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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO SINGULARLY PERTURBED ODE OF SINE-GORDON TYPE

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1. Introduction and result. This paper is concerned with the perturbed elliptic Sine-Gordon equation on an interval

$$\begin{aligned} -u''(t)+\lambda\sin u(t) &= \mu f(u(t)), \quad t\in I:=(-T,T),\\ u(t) &> 0, \quad t\in I, \\ u(\pm T) &= 0, \end{aligned} \tag{1.1}$$

where $\lambda,\mu>0$ are parameters and $T>0$ is a constant. The following assumptions (A.1)-(A.4) are imposed on f:

(A.1) f is locally Lipschitz continuous, odd in u. Furthermore, $f(u)>0$ for $u>0$.

(A.2) There exist constants $C_1>0$ and $p>1$ such that for $u\in \mathbf{R}$

$$|f(u)|\le C_1(1+|u|^p). \tag{1.2}$$

(A.3) $f(u)=o(u)$ for $0<u\ll 1$.

(A.4) There exists a constant $1<m\le p+1$ such that for $u\in\mathbf{R}$

$$f(u)u\ge mF(u), \tag{1.3}$$

where $F(u)=\int_0^u f(s)ds$. The equation (1.1) is motivated by the perturbed Sine-Gordon equation

$$u_{tt}=u_{xx}-\sin u+f(u), \qquad 0<x<\pi, \tag{1.4}$$

which was recently studied by Bobenko and Kuksin [B-K]. They studied small amplitude solutions of nonlinear Klein-Gordon equation which was regarded as a perturbation of (1.4).

In this paper, regarding $\lambda > 0$ as a given parameter, and using a variational method, we study the asymptotic behavior of the solutions u_λ of (1.1) with prescribed energy as $\lambda \to \infty$. More precisely, we determine the location of interior transition layers of u_λ as $\lambda \to \infty$. We note that the solutions u_λ considered here are not small amplitude solutions.

We explain the variational framework. In order to obtain solution trios $(\lambda, \mu, u) \in \mathbf{R}_+^2 \times C^2(\bar{I})$ of (1.1), we apply the variational problem (M) subject to the constraint depending on λ:

(M) Minimize

$$L(u) := \frac{1}{2}\int_I |u'(t)|^2 dt + \lambda \int_I (1 - \cos u(t))dt$$

under the constraint

$$u \in M_\alpha := \left\{ u \in H_0^1(I) : K(u) := \int_I F(u(x))dx = 2TF(\alpha) \right\},$$

where $\alpha > 0$ is a *fixed constant*, $H_0^1(I)$ is the usual real Sobolev space. Then by the Lagrange multiplier theorem, we obtain solution trios $(\lambda, \mu(\lambda), u_\lambda) \in \mathbf{R}_+^2 \times M_\alpha$ of (1.1) (and consequently $u_\lambda \in C^2(\bar{I})$ by a standard regularity theorem) corresponding to the problem (M).

Now we state our theorem.

Theorem. *Assume (A.1)–(A.4). Suppose that* $2F(\alpha) < F(2\pi)$. *Then:*

(1) $\|u_\lambda\|_\infty < 2\pi$ *for* $\lambda \gg 1$.

(2) $u_\lambda \to 2\pi$ *uniformly on any compact interval in* $(-t_\alpha, t_\alpha)$ *as* $\lambda \to \infty$, *where* $t_\alpha := \frac{F(\alpha)}{F(2\pi)}T$.

(3) $u_\lambda \to 0$ *uniformly on any compact interval in* $I \setminus [-t_\alpha, t_\alpha]$ *as* $\lambda \to \infty$.

(4) $\mu(\lambda) \to 0$ *as* $\lambda \to \infty$.

2. Proof of Theorem. We know that a solution u of (1.1) satisfies the following properties:

(2.1) u is symmetric with respect to the origin, that is, $u(t) = u(-t)$ for $t \in [0, T]$.

(2.2) $u'(t) < 0$ for $t \in (0, T]$.

(2.3) $u'(0) = 0, \quad u(0) = \|u\|_\infty$.

