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On a theta integral

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1 Introduction.

Around the begining of 70's,Doi-Naganuma and Shimura discovered correspondences between certain spaces of modular forms,being compatible with the Hecke operators[Sh,DN].Shimura's correspondence was investigated by Shintani and Niwa, using the Weil representations[Sn,Nw].

The construction of holomorphic cusp forms by Niwa was generalized by many authers [Kd,Za,Od,RS,Kj],namely Oda and Rallis-Shiffman independently considered the case of orthogonal groups $O(2, N)$ for general N.(Zagier constructed the holomorphic kernel function for Doi-Naganuma's correspondence.)

Recently, Borcherds discoverd a family of automorphic forms with infinite product on orthogonal groups of signature $(2, N)$. At the same time, Physicists [AFGNT,HM] started to study and developed a variant of theta correspondence from another direction. Especially, Harvey and Moore found certain theta integral express the automorphic form of Borcherds[HM,Bo2].

The purpose of this note is to give a construction of meromorphic automorphic forms on orthogonal groups of the signature $(2, N)$,$N \geq 2$,using the same kind of theta correspondence ,(in section 4,) without proofs. This generalizes a result of Antoniadis,Ferrara,Gava,Narain and Taylor [AFGNT],which dealed with the case $N = 1, 2$.

We note that,in the classical(positive weight) case,Maass' lifting [Ma,Gr,Su] and theta correspondence are known to be coincide up to non-zero scalar multiplication (,at least S is maximal even, as far as I know.) In the negative weight case,the situation is different.

The following facts for $SL_2(\mathbf{R})$ are well known ($k \in \mathbf{Z}, k \geq 2, D_\tau = \frac{\partial}{\partial\tau}$,and $\mathcal{H}_1$ is the upper half plane):

(1.1)for $f \in C^\infty(\mathcal{H}_1)$ and $g \in SL_2(\mathbf{R})$,

$$D_\tau^{k-1}(f|_{2-k}g) = (D_\tau^{k-1}f)|_k g.$$

(1.2)A holomorphic function f on $\mathcal{H}_1$ satisfies $D_\tau^{k-1}f = 0$ if and only if f is a polynomial in τ of degree at most $k-2$.

(1.3)For a holomorphic function F on $\mathcal{H}_1$,set

$$f=\int_{\tau_0}^{\tau} F(\tau')\,\frac{(\tau-\tau')^{k-2}}{(k-2)!}\,d\tau'.$$

Then it satisfies $D_\tau^{k-1}f=F$.

In section 3, we ,will find analogous statements in the case of orthogonal groups $O(2,N)$.Then ,by integrating the form constructed in section 4,one can obtain a function, which behaves like an automorphic form of negative weight,but is multi-valued, and analytic function with logarithmic singularities.This form is obtained from Maass' lifting. The same construction for $N=2$ already appears in [AFGNT](,See also [FS]).Note that ,[HM] and [Kw] also conciderd the case $N>2$,but take slightly different construction .

2 Basic definitions.

Let $N\geq 2$, and S be an even integral symmetric matrix of degree $N+2$ with signature $(2,N)$,and of the following form:

$$S=\begin{pmatrix}0 & 0 & 1\\ 0 & S_0 & 0\\ 1 & 0 & 0\end{pmatrix}. \tag{1}$$

The real orthogomal group $O(S)$ of S acts on the domain $H^*=\{z\in \mathbf{C}^N|\ \ \eta(z):=2^t Imz S_0 Imz>0\}$ as follows:If we set

$$p(z)={}^t\left(-{}^tzS_0z/2\quad {}^tz\quad 1\right)\qquad (z\in H^*), \tag{2}$$

then for any $g\in O(S)$ and $z\in H^*$,there exists unique $gz\in H^*$ and $\mu(g,z)\in \mathbf{C}^\times$ satisfying

$$g\ p(z)=p(gz)\mu(g,z).$$

Take one of the two connected components $H\subset H^*$, and we denote by $O(S)^+$ the set of the elements in $O(S)$, fixing H.Then $O(S)^+$ holomorphicaly acts on H. The action is transitive and the stabilizer of a point is isomorphic to $SO(2)\times O(N)$.

