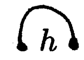
By the combinatorial argument, we can see that the limit m_p of the moments can be calculated by the combinatorial formula

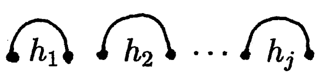
$m_p = \sum \langle$  $\rangle$

Here the summation of the values $\langle\, g \,\rangle$ is taken over all such admissible diagrams g as

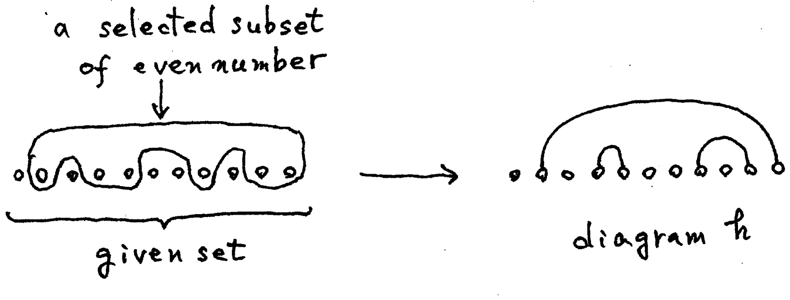
$g =$ 

Here we omit the formal definition of the *admissible diagram g*. But, in the pictorial language, it is the object g defined as follows.

(1) The diagram g consists of some connected components of the form  (in this case $\sharp$\{connected components\} $= j$).

$g =$ 

(2) The diagram h is defined by specifying the following two objects
   (a) a subset of even number from the given set of points in linear order,
   (b) a noncrossing pair partition of the selected set of even number.



The value $\langle\, g \,\rangle$ for an admissible diagram g is calculated by the following rule.

(a) $$\langle\, g \,\rangle = \frac{\langle\, h_1 \,\rangle}{\sharp\{\text{lines in } h_1\}+1}\,\frac{\langle\, h_2 \,\rangle}{\sharp\{\text{lines in } h_2\}+1}\cdots\frac{\langle\, h_j \,\rangle}{\sharp\{\text{lines in } h_j\}+1}$$
$$\left(\text{if}\quad g = \widehat{\bullet\, h_1 \bullet}\;\widehat{\bullet\, h_2 \bullet}\cdots\widehat{\bullet\, h_j \bullet}\right)$$

(b) $\langle\, h \,\rangle = c^{q-2k}\langle\, h' \,\rangle$
(if h splits into a noncrossing pair partition h' with $2k$ points and $q-2k$ singletons, where $q = \sharp\{$ points in $h\}$)

(c) For a noncrossing pair partition h',

(c1) $\langle\, h' \,\rangle = \langle\, h_1' \,\rangle\langle\, h_2' \,\rangle\cdots\langle\, h_j' \,\rangle$
(if h splits into the connected components $h_1', h_2', \cdots, h_l'$)

(c2) $$\langle\, h' \,\rangle = \frac{\langle\, h'' \,\rangle}{\sharp\{\text{lines in } h''+1\}} \qquad \left(\text{if } h' = \widehat{\bullet\, h'' \bullet}\right)$$

(c3) $\langle$ empty diagram $\rangle = 1.$

5 Moment Generating Function

Let us investigate the moment generating function
$$f(s) = \sum_{p=0}^{\infty} m_p s^p$$
for the moment sequence $m_p = \lim_{n\to\infty}\langle\left(\frac{X_n}{\sqrt{n}}\right)^p\rangle$. By the result of the previous section, the moment m_p can be expressed by

$$m_p = \sum_{g\,:\,\text{admissible}}\Big\langle \underbrace{\widehat{\bullet\, h_1 \bullet}\;\widehat{\bullet\, h_2 \bullet}\cdots\widehat{\bullet\, h_j \bullet}}_{\#\{\text{points}\}=p}\Big\rangle$$
$$= \sum_{j=1}^{[\frac{p}{2}]}\ \sum_{\substack{p_1+\cdots+p_j=p\\ p_1\ge 2,\cdots,p_j\ge 2}}\ \sum_{h_1,h_2,\cdots,h_j}\langle \underbrace{\widehat{\bullet\, h_1 \bullet}}_{p_1}\rangle\langle \underbrace{\widehat{\bullet\, h_2 \bullet}}_{p_2}\rangle\cdots\langle \underbrace{\widehat{\bullet\, h_j \bullet}}_{p_j}\rangle.$$

