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ON DUCKS IN THE MINIMAL SYSTEM

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Abstract. In the linearized local model of the generalized BVP system, the winding number for a duck solution is not defined well. By choosing an adequate regular transformation, which contains a new parameter, it can be proved that this number becomes well defined after that and tends to infinity as a regular limit for the new parameter. In case the new one is fixed sufficiently small, the number becomes large as the absolute value of a coparameter embedded originally becomes small.

1. INTRODUCTION.

The modified Bonhoeffer-van der Pol(BVP) equations were proposed by H.Kawakami et al.[5] in 1999. Their result of the simulation for this system shows that there exist winding orbits on some projected phase space. Furthermore, the winding number increases when some parameter contained originally in this system decreases.

The BVP equations are described as follows:

$$\begin{aligned} L_1 di_1/dt &= E_1 - R_1 i_1 - v, \\ L_2 di_2/dt &= E_2 - R_2 i_2 - v, \\ C dv/dt &= i_1 + i_2 + \rho(v), \end{aligned} \tag{1.1}$$

where i_1, i_2 are the currents through the inductors L_1, L_2 and the registors R_1, R_2, respectively. Moreover, E_1, E_2 are the constant voltages, v represents at the non-linear registor ρ ($\rho(v) = v - v^3/3$) and C is a capacitor with very small capacitance. Let consider the following generized BVP system:

$$\begin{aligned} dx/dt &= -ax - az, \\ dy/dt &= -by - bz, \\ \epsilon dz/dt &= x + y + z - z^3/3, \end{aligned} \tag{1.2}$$

where ϵ is infinitesimally small. In the equation (1.1), put $i_1 = x$, $i_2 = y$, $v = z$, $C = \epsilon$ and then assume that $E_1 = E_2 = 0$, $R_1 = R_2 = 1$, $1/L_1 = a$, $1/L_2 = b$.

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As the generalized BVP system satisfies the generic conditions; A1, ... ,A5(see Section2), some orbit expresses a jumping state with delay (duck solution, or simply duck) in the system of the minimal polynomials (or *in the minimal system*). By giving a relation between two parameters as $a-b=1$, this system can be reduced to the problem of one parameter family with incomplete ducks. See Section3. In this paper, under the above assumptions, we will describe the following two theorems.

Theorem1. If the parameter b satisfies $-1<b<0$, then the minimal system has a duck near the pseudo singular saddle point.

In the BVP system, the winding number for a duck is not defined well by itself. By introducing some regular transformation, which contains a parasitic parameter, the winding number for the duck turns to be well defined as a regular limit in the regularized BVP system. In another point of view, there exists a regular coordinate transformation containing certain parameter to realize it in the minimal system. As a result, the following theorem is obtained:

Theorem2. If the parameter b satisfies $-1/2-\sqrt{8/5}/2 < b < -1$, then the minimal system has a duck near the pseudo singular node point and the winding number of the duck tends to infinity as b tends to -1.

2.Preliminaries

Let consider a constrained system(2.1):

$$\begin{aligned} dx/dt &= f(x,y,z,u), \\ dy/dt &= g(x,y,z,u), \\ h(x,y,z,u) &= 0, \end{aligned} \tag{2.1}$$

where u is a parameter (any fixed) and f,g,h are defined in $R^3\times R^1$. Furthermore, let consider the singular perturbation problem of the system (2.1):

$$\begin{aligned} dx/dt &= f(x,y,z,u), \\ dy/dt &= g(x,y,z,u), \\ \epsilon dz/dt &= h(x,y,z,u), \end{aligned} \tag{2.2}$$

where ϵ is infinitesimally small.

We assume that the system (2.1) satisfies the following conditions $(A1)-(A5)$:

$(A1)$ f and g are of class C^1 and h is of class C^2.

$(A2)$ The set $S=\{(x,y,z)\in R^3 : h(x,y,z,u)=0\}$ is a 2-dimensional differentiable manifold and the set S intersects the set $T=\{(x,y,z)\in R^3 : \partial h(x,y,z,u)/\partial z=0\}$ transversely so that the set $PL=\{(x,y,z)\in S\cap T\}$ is a 1-dimensional differentiable manifold.