Lemma 2.1. *Assume that* $(\lambda, \mu, u) \in \mathbf{R}_+ \times \mathbf{R} \times C^2(\bar{I})$ *satisfies (1.1). Then:*

(1) *For* $t \in \bar{I}$,

$$\frac{1}{2}u'(t)^2 + \mu F(u(t)) + \lambda \cos u(t) = \frac{1}{2}u'(T)^2 + \lambda = \mu F(\|u\|_\infty) + \lambda \cos \|u\|_\infty. \tag{2.4}$$

(2) $\mu > 0$.

Proof. (1) Multiply the equation in (1.1) by $u'(t)$. Then for $t \in \bar{I}$, we obtain

$$\{u''(t) + \mu f(u(t)) - \lambda \sin u(t)\}u'(t) = 0.$$

This implies that for $t \in \bar{I}$,

$$\frac{d}{dt}\left\{\frac{1}{2}u'(t)^2 + \mu F(u(t)) + \lambda \cos u(t)\right\} = 0. \tag{2.5}$$

Hence, for $t \in \bar{I}$,

$$\frac{1}{2}u'(t)^2 + \mu F(u(t)) + \lambda \cos u(t) \equiv \text{constant}. \tag{2.6}$$

By putting $t = 0, T$ in (2.6), we obtain (2.4) by (2.3).

(2) By (2.4), we obtain

$$\mu F(\|u\|_\infty) = \frac{1}{2}u'(T)^2 + \lambda(1 - \cos \|u\|_\infty) > 0. \tag{2.7}$$

Since $F(\|u\|_\infty) > 0$ by (A.1), $\mu > 0$ follows from (2.7). $\square$

Let $\beta(\lambda) := \inf_{u \in M_\alpha} L(u) \geq 0$. By a standard argument of Lagrange multiplier theorem, we can prove the following lemma.

Lemma 2.2. *For a fixed* $\lambda > 0$, *there exists* $(\mu(\lambda), u_\lambda) \in \mathbf{R}_+ \times (M_\alpha \bigcap C^2(\bar{I}))$ *which satisfies (1.1) and* $L(u_\lambda) = \beta(\lambda)$.

Lemma 2.3. *For* $\lambda \gg 1$,

$$L(u_\lambda) \le C\lambda^{\frac{m+2}{2(m+1)}}. \tag{2.8}$$

Proof. We put

$$w_\lambda(t) = \begin{cases} -\lambda^{1/2}|t| + \lambda^{1/(2(m+1))}, & 0 \le |t| \le \lambda^{-m/(2(m+1))}, \\ 0, & \lambda^{-m/(2(m+1))} < |t| \le T. \end{cases}$$

For a fixed $\lambda > 0$, there exists $c_\lambda > 0$ such that $V_\lambda := c_\lambda w_\lambda \in M_\alpha$. Then by direct calculation, we easily see that $c_\lambda \le C$ for $\lambda \gg 1$. Furthermore,

$$\|V_\lambda'\|_2^2 = 2c_\lambda^2 \int_0^{\lambda^{-m/(2(m+1))}} \lambda dt = 2c_\lambda^2 \lambda^{\frac{m+2}{2(m+1)}},$$

$$\lambda \int_I (1 - \cos V_\lambda(t))dt \le 2\lambda \int_0^{\lambda^{-m/(2(m+1))}} 2dt \le 4\lambda^{\frac{m+2}{2(m+1)}}.$$

By this, we obtain

$$\beta(\lambda) = L(u_\lambda) \le L(V_\lambda) \le C\lambda^{\frac{m+2}{2(m+1)}}.$$

This implies (2.8). □

Lemma 2.4. $\mu(\lambda) = o(\lambda)$ *for* $\lambda \gg 1$.