Hence the moment generating function $f(s)$ is given by
$$\begin{aligned} f(s) &= m_0 s^0 + m_1 s^1 \\ &\quad + \sum_{p=2}^{\infty}\left\{\sum_{j=1}^{[\frac{p}{2}]}\ \sum_{\substack{p_1+\cdots+p_j=p\\ p_1\ge 2,\cdots,p_j\ge 2}}\ \sum_{h_1,h_2,\cdots,h_j}\langle \widehat{\bullet\, h_1 \bullet}\rangle\langle \widehat{\bullet\, h_2 \bullet}\rangle\cdots\langle \widehat{\bullet\, h_j \bullet}\rangle\right\} s^p \\ &= 1 + \sum_{p=2}^{\infty}\left\{\sum_{j=1}^{[\frac{p}{2}]}\ \sum_{\substack{p_1+\cdots+p_j=p\\ p_1\ge 2,\cdots,p_j\ge 2}}\left(\sum_{h_1}\langle \widehat{\bullet\, h_1 \bullet}\rangle s^{p_1}\right)\cdots\left(\sum_{h_j}\langle \widehat{\bullet\, h_j \bullet}\rangle s^{p_j}\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&= 1+\sum_{j=1}^{\infty}\left\{\sum_{p_1=2}^{\infty}\left(\sum_{h_1}\langle \overset{\frown}{\bullet h_1 \bullet} \rangle\right) s^{p_1}\right\}^{j} \\
&= \frac{1}{1-g(s)}.
\end{aligned}
$$

Here the function $g(s)$ above is defined by

$$
g(s)=\sum_{p=2}^{\infty}\left(\sum_{h}\langle \overset{\frown}{\bullet h \bullet} \rangle\right) s^{p}.
$$

Now, let us calculate the function $g(s)$.

$$
\begin{aligned}
g(s) &= \sum_{p=2}^{\infty}\left(\sum_{h}\langle \overset{\frown}{\bullet h \bullet} \rangle\right) s^{p} \\
&= \sum_{p=2}^{\infty}\left\{\sum_{k=0}^{[\frac{p-2}{2}]}\binom{p-2}{2k}\frac{1}{k+1}c^{p-2-2k}a_{2k}\right\} s^{p} \\
&= s^{2}\sum_{q=0}^{\infty}\left\{\sum_{k=0}^{[\frac{q}{2}]}\binom{q}{2k}\frac{a_{2k}}{k+1}c^{q-2k}\right\} s^{q}.
\end{aligned}
$$

Here $a_{2k}=\frac{1}{2^k}\binom{2k}{k}$ is the $2k$-th moment of the standard arcsine law [1]. Using the formula

$$
\frac{a_{2k}}{k+1}=\frac{w_{2k}}{2^k}.
$$

between the arcsine moments $\{a_{2k}\}$ and the semicircular moments $\{w_{2k}\}$, we can rewrite the function $g(s)$ as

$$
g(s) = s^{2}\sum_{q=0}^{\infty}\left\{\sum_{k=0}^{[\frac{q}{2}]}\binom{q}{2k}\frac{w_{2k}}{2^k}c^{q-2k}\right\} s^{q}.
$$

By the way, the moment generating function $f_{free}(s)$ for the free Poisson distribution [6] is given by $f_{free}(s)=\frac{1}{1-g_{free}(s)}$ with

$$
\begin{aligned}
g_{free}(s) &= \sum_{p=2}^{\infty}\left(\sum_{h}\langle \overset{\frown}{\bullet h \bullet} \rangle'\right) s^{p} \\
&= s^{2}\sum_{q=0}^{\infty}\left\{\sum_{k=0}^{[\frac{q}{2}]}\binom{q}{2k}w_{2k}c^{q-2k}\right\} s^{q}.
\end{aligned}
$$