$(A3)$ Either the value of f or that of g is nonzero at any point $p\in PL$.

Let $(x(t,u),y(t,u),z(t,u))$ be a solution of (2.1). By differentiating $h(x,y,z,u)$ with respect to the time t, the following equation holds:

$$h_x(x,y,z,u)f(x,y,z,u)+h_y(x,y,z,u)g(x,y,z,u)+h_z(x,y,z,u)dz/dt=0, \tag{2.3}$$

where $h_i(x,y,z,u) = \partial h(x,y,z,u)/\partial i$, $i = x,y,z$. The above system (2.1) becomes the following system:

$$(2.4)\qquad \begin{aligned} dx/dt &= f(x,y,z,u), \\ dy/dt &= g(x,y,z,u), \\ dz/dt &= -\{h_x(x,y,z,u)f(x,y,z,u)+ \\ &\quad h_y(x,y,z,u)g(x,y,z,u)\}/h_z(x,y,z,u), \end{aligned}$$

where $(x,y,z) \in S \setminus PL$. The system (2.1) coincides with the system (2.4) at any point $p \in S \setminus PL$. In order to study the system (2.4), let consider the following system:

$$(2.5)\qquad \begin{aligned} dx/dt &= -h_z(x,y,z,u)f(x,y,z,u), \\ dy/dt &= -h_z(x,y,z,u)g(x,y,z,u), \\ dz/dt &= h_x(x,y,z,u)f(x,y,z,u) + h_y(x,y,z,u)g(x,y,z,u). \end{aligned}$$

As the system(2.5) has well posedness at any point of R^3, it has well posedness indeed at any point of PL. The solutions of (2.4) coincide with those of (2.1) on $S \setminus PL$ except the velocity when they start from the same initial points.

$(A4)$ For any $(x,y,z) \in S$, the implicit function theorem holds;

$$(2.6)\qquad h_y(x,y,z,u) \neq 0, h_x(x,y,z,u) \neq 0,$$

that is, the surface S can be expressed by using $y = \varphi(x,z,u)$ or $x = \psi(y,z,u)$ in the neighborhood of PL. Let $y = \varphi(x,z,u)$ exist, then the projected system, which restricts the system (2.5) is obtained:

$$(2.7)\qquad \begin{aligned} dx/dt &= -h_z(x,\varphi(x,z,u),z,u)f(x,\varphi(x,z,u),z,u), \\ dz/dt &= h_x(x,\varphi(x,z,u),z,u)f(x,\varphi(x,z,u),z,u)+ \\ &\quad h_y(x,\varphi(x,z,u),z,u)g(x,\varphi(x,\varphi(x,z,u),z,u). \end{aligned}$$

$(A5)$ All the singular points of (2.7) are nondegenerate, the matrix induced from the linearized system of (2.7) at a singular point has two nonzero eigenvalues. Note that all the points contained in $PS = \{(x,y,z) \in PL : dz/dt = 0\}$, which is called *pseudo singular points* are the singular points of (2.5).

Definition2.1. Let $p \in PS$ and $\mu_1(u)$, $\mu_2(u)$ be two eigenvalues of the linearized system of (2.7), then the point p is called *pseudo singular saddle* if $\mu_1(u) < 0 < \mu_2(u)$ and called *pseudo singular node* if $\mu_1(u) < \mu_2(u) < 0$ or $\mu_1(u) > \mu_2(u) > 0$.