Proof. By Lemma 2.3, we have

$$\|u_\lambda'\|_2^2 \le C\lambda^{(m+2)/(2(m+1))}, \tag{2.10}$$

$$\int_I (1 - \cos u_\lambda(t))dt \le C\lambda^{-m/(2(m+1))}. \tag{2.11}$$

Multiply the equation in (1.1) by u_λ and integrate it over I. Then by (1.3), we obtain

$$\begin{aligned} 2mTF(\alpha)\mu(\lambda) &= \mu(\lambda)\int_I mF(u_\lambda(t))dt \le \mu(\lambda)\int_I f(u_\lambda(t))u_\lambda(t)dt \\ &= \|u_\lambda'\|_2^2 + \lambda\int_I u_\lambda(t)\sin u_\lambda(t)dt. \end{aligned} \tag{2.12}$$

We estimate $\int_I u_\lambda(t)\sin u_\lambda(t)dt$. By (2.9), we have

$$\int_I u_\lambda(t)^m dt \le \frac{1}{C_2}\left\{\int_I F(u_\lambda(t))dt + 2C_3T\right\} = C_4^m := \frac{1}{C_2}(2TF(\alpha) + 2C_3T).$$

By this and Hölder's inequality, we obtain

$$\begin{aligned}\left|\int_I u_\lambda(t)\sin u_\lambda(t)dt\right| &\le \left(\int_I |\sin u_\lambda(t)|^q dt\right)^{1/q}\left(\int_I u_\lambda(t)^m dt\right)^{1/m} \\ &= C_4\left(\int_I |\sin u_\lambda(t)|^q dt\right)^{1/q},\end{aligned} \tag{2.13}$$

where $1/q+1/m=1$. Let $0<\epsilon\ll 1$ be fixed. Then

$$2TF(\alpha)=2\int_0^T F(u_\lambda(t))dt \ge 2\int_0^{\epsilon/2} F(u_\lambda(t))dt \ge \epsilon F(u_\lambda(\epsilon/2)). \tag{2.14}$$

By (2.9) and (2.14), we see that $u_\lambda(\epsilon/2)\le C_\epsilon$ for $\lambda \gg 1$. We choose $k_\epsilon \in \mathbf{N}$ such that $C_\epsilon < 2k_\epsilon\pi$. For $0<\delta\ll 1$ and $k\in\mathbf{N}$, we put

$$J_{\lambda,k,\delta} := \{t\in I : 2(k-1)\pi+\delta < u_\lambda(t) < 2k\pi-\delta\}.$$

Then by (2.11), we obtain that as $\lambda\to\infty$,

$$|J_{\lambda,k,\delta}| \le \frac{1}{1-\cos\delta}\int_{J_{\lambda,k,\delta}}(1-\cos u_\lambda(t))dt \to 0. \tag{2.15}$$

We choose $\delta>0$ so small that $|\sin u_\lambda(t)|<\epsilon/2$ for $t\in(\epsilon/2,T)\setminus(\sum_{k=1}^{k_\epsilon} J_{\lambda,k,\delta})$. Then for $\lambda\gg 1$, by (2.15), we obtain

$$\begin{aligned}\int_I |\sin u_\lambda(t)|^q dt \le 2\int_0^T |\sin u_\lambda(t)|dt &= 2\int_0^{\epsilon/2} |\sin u_\lambda(t)|dt \\ &+2\int_{(\epsilon/2,T)\setminus(\sum_{k=1}^{k_\epsilon} J_{\lambda,k,\delta})} |\sin u_\lambda(t)|dt \\ &+2\int_{(\epsilon/2,T)\bigcap(\sum_{k=1}^{k_\epsilon} J_{\lambda,k,\delta})} |\sin u_\lambda(t)|dt \\ &\le \epsilon + T\epsilon + |\sum_{k=1}^{k_\epsilon} J_{\lambda,k,\delta}| < C\epsilon.\end{aligned}$$

This along with (2.13) implies that

$$\left|\int_I u_\lambda(t)\sin u_\lambda(t)dt\right| \to 0 \tag{2.16}$$

as $\lambda\to\infty$. By (2.10), (2.12), and (2.16), we obtain our conclusion. □

For $0\le r\le \|u_\lambda\|_\infty$, let $t_{\lambda,r}\in[0,T]$ satisfy $u_\lambda(t_{\lambda,r})=r$, which exists uniquely by (2.2). Since $u_\lambda\in M_\alpha$, we see that $u_\lambda(0)=\|u_\lambda\|_\infty\ge\alpha$. Therefore, there exists a unique $t_{\lambda,\epsilon}\in[0,T]$ for $0<\epsilon\ll 1$.