Write $D_z={}^t\left(\frac{\partial}{\partial z_1}\ ,\cdots,\ \frac{\partial}{\partial z_N}\right)$ for $z={}^t\left(z_1\ ,\cdots,\ z_N\right)\in H$. Following Shimura[Sh1],we define

$$q(z)={}^t(\eta(z)D_z(\eta(z)^{-1}\,{}^tp(z))).\qquad (z\in H) \tag{3}$$

Then for any $g \in O(S)^+$ and $z \in H$,there exists $\lambda(g,z) \in O(S_0)_{\mathbf{C}}$ which satisfies the following:

$$g\ q(z) = q(gz)\lambda(g,z), \tag{4}$$

$$D_z(f \circ g) = {}^t\lambda(g,z)((D_z f)\circ g)\mu(g,z)^{-1}$$
$$(f \in C^\infty(H), g \in O(S)^+, z \in H), \tag{5}$$

and we find that, $\mu(g,z)$ and $\lambda(g,z)$ are the holomorphic automorphic factor.

3 Differential calculus.

Set $V = \mathbf{C}^N$ and $V_0 = \{a \in \mathbf{C}^N |\ {}^t a S_0 a = 0\}$. The symmetric algebra $S(V) = \bigoplus_{l=0}^{\infty} S(V)_l$ of V possess a bilinear form $<,>$ satisfying $< a^l, b^l >= ({}^t ab)^l$. We denote by $H(V)_l$ the subspace in $S(V)_l$ generated by a^l $(a \in V_0)$ over $\mathbf{C}$.Then the representation of $O(S_0)_{\mathbf{C}}$ on $H(V)_l$ is irreducible,equivalent to the one on the space of harmonic polynomial of degree l ,under the isomorphism between $S(V)$ and the ring of polynomial map on V. The latter is identified with the symmetric algebra $S(V^*)$ of the dual space of V.We have an isomorphism

$$H(V)_l = S(V)_l / Q\ S(V)_{l-2} \quad (l \geq 0),$$

where Q is the element in $S(V)_2$ corresponding to ${}^t x S_0 x \in S(V^*)_2$. We denote by π_l the projection of $S(V)_l$ on $H(V)_l$.

Proposition3.1. Assume $r > 0$.

(1)For $f \in C^\infty(H)$, $g \in O(S)^+$ and $a \in V_0$,

$$< a^r, D_z^r(\mu(g,z)^{r-1} f(gz)) >= \mu(g,z)^{-1} < \lambda(g,z)a^r, (D_z^r f)(gz) > \tag{6}$$

(2) A holomorphic function $f \in C^\infty(H)$ satisfies $({}^t a D_z)^r f(z) = 0$ for all $a \in V_0$,if and only if f is a polynomial in $z_1, \cdots, z_N$, ${}^t z S_0 z$ of degree at most $r-1$ (as a polynomial in $N+1$ variable.)

(3)Suppose $\Phi : H \to Hom_{\mathbf{C}}(H(V)_r \bigoplus_{l=0}^{r} S(V)_l, \mathbf{C})$ is a holomorphic function satisfying the following condition:

$${}^t b D_z \Phi(z,h) = \Phi(z,bh) \quad (b \in V_0, h \in H(V)_r \bigoplus_{l=0}^{r-1} S(V)_l). \tag{7}$$

Then there exists a holomorphic function f on H,such that

$$({}^t a D_z)^r f(z) = \Phi(z, a^r) \quad (a \in V_0). \tag{8}$$

In fact,if we set

$$\omega(z,z') = \sum_{l=0}^{r-1} b_l \frac{(Q/2\ {}^t(z-z')S_0(z-z')/2)^l}{l!} \pi_{r-l}\left(\frac{(z-z')^{r-l-1}}{(r-l-1)!} dz'\right), \tag{9}$$

$$b_l = \prod_{j=1}^{l} (r-j+N/2-1)^{-1} \quad (r > l \geq 0) \tag{10}$$

then ω satisfies $d\omega = -dz' \wedge \omega$, $\Phi(z', \omega)$ is a closed 1-form ,and the function

$$f(z) = \int_{z_0}^{z} \Phi(z', \omega(z,z')). \tag{11}$$

hold the required properties.

4 The construction of meromorphic automorphic forms.

Hereafter, we assume that S is unimodular, for simplicity (,so it follows $N \equiv 2 \pmod 8$).Set $M = \mathbf{Z}^{N+2} = {}^t\left(\mathbf{Z} \quad L \quad \mathbf{Z}\right)$, and $O(M)^+ = O(S)^+ \bigcap GL_{N+2}(\mathbf{Z})$.