Definition2.2. A solution $(x(t,u), y(t,u), z(t,u))$ of the system(2.2) is called *a duck*, if there exist standard $t_1 < t_0 < t_2$ such that

(1) ${}^*(x(t_0,u), y(t_0,u), z(t_0,u)) \in S$, where ${}^*(X)$ denotes the standard part of X,
(2) for $t \in (t_1,t_0)$ the segment of the trajectory $(x(t,u), y(t,u), z(t,u))$ is infinitesimally close to the attracting part of the slow curves,
(3) for $t \in (t_0,t_2)$, it is infinitesimally close to the repelling part of the slow curves, and
(4) the attracting and repelling parts of the trajectory are not infinitesimally small.
If a duck exists as a part of a limit cycle, it is called *a proper duck.*

Definition2.3. Let E be a set in R^3. We call a point p is a δ- micro-galaxy of E when the distance from p to E is less than $exp(-n/\delta)$, where n is some positive integer and $\delta = \epsilon/\alpha^2$ (α is infinitesimally small).

Definition2.4. Let θ is an angle of the polar coordinate after changing the coordinates in the "local model" such as the orbit passing through the pseudo singular point becomes the z-axis itself as the below. See [3],[4]. Then, the winding number $N(\psi)$ of a duck ψ is defined as follows:

$$N(\psi) = (1/2\pi)\int_{\psi} d\theta, \tag{2.8}$$

where ψ is contained partially in the δ-micro- galaxy of γ_{μ}.

Theorem2.1(Benoit). In the system(2.1), if the following two conditions at a pseudo singular saddle or node point;

(1) $f(O,u) \simeq h(O,u) \simeq h_y(O,u) \simeq h_z(O,u) \simeq 0$,

(2) $g(O,u) \not\simeq 0, h_x(O,u) \not\simeq 0, h_{zz}(O,u) \not\simeq 0$, where $O = (0,0,0) \in PS$,

are satisfied, the explicit duck solutions $\gamma_{\mu_i(u)}$ in the first approximation of the local model can be constructed:

$$\gamma_{\mu_i(u)}(t) = (-\mu_i(u)^2t^2 - \delta\mu_i(u), t, \mu_i(u)t)(i = 1,2), \tag{2.9}$$

wehere δ is an infinitesimally small constant.

The above Definition2.3 is based on the following fact. If ϵ is fixed arbitrarily and $\gamma(t)$ is a duck near $\gamma_{\mu(u)}(t)$ is within $exp(-n/\delta)$ in some neighborhood of the pseudo singular point. See[10].

In the system(2.2), under the conditions (1) and (2) in the Theorem2.1, making the following coordinate transformations (2.10) and (2.11) successively;

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha^2\tilde{x} \\ \alpha\tilde{y} \\ \alpha\tilde{z} \end{pmatrix}, (\alpha \simeq 0, \epsilon/\alpha^2 \simeq 0) \tag{2.10}$$

(2.11)

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} h_x(0,u)h_{zz}(0,u)\tilde{x}/2 + (h_{yy}(0,u)h_{zz}(0,u) - h_{yz}(0,u)^2)\tilde{y}^2/4 \\ \tilde{y}/g(0,u) \\ -h_{yz}(0,u)\tilde{y}/2 - h_{zz}(0,u)\tilde{z}/2 \end{pmatrix},$$

the following local model (2.12) is obtained:

$$\begin{aligned} dX/dt &= pY + qZ + \xi(X,Y,Z,u), \\ dY/dt &= 1 + \eta(X,Y,Z,u), \\ \delta dZ/dt &= -(Z^2 + X) + \zeta(X,Y,Z,u), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} p &= g(0,u)h_x(0,u)(f_y(0,u)h_{zz}(0,u) - f_z(0,u)h_{yz}(0,u))/2 \\ &+ g(0,u)^2(h_{yy}(0,u)h_{zz}(0,u) - h_{yz}(0,u)^2)/2, \\ q &= -h_x(0,u)f_z(0,u), \\ \delta &= \epsilon/\alpha^2. \end{aligned} \tag{2.13}$$

Here $\xi(X,Y,Z,u)$,$\eta(X,Y,Z,u)$ and $\zeta(X,Y,Z,u)$ are infinitesimal when X,Y and Z are limited. Note that the solutions (2.9) are in the first approximation system of the system(2.12).