Lemma 2.5. $|u'_\lambda(T)| \to 0$ *as* $\lambda \to \infty$.

Proof. We assume that there exists $\delta_1 > 0$ and a subsequence of $\{\lambda\}$ such that $|u'_\lambda(T)| \ge \delta_1$ and derive a contradiction. We know that if $0 < \epsilon \ll 1$, then

$$(1-\epsilon)u \le \sin u \qquad \text{for } 0 \le u \le \epsilon. \tag{2.17}$$

For $t \in [t_{\lambda,\epsilon}, T]$ and $\lambda \gg 1$, by (A.3) and Lemma 2.4, we obtain

$$\frac{f(u_\lambda(t))}{u_\lambda(t)} \le \frac{(1-\epsilon)\lambda}{\mu(\lambda)}. \tag{2.18}$$

Then for $t \in [t_{\lambda,\epsilon}, T]$ and $\lambda \gg 1$, by (1.1), (2.17) and (2.18), we obtain

$$\begin{aligned} -u''_\lambda(t) &= \mu(\lambda)f(u_\lambda(t)) - \lambda \sin u_\lambda(t) \\ &\le \mu(\lambda)f(u_\lambda(t)) - (1-\epsilon)\lambda u_\lambda(t) \\ &= u_\lambda(t)\left(\mu(\lambda)\frac{f(u_\lambda(t))}{u_\lambda(t)} - (1-\epsilon)\lambda\right) \le 0. \end{aligned} \tag{2.19}$$

We show that $t_{\lambda,\epsilon} \to T$ as $\lambda \to \infty$. To do this, we assume that there exists a subsequence of $\{\lambda\}$ and $C_5 > 0$ such that $T - t_{\lambda,\epsilon} > C_5$. By (2.19), we see that $|u'_\lambda(t)| \ge |u'_\lambda(T)| \ge \delta_1$ for $t \in [t_{\lambda,\epsilon}, T]$. Then we obtain that for $t \in [T - C_5, T - C_5/2] \subset [t_{\lambda,\epsilon}, T]$ and $\lambda \gg 1$,

$$\epsilon \ge u_\lambda(t) = \int_t^T (-u'_\lambda(s))ds \ge \delta_1(T-t) \ge \epsilon_1 := \frac{\delta_1 C_5}{2} > 0.$$

Therefore, $[T - C_5, T - C_5/2] \subset J_{\lambda,1,\epsilon_1}$ for $\lambda \gg 1$. This contradicts (2.15). Hence $t_{\lambda,\epsilon} \to T$ as $\lambda \to \infty$. This implies that $u_\lambda(t) \ge \epsilon$ for $t \in I_\eta = [0, T-\eta] \subset [0,T)$ ($0 < \eta < T$: arbitrary), and so

$$I_\eta = J_{\lambda,1,\epsilon} \bigcup \{t \in I_\eta : u_\lambda(t) \ge 2\pi - \epsilon\}.$$

Since $F(u)$ is increasing for $u > 0$ by (A.1), this along with (2.15) yields that for $\lambda \gg 1$,

$$\begin{aligned} 2TF(\alpha) &= \int_I F(u_\lambda(t))dt \ge 2\int_{I_\eta \setminus J_{\lambda,1,\epsilon}} F(u_\lambda(t))dt \ge 2F(2\pi - \epsilon)(T - \eta - |J_{\lambda,1,\epsilon}|) \\ &\ge 2F(2\pi-\epsilon)(T - 2\eta). \end{aligned}$$

Since $\epsilon, \eta > 0$ are arbitrary, this contradicts the assumption of Theorem. Thus the proof is complete. □

Lemma 2.6. *Let $0 < \epsilon \ll 1$ be fixed. Furthermore, let $\delta_{\lambda,\epsilon} := T - t_{\lambda,\epsilon}$. Then there exists $\lambda_\epsilon \gg 1$ and a constant $C_6 > 0$ such that for $\lambda > \lambda_\epsilon$*

$$u_\lambda'(T)^2 \le C_6 \lambda e^{-2\delta_{\lambda,\epsilon}\sqrt{(1-2\epsilon)\lambda}}. \tag{2.20}$$