Here the calculation of $\langle \overset{\frown}{\bullet h \bullet} \rangle'$ is done based on the vacuum expectation $\langle \cdot \rangle'$ on the full Fock space. If we rewrite the moment generating function $g(s)$ as the form

$$
g(s)=2\left(\frac{s}{\sqrt{2}}\right)^{2}\sum_{q=0}^{\infty}\left\{\sum_{k=0}^{[\frac{q}{2}]}\binom{q}{2k}w_{2k}(\sqrt{2}c)^{q-2k}\right\}\left(\frac{s}{\sqrt{2}}\right)^{q},
$$

we can recognize the relation between $g(s)$ and $g_{free}(s) = g_{free}(s; c)$:

$$g(s) \;=\; 2\, g_{free}\left(\frac{s}{\sqrt{2}}; \sqrt{2}c\right).$$

By the way, the function $g_{free}(s)$ is calculated as follows.

$$\begin{aligned}
g_{free}(s) &= \sum_{p=2}^{\infty}\left(\sum_{h}\langle \widehat{h} \rangle'\right) s^p \\
&= s^2 \sum_{q=0}^{\infty}\left\{\sum_{k=0}^{[\frac{q}{2}]}\binom{q}{2k} w_{2k} c^{q-2k}\right\} s^q \\
&= s^2 \sum_{k=0}^{\infty} w_{2k} c^{-2k} \sum_{q=2k}^{\infty}\binom{q}{2k}(cs)^q \\
&= \frac{s^2}{1-cs}\sum_{k=0}^{\infty} w_{2k}\left(\frac{s}{1-cs}\right)^{2k} \\
&= \frac{(1-cs) - \sqrt{(1-cs)^2 - 4s^2}}{2}.
\end{aligned}$$

Here, in the last two equalites, we used two formulas of generating functions [7]:

$$\begin{cases}
\dfrac{z^n}{(1-z)^{n+1}} = \displaystyle\sum_{k=0}^{\infty}\binom{k}{n} z^k, \\[2ex]
\displaystyle\sum_{k=0}^{\infty} w_{2k}\, t^k = \dfrac{1-\sqrt{1-4t}}{2t}.
\end{cases}$$

By the relation $g(s) = 2\, g_{free}\left(\frac{s}{\sqrt{2}}; \sqrt{2}c\right)$, we obtain the explicit form of $g(s)$:

$$g(s) \;=\; (1-cs) - \sqrt{(1-cs)^2 - 2s^2}.$$

Hence we finally get the explicit form of the generating function $f(s) = \frac{1}{1-g(s)}$ for the moment sequence $\{m_p\}$:

$$f(s) = \frac{1}{cs + \sqrt{(1-cs)^2 - 2s^2}}.$$

6 Density

The probability measure ν, associated to the the limit process

$$\frac{q_1 + q_2 + \cdots + q_n}{\sqrt{n}} + c(\delta_1^\circ + \delta_2^\circ + \cdots + \delta_n^\circ) \qquad\qquad (n \to \infty)$$

under the vacuum state ϕ, is an analogue of Poisson distribution in the case of monotone Fock space. This limit measure (= monotonic "Poisson distribution") can be explicitely determined through the Cauchy transform [8] of the generating function $f(s)$, which is already calculated in the previous section.

Monotonic "Poisson distribution" ν is given by

$$\nu = p \cdot \lambda + A\varepsilon_{c+\sqrt{2+c^2}} + B\varepsilon_{c-\sqrt{2+c^2}},$$

with its density of the absolutely continuous part

$$p(x) = \frac{1}{\pi} \frac{\sqrt{2-(x-c)^2}}{c^2+2-(x-c)^2} \qquad (c-\sqrt{2} < x < c+\sqrt{2}).$$

Here, the density $p(x)$ satisfies

$$\int_{c-\sqrt{2}}^{c+\sqrt{2}} p(x)dx = 1 - \frac{|c|}{\sqrt{2+c^2}},$$

λ is the Lebesgue measure on the real line, ε_x denotes the Dirac mesure at a point x, and A and B are the normalization constants.

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