By applying the following transformations of the coordinates as mentioned above, in Definition2.4, successively;

$$
\begin{aligned}
u &= X + Z^2 + \delta\mu, \\
v &= Y - Z/\mu, \\
z &= Z,
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
u &= r cos\theta, \\
v &= r sin\theta,
\end{aligned} \tag{2.15}
$$

the Hermite equation (2.16) is obtained. This equation associated with $\gamma_{\mu_i(u)}$ $(i = 1,2)$ is the following:

$$\delta\ddot{z} - \tau\dot{z} + K_i z = 0, \dot{z} = dz/d\tau, t = \tau/\alpha, (i = 1,2), \tag{2.16}$$

where K_i is a positive integer and $K_1 = 1 + \mu_2(u)/\mu_1(u)$, $K_2 = 1 + \mu_1(u)/\mu_2(u)$. See [3].

It is said that a duck $\psi(t)$ has *a jump* if the shadow of it contains a vertical segment and that $\psi(t)$ is *long* if it is in an infinitesimally small neighborhood at the pseudo singular point. It can be proved that if ψ is not long, the standard part of the winding number $N(\psi_i)$ associated with μ_i is an integer. If the pseudo singular point is node, it is positive. If the point is saddle, it needs some conditions such as K_i is poitive. The relation between $N(\psi_i)$ and K_i $(i = 1,2)$ is as follows.

Theorem2.2(Benoit). If the duck ψ_1, which is not long has 2 jumps, $N(\psi_1) \approx -[K_1/2]$, and if the duck ψ_2 has 2 jumps, $N(\psi_2) \approx 0$.

3.Incomplete ducks.

Definition3.1. In the system(2.12), if the followings (1) and (2):

(1) for any limited parameter u,
it satisfies the conditions ($A1$)-($A5$) and has a duck,

(2) when the parameter u tends to infinity, one of the winding numbers tends to infinity and the other tends to zero,
and the system does not have a duck as a singular limit,
are established, this solution is called an *ω-incomplete duck* .

Definition3.2. A solution $\psi(x,u)$ is called S^1 *at a* ,
if there exists a real number b such that

$$\frac{\psi(x,u) - \psi(y,u)}{x - y} \approx b, \tag{3.1}$$

for any $x, y (x \approx a, y \approx a)$.
A duck is called an S^1*duck* if it is S^1 in some neighborhood of the pseudo singular point.

Theorem3.1(Benoit). In the first approximation of the system(2.12), if $\mu_1(u)/\mu_2(u)$ is positive (> 3) but no an integer, then all the S^1 ducks are exponentially close to one of the two explicit ducks and there exists non S^1 ducks.

In the system(2.12), we assume that

$$f_y(0,u) = g_u(0,u) = h_{yz}(0,u) = h_{yyu}(0,u) = h_{zzu}(0,u) = 0, \tag{3.2}$$

and that the following (1) or (2):
(1) $h_x(0,u) = O(u)$ and $f_z(0,u) = O(1)$,
(2) $f_z(0,u) = O(u)$ and $h_x(0,u) = O(1)$,
where all the coefficients of higher order (more than 2) for u is negligible, that is, only the coefficient q can take an unlimited number:

$$q = c_1 u + o(1), c_1 \not\simeq 0. \tag{3.3}$$

Then, blowing up only the variable Z again;

$$Z = (1/u)\tilde{Z}, \tag{3.4}$$

the first approximation of the system(2.12) becomes the following:

$$\begin{aligned} dX/dt &= pY + c_1\tilde{Z}, \\ dY/dt &= 1, \\ (\delta/u)d\tilde{Z}/dt &= -(\tilde{Z}^2/u^2 + X), \end{aligned} \tag{3.5}$$

where c_1 is limited (does not contain u) and $\delta/u \simeq 0$. The explicit solutions in the system$(3,5)$ are

$$\gamma_{\mu_i(u)}(t) = (-\mu_i(u)^2t^2 - \delta\mu_i(u), t, u\mu_i(u)t)(i = 1,2), \tag{3.6}$$

where $\mu_1(u)$, $\mu_2(u)$ ($\mu_1(u) > \mu_2(u)$) are the solutions of the characteristic equation of the system(3.5) in case $\delta/u \simeq 0$.