Proof. By (1.1) and (2.17), for $I_{\lambda,\epsilon} := [t_{\lambda,\epsilon}, T]$, we obtain

$$u_\lambda''(t) + \mu(\lambda) f(u_\lambda(t)) = \lambda \sin u_\lambda(t) \ge (1-\epsilon)\lambda u_\lambda(t) \qquad \text{for } t \in I_{\lambda,\epsilon}. \tag{2.21}$$

Since $u_\lambda'(t) \le 0$ in $[0, T]$, we obtain

$$\{u_\lambda''(t) + \mu(\lambda) f(u_\lambda(t)) - (1-\epsilon)\lambda u_\lambda(t)\} u_\lambda'(t) \le 0 \qquad \text{for } t \in I_{\lambda,\epsilon}.$$

That is,

$$\frac{dS_\lambda(t)}{dt} := \frac{d}{dt}\left\{\frac{1}{2}u_\lambda'(t)^2 + \mu(\lambda)F(u_\lambda(t)) - \frac{(1-\epsilon)\lambda u_\lambda(t)^2}{2}\right\} \le 0 \qquad \text{for } t \in I_{\lambda,\epsilon}.$$

This implies that $S_\lambda(t)$ is non-increasing on $I_{\lambda,\epsilon}$. Hence,

$$\frac{1}{2}u_\lambda'(t)^2 + \mu(\lambda)F(u_\lambda(t)) - \frac{(1-\epsilon)\lambda u_\lambda(t)^2}{2} \ge \frac{1}{2}u_\lambda'(T)^2 \qquad \text{for } t \in I_{\lambda,\epsilon}.$$

Then for $t \in I_{\lambda,\epsilon}$,

$$-u_\lambda'(t) \ge \sqrt{u_\lambda'(T)^2 + (1-\epsilon)\lambda u_\lambda(t)^2 - 2\mu(\lambda)F(u_\lambda(t))}. \tag{2.22}$$

Since the inequality

$$\epsilon \lambda u_\lambda(t)^2 \ge 2\mu(\lambda)F(u_\lambda(t)) \tag{2.23}$$

is equivalent to

$$\frac{F(u_\lambda(t))}{u_\lambda(t)^2} \le \frac{\epsilon\lambda}{2\mu(\lambda)}, \tag{2.24}$$

by (A.3) and Lemma 2.4, we see that (2.23) and (2.24) are valid for $t \in I_{\lambda,\epsilon}$ and $\lambda \gg 1$. Then by (2.22) and (2.23), we obtain

$$-u_\lambda'(t) \ge \sqrt{u_\lambda'(T)^2 + (1-2\epsilon)\lambda u_\lambda(t)^2} \qquad \text{for } t \in I_{\lambda,\epsilon}. \tag{2.25}$$

Therefore, by (2.25)

$$\begin{aligned}\delta_{\lambda,\epsilon} = T - t_{\lambda,\epsilon} &= \int_{I_{\lambda,\epsilon}} 1 dt \le \int_{I_{\lambda,\epsilon}} \frac{-u_\lambda'(t)}{\sqrt{u_\lambda'(T)^2 + (1-2\epsilon)\lambda u_\lambda(t)^2}} dt \\ &= \int_0^\epsilon \frac{ds}{\sqrt{u_\lambda'(T)^2 + (1-2\epsilon)\lambda s^2}} \\ &= \frac{1}{\sqrt{(1-2\epsilon)\lambda}} \log\left(\frac{\left|\epsilon + \sqrt{\epsilon^2 + X_\lambda^2}\right|}{X_\lambda}\right),\end{aligned} \tag{2.26}$$

where $X_\lambda^2 := \frac{u_\lambda'(T)^2}{(1-2\epsilon)\lambda}$. Since $X_\lambda \to 0$ as $\lambda \to \infty$ by Lemma 2.5, (2.26) implies that $X_\lambda e^{\delta_{\lambda,\epsilon}\sqrt{(1-2\epsilon)\lambda}} \le 3\epsilon$ for $\lambda \gg 1$. By this, we obtain

$$u_\lambda'(T)^2 \le 9\epsilon^2(1-2\epsilon)\lambda e^{-2\delta_{\lambda,\epsilon}\sqrt{(1-2\epsilon)\lambda}} \le C_6 \lambda e^{-2\delta_{\lambda,\epsilon}\sqrt{(1-2\epsilon)\lambda}}.$$