Define the Siegel theta function by

$$\theta_M(\tau, z, a^r) = \sum_{\lambda \in M} {}^t\lambda S \frac{p(z)}{\eta(z)} ({}^t\lambda S \overline{q(z)} a)^r \mathbf{e}\left(\overline{\tau}\ {}^t\lambda S \lambda/2 + 2iy \frac{|{}^t\lambda S p(z)|^2}{\eta(z)}\right) \tag{12}$$

for $\tau = x + iy \in \mathcal{H}_1, z \in H$,and $a \in V_0$ $(r > 0)$. Here, we denote by $\mathcal{H}_1$ the upper half plane.

Let $C(\tau)$ be a modular form of weight $k = 2 - r - N/2$ for $SL_2(\mathbf{Z})$, holomorphic on $\mathcal{H}_1$,and meromorphic at the cusp $i\infty$. The Fourier expansion at $i\infty$ is given by

$$C(\tau) = \sum_{n \in \mathbf{Z}, n \geq -N_0} c(n) \mathbf{e}(n\tau)$$

for some $c(n) \in \mathbf{C}$,and $N_0 \geq 0$. We set $F = \{\tau \in \mathcal{H}_1 |\ |\tau| \geq 1, |Re(\tau)| \leq 1/2\}$,and $F_w = \{\tau \in F|\ Im(\tau) \geq w\}\ (w \geq 1)$.

Further we assume that $r > 0$ is odd, and $c(0) = 0$ in case $r = 1$.

Theorem 4.1.

1.The integral

$$\Phi(z, a^r) = \lim_{w\to\infty} \int_{F_w} \frac{dx\, dy}{y^2}\, y^2 C(\tau)\overline{\theta_M(\tau, z, \overline{a}^{\,r})} \quad (z \in H, a \in V_0) \tag{13}$$

converges outside the quadratic divisors, and defines a meromorphic function on H,satisfying

$$\Phi(z, a^r) = \mu(g, z)^{-1}\Phi(gz, \lambda(g, z)a^r) \tag{14}$$

for all $g \in O(M)^+$.

2.For any compactly supported open set U in H,the singularities of Φ on U are given by the finite sum

$$\frac{1}{4\pi} \sum_{\lambda\in M, {}^t\lambda S\lambda<0, U\cap H_\lambda\neq\emptyset} c({}^t\lambda S\lambda/2)\ ({}^t a D_z^t\lambda Sp(z))^r/\ {}^t\lambda Sp(z) \tag{15}$$

where $H_\lambda = \{z \in H|\ {}^t\lambda Sp(z) = 0\}$.(Precise meaning of the word "singularity" is that,the difference of two functions is extended to the C^∞ function on U.) Note that the inner expression of the sum can be rewritten as

$$({}^t aD_z)^r \left\{ \frac{({}^t\lambda Sp(z))^{r-1}}{(r-1)!} \log({}^t\lambda Sp(z)) \right\}.$$

3.Set $C = H \cap \mathbf{R}^N$,and take a connected component W_L of

$$C - \bigcup_{\mu\in L, {}^t\mu S_0\mu<0, c({}^t\mu S_0\mu/2)\neq 0} \{v \in C|\ {}^t\mu S_0 v = 0\}.$$

The Fourier expansion of Φ is given by

$$\Phi = -i\left(\frac{{}^t aD_z}{2\pi i}\right)^r \left\{ h(z) + \sum_{\mu\in L, {}^t\mu S_0 W_L>0} c({}^t\mu S_0\mu/2) \sum_{n>0} n^{-r}\mathbf{e}(n\,{}^t\mu S_0 z) \right\} \tag{16}$$

for sufficiently large $\eta(z) > 0, Imz \in W_L$, and a harmonic polynomial $h(z)$ of degree r.

Example 4.2.[E,AFGNT]: Let $N = 2, S_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, r = 3, k = 2 - 3 - 2/2 = -2, C(\tau) = E_{10}(\tau)/\Delta(\tau)$. Then

$$\Phi\left(\begin{pmatrix} \sigma \\ \tau \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}^3\right) = i\frac{j'(\sigma)}{j(\sigma) - j(\tau)} \frac{C(\tau)}{C(\sigma)}. \tag{17}$$

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