The above system satifies the conditions $(A1)$-$(A5)$ and the solutions(3.6) satisfy the condition (1) and satisfies the condition (2) when $u \to \omega$ in Definition3.1. In fact, if $q = \epsilon^{-1/3}$, then the existence of such a duck is ensured. We choose $\epsilon = 1/n$ $(n = 2, 3, \dots)$ $(u = 1/n^{1/3})$, then $1/n^{1/3} \gg exp(-n/\delta)$ for any n $(n \geq 2)$. In the system for each any fixed n, let $J = [AB]$ be a connected segment in R^3, where the solution which starts at A(or B) belongs to the family of the duck γ_{μ_1} (or γ_{μ_2}). It can be proved that if any solution starting at $p \in J$ is not long, then it has the same winding number. From Theorem3.1, a duck passing through the pseudo singular node point belongs to one of two families of the above ducks. On the other hand, there exists a segment $[CD] \subset J$ such that any solution starting at $p \in [CD]$ is not long and the solutions passing through C or D are ducks. This fact ensures the existence of a non S^1 duck. Note that $\mu_1(u)/\mu_2(u)$ is positive but no an integer. If it is a positive integer k, it indicates the fact that the slow vector field has two C^1 trajectories but only one of them is C^k. Then, it is not possible to have an asymptotic expansion in powers of ϵ with the coefficients analytic in

t. Furthermore, one of the solutions(3.6) may tend tangent to the X-axis, since $\mu_2(u) \to -\omega/2$ as $u \to \omega$. In fact, for the first component of the solutions, the following

$$\frac{-(\omega/2)^2(2/\omega)^2 + (\omega/2)^2(1/\omega)^2}{2/\omega - 1/\omega} \to -3\omega/4, \tag{3.7}$$

establishes. In this state, the winding number $N(\psi_2)$ associated with μ_2 tends to infinity and the other $N(\psi_1)$ associated with μ_1 tends to infinitesimal. When $u \to \omega = \epsilon^{-1/3}$, the eigen space of the linear part of the slow vector field for $\mu_2 \simeq -\epsilon^{-1/3}$ is $z \simeq \epsilon^{1/3}y$ ($z \simeq -\epsilon^{1/3}y$ for $\mu_1 \simeq -\epsilon^{1/3}$). The ducks are almost tangent to the eigen spaces and therefore the ω-limit of the duck with respect to the parameter u (ω-incomplete duck) is not S^1.

Let $v = 1/u$, then $\partial_u = -v^2\partial_v$ holds and then the following conditions are assumed; $f(x,y,z,u) = \tilde{f}(x,y,z,v) \in C^3$, $g(x,y,z,u) = \tilde{g}(x,y,z,v) \in C^1$ and $h(x,y,z,u) = \tilde{h}(x,y,z,v) \in C^3$ at almost every where but $v = v_0 = 0$. From the assumptions, the relation $q = -h_x(0,u)f_z(0,u) = c_1 u$ holds. Differentiating the both side of this equation by the parameter v, we can lead to the following theorem.

Theorem3.2. In the first approximation of the system(2.12), if $\mu_1(u)/\mu_2(u)$ is positive but no integer under the condition (3.2) and if $\tilde{h}_{xv}(0,v)\tilde{f}_{zv}(0,v) = 0$ when either the condition (1) or (2) ;
(1) $\tilde{f}_z(0,v) = 0$, and $\tilde{h}_x(0,v)\tilde{f}_{zvv}(0,v) = 0$,
(2) $\tilde{h}_x(0,v) = 0$, and $\tilde{h}_{xvv}(0,v)\tilde{f}_z(0,v) = 0$,
where all the coefficients of higher order (more than 2) for u is negligible are satisfied, then this system has an ω-incomplete duck.