This implies (2.20). □

Lemma 2.7. *Assume that there exists a subsequence $\{\lambda_j\}$ of $\{\lambda\}$ ($\lambda_j \to \infty$ as $j \to \infty$) such that*

$$\|u_{\lambda_j}\|_\infty \ge 2\pi. \tag{2.27}$$

Let $0 < \epsilon \ll 1$ be fixed. Then

$$t_{\lambda_j,2\pi-\epsilon} - t_{\lambda_j,2\pi} \ge \sqrt{1-2\epsilon}\,\delta_{\lambda_j,\epsilon} - o(1) \qquad \text{for } \lambda_j \gg 1. \tag{2.28}$$

Proof. By (2.27), $t_{\lambda_j,2\pi} \in [0,T]$ exists. Let $J_{j,\epsilon} := (t_{\lambda_j,2\pi}, t_{\lambda_j,2\pi-\epsilon})$. By the inequality $1 - \cos\theta \le \theta^2/2$ for $\theta \ge 0$ and noting that $\cos(2\pi - u_{\lambda_j}(t)) = \cos u_{\lambda_j}(t)$, we obtain by (2.4) that for $t \in J_{j,\epsilon}$,

$$\begin{aligned}\frac{1}{2}u_{\lambda_j}'(t)^2 &= \frac{1}{2}u_{\lambda_j}'(T)^2 + \lambda_j(1 - \cos u_{\lambda_j}(t)) - \mu(\lambda_j)F(u_{\lambda_j}(t)) \\ &\le \frac{1}{2}u_{\lambda_j}'(T)^2 + \frac{1}{2}\lambda_j(2\pi - u_{\lambda_j}(t))^2.\end{aligned}$$

This implies

$$-u_{\lambda_j}'(t) \le \sqrt{u_{\lambda_j}'(T)^2 + \lambda_j(2\pi - u_{\lambda_j}(t))^2} \qquad \text{for } t \in J_{j,\epsilon}. \tag{2.29}$$

By (2.29), we obtain

$$t_{\lambda_j,2\pi-\epsilon} - t_{\lambda_j,2\pi} \geq \int_{J_{j,\epsilon}} \frac{-u'_{\lambda_j}(t)}{\sqrt{u'_{\lambda_j}(T)^2 + \lambda_j(2\pi - u_{\lambda_j}(t))^2}}dt = \int_0^{\epsilon} \frac{1}{\sqrt{u'_{\lambda_j}(T)^2 + \lambda_j s^2}}ds$$
$$= \frac{1}{\sqrt{\lambda_j}} \log\left(\frac{\epsilon + \sqrt{B_j^2 + \epsilon^2}}{B_j}\right), \tag{2.30}$$

where $B_j^2 = u'_{\lambda_j}(T)^2/\lambda_j$. By this and (2.20), we obtain

$$\begin{aligned}
t_{\lambda_j,2\pi-\epsilon} - t_{\lambda_j,2\pi} &\geq \frac{1}{\sqrt{\lambda_j}} \log\left(\frac{2\epsilon}{|u'_{\lambda_j}(T)|/\sqrt{\lambda_j}}\right) \\
&= \frac{1}{\sqrt{\lambda_j}}\left\{\log(2\epsilon) - \log|u'_{\lambda_j}(T)| + \frac{1}{2}\log\lambda_j\right\} \\
&= \frac{1}{\sqrt{\lambda_j}}\left\{\log(2\epsilon) - \frac{1}{2}\log(C_6\lambda_j) + \delta_{\lambda_j,\epsilon}\sqrt{(1-2\epsilon)\lambda_j} + \frac{1}{2}\log\lambda_j\right\} \\
&= \sqrt{(1-2\epsilon)}\delta_{\lambda_j,\epsilon} - o(1).
\end{aligned}$$