Corollary3.3. In the system(2.12), if the coefficient q satisfies $q = c_1u + O(1)$, that is, $q = c_1u + c_2$ where $c_1, c_2 \not\simeq 0$ and $p > 0$ or $0 > p \geq -1/32$, then there exists a finite value u_0 which makes the winding number infinite when u tends to u_0; the corresponding duck is called *incomplete* , simply.

Remark. In this situation, the singular perturbation problem is equivalent to the following system of two parameters family with ϵ_1 and ϵ_2:

$$\begin{aligned} \epsilon_1 dX/dt &= \epsilon_1 pY + qZ + \xi(X,Y,Z,\epsilon_1,\epsilon_2), \\ dY/dt &= 1 + \eta(X,Y,Z,\epsilon_1,\epsilon_2), \\ \epsilon_2 dZ/dt &= -(Z^2 + X) + \zeta(X,Y,Z,\epsilon_1,\epsilon_2), \end{aligned} \tag{3.8}$$

where ϵ_1 and ϵ_2 are infinitesimally small.

4. The regularized BVP system

Let consider the following equations (regularized BVP):

$$\begin{aligned} L_1 di_1/dt &= E_1 - R_1 i_1 - R(i_1 + i_2) - v, \\ L_2 di_2/dt &= E_2 - R_2 i_2 - R(i_1 + i_2) - v, \\ C dv/dt &= i_1 + i_2 + v - v^3/3, \end{aligned} \tag{4.1}$$

where R is a linear reistor and C has a very small capacitance. The equation (4.1) is composed of a parasitic resistor R (as a parameter) to the BVP system. When R tends to zero, the system (4.1) tends to quite the same one (1.1) as a regular limit. In the linearized local model of (4.1), the winding numbers for the ducks are well defined since the matrix associated with the model at the pseudo singular point are nondegenerate; it is called *regularized BVP system* . The parameter R holds the relation (3.3) as $u = R$. Thus, this system has an incomplete duck from Corollary3.3. When R tends to zero again, only one of the eigenvalues tends to zero, that is, the winding number for the duck tends to infinity.

The linearized local model of the system(1.2) could not make one of the winding numbers defined well because the first equation does not contain the variable y. This fact causes that the value of p in the equation (2.13) takes zero, therefore one of the eigenvalues takes zero. There exists a problem how to avoid this trouble. The mathematical model which it may concern is obtained as follows.

Lemma4.1. *In the system(1.2), there exists a regular coordinate transformation such that the winding numbers become well defined.*

(proof)
Choosing the following transformation:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{4.2}$$

where $u(\neq 1)$ is a parasitic parameter, the system(1.2) becomes

$$\begin{aligned} dX/dt &= -aX + u(a-b)Y - (a+bu)Z, \\ dY/dt &= -bY - bZ, \\ \epsilon dZ/dt &= X + (1-u)Y + Z - Z^3/3. \end{aligned} \tag{4.3}$$

Considering the constrained system of the equation(4.3) and the relation $a - b = 1$, the following

$$\begin{aligned} dX/dt &= (-(b+1)X + uY - (bu+b+1)Z)(1-Z^2), \\ dZ/dt &= (b+1)X - (bu+u-b)Y + (2b+1)Z, \end{aligned} \tag{4.4}$$

establishes, where b, u are parameters. Substituting the following

$$Y = (X + Z - Z^3/3)/(u-1), \tag{4.5}$$

for the above, the equation(4.4) becomes

$$\begin{aligned} dX/dt &= (-(b+1-u/(u-1))X - (bu+b+1-u/(u-1))Z \\ &\quad - uZ^3/3(u-1))(1-Z^2), \\ dZ/dt &= (1-u/(u-1))X + (b+1-u/(u-1))Z \\ &\quad + (b+u/(u-1))Z^3/3. \end{aligned} \tag{4.6}$$

The pseudo singular points are

$$
(4.7)\qquad \begin{aligned}
PS_{-1} &= (X_{0-}, Y_{0-}, Z_{0-}),\\
X_{0-} &= (-(4b+1)u + 4b + 3)/3,\\
Y_{0-} &= (-(4b+1)u + 4b + 1)/3(u-1),\\
Z_{0-} &= -1,
\end{aligned}
$$

and

$$
(4.8)\qquad \begin{aligned}
PS_{1} &= (X_{0+}, Y_{0+}, Z_{0+}),\\
X_{0+} &= ((4b+1)u - 4b - 3)/3,\\
Y_{0+} &= ((4b+1)u - 4b - 1)/3(u-1),\\
Z_{0+} &= 1.
\end{aligned}
$$

The values of p, q in the equation(2.13) at these points are

$$
(4.9)\qquad \begin{aligned}
p_{-} &= uZ_{0-}(bY_{0-} - b),\\
q_{-} &= b(u+1) + 1,\\
p_{+} &= u(-bY_{0+} + b),\\
q_{+} &= q_{-}.
\end{aligned}
$$

Since the value of p is not zero, it becomes clear that the eigenvalues at these points are nondegenerate, so the winding number is well defined. This completes the proof.

Lemma4.2. *If a duck of the system(4.3) is proper, that is, there exists a limit cycle, which has the duck as a part of the solution, the right hand of this system has minimal degree of the polynomials (minimal system) for the proper duck.*

(proof)
If the degree of the polynomials are smaller than 3, there is not a jumping orbit to return to an initial point. Therefore, there is not a limit cycle, which contains a duck. This completes the proof.

In this paper, the condition for the existence of the proper duck does not be treated.

5. The proofs of Theorem1 and Theorem2

In the generalized BVP system(1.2), there are three pseudo singular points PS_z ($z = -1, 0, 1$); $PS_{-1} = (1 + 4b/3, -1/3 - 4b/3, -1)$, $PS_0 = (0, 0, 0)$, $PS_1 = (1 + 4b/3, -5/3 - 4b/3, 1)$, since we assumed $a - b = 1$. In fact, the constrained system for the system(4.3) becomes the following; corresponding to the equation(2.7):

$$
(5.1)\qquad \begin{aligned}
dx/dt &= -(1+b)(x+z)(1-z^2),\\
dz/dt &= x + (1+b)z + bz^3/3,
\end{aligned}
$$

where only b is a parameter. However, the point PS_0 does not satisfy the generic condions, especially the condition($A3$) in Section2. So, let consider other two points PS_{-1} and PS_1.

In both cases, the characteristic equations associated with the linearized system of the equation(5.1) are quite the same as follows:

$$\lambda^2 - (1+2b)\lambda + 8b(1+b)/3 = 0. \tag{5.2}$$

If b satisfies $-1 < b < 0$, PS_{-1} and PS_1 are the pseudo singular saddle points, that is, there exist the ducks. This completes the proof of Theorem1.

In the regularized system(4.3), if the parameter b satisfies $-1/2 - \sqrt{8/5}/2 < b < -1$, the pseudo singular points PS_{-1}, PS_1 are the node type when coparameter $u = R$ tends to zero. The characteristic equation of the linearized local model under the condition (2.13) is

$$2\lambda^2 - q\lambda + p = 0, \tag{5.3}$$

where p, q are in the equation(4.9). From Lemma4.1, the winding number is well defined. Furthermore, the constant p satisfying the equation(4.9) ensures the relation(3.3). Therefore, this regularized system has an incomplete duck. Then, one of the eigenvalues tends to zero as the parameter u tends to zero. At that time, the corresponding index K in the equation(2.16) tends to infinity, so the winding number associated with the duck tends to infinity by Theorem2.2. Let the parameter $u(\neq 0)$ is fixed sufficiently small, then the winding number tends to infinity as the coparameter b tends to -1. In fact, it should be available if u holds $u = O(\epsilon^{1/2})$ whenever b satisfies $b + 1 = \epsilon$. This completes the proof of Theorem2.

Remark. When b tends to -1, the absolute value of b becomes small as possible in the above restriction. This fact coinsides with the reults of the simulation [5]: when $1/L_1$ becomes smaller, the winding number becomes larger.

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