Thus the proof is complete. □

Proof of Theorem. First, we shall prove (1). To do this, we assume (2.27) and derive a contradiction. We fix $0 < \epsilon \ll 1$. Since $|t_{\lambda_j,2\pi-\epsilon} - t_{\lambda_j,\epsilon}| = |J_{\lambda_j,1,\epsilon}| \to 0$ as $j \to \infty$ by (2.15), we obtain

$$\begin{aligned}
2TF(\alpha) &= 2\int_0^T F(u_{\lambda_j}(t))dt \geq 2\int_0^{t_{\lambda_j,2\pi-\epsilon}} F(u_{\lambda_j}(t))dt \\
&\geq 2F(2\pi-\epsilon)(T - \delta_{\lambda_j,\epsilon} + t_{\lambda_j,2\pi-\epsilon} - t_{\lambda_j,\epsilon}) \qquad (2.31)\\
&= 2F(2\pi-\epsilon)(T - \delta_{\lambda_j,\epsilon}) + o(1).
\end{aligned}$$

This implies

$$\delta_{\lambda_j,\epsilon} \geq \left(1 - \frac{F(\alpha)}{F(2\pi-\epsilon)}\right)T + o(1). \tag{2.32}$$

On the other hand, by Lemma 2.7, we obtain

$$T \geq t_{\lambda_j,2\pi-\epsilon} - t_{\lambda_j,2\pi} + \delta_{\lambda_j,\epsilon} \geq \sqrt{(1-2\epsilon)}\delta_{\lambda_j,\epsilon} - o(1) + \delta_{\lambda_j,\epsilon}.$$

This implies that for $j \gg 1$

$$\delta_{\lambda_j,\epsilon} \leq \frac{T}{\sqrt{1-2\epsilon}+1} + o(1). \tag{2.33}$$

Therefore, for $j \gg 1$, by (2.32) and (2.33), we obtain

$$\frac{F(\alpha)}{F(2\pi - \epsilon)} + o(1) \geq \frac{\sqrt{1-2\epsilon}}{\sqrt{1-2\epsilon}+1}.$$

By letting $j \to \infty$ and $\epsilon \to 0$, we obtain $F(\alpha)/F(2\pi) \geq 1/2$. This contradicts the assumption of Theorem. Consequently, we find that the assumption (2.27) is false. Thus the proof of Theorem (1) is complete.

Since Theorem (2) and (3) are easily derived from Theorem (1), we omit the proofs.

Finally, we prove the assertion (4). Let $J := [t_1, t_2] \subset (0, t_\alpha)$ be fixed ($t_1 < t_2$). We choose an arbitrary $0 < \epsilon \ll 1$. Note that $\sin u_\lambda(t) < 0$ for $t \in J$. Therefore, by the equation in (1.1) and Lemma 2.1 (2), we see that

$$-u_\lambda''(t) = \mu(\lambda) f(u_\lambda(t)) - \lambda \sin u_\lambda(t) > 0.$$

Hence $-u_\lambda'(t)$ is increasing on J. Then

$$C\epsilon \geq |u_\lambda(t_1) - u_\lambda(t_2)| = \left| \int_{t_1}^{t_2} (-u_\lambda'(t))dt \right| \geq (t_2 - t_1)|u_\lambda'(t_1)|.$$

Now, by this and the equation (1.1), for $\lambda \gg 1$, we obtain

$$\begin{aligned} C\epsilon/(t_2 - t_1) \geq |u_\lambda'(t_1)| = -u_\lambda'(t_1) &= \int_0^{t_1} -u_\lambda''(s)ds \\ &= \mu(\lambda) \int_0^{t_1} f(u_\lambda(t))dt - \lambda \int_0^{t_1} \sin u_\lambda(t)dt \\ &\geq \mu(\lambda) \left(\min_{2\pi-\epsilon \leq u \leq 2\pi} f(u) \right) t_1. \end{aligned}$$

Thus the proof of (4) is complete. □

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