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TORIC MODIFICATIONS OF LINE SINGULARITIES ON SURFACES

GUANGFENG JIANG(姜广峰) AND MUTSUO OKA(岡 睦雄)

ABSTRACT. We study the topology of the Milnor fibre F of a function f with critical locus a smooth curve L on a surface X, where X has an isolated complete intersection singularity and contains L. We use toric modification to resolve the non-isolated singularity $V = X \cap f^{-1}(0)$. Then we compute the Euler-Poincaré characteristic of F. Some examples are worked out.

INTRODUCTION

Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be a germ of an *icis* (isolated complete intersection singularity) and contain a smooth curve L, which will be called a line in this article. We are interested in the topology of the Milnor fibre F_f of a function f whose zero level hypersurface passes L or is tangent to the regular part of X along a line L. Hence the critical locus of f contains L if its zero hypersurface is tangent to $X_{\rm reg}$ along L. Since toric modification is used, we assume that f together with the defining equations $h_1, \dots, h_{n-1}$ of X form a non-degenerate complete intersection.

Let L be the x-axis in a local coordinate system defined by the ideal $\mathfrak{g} = (y_1, \dots, y_n)$. Since $L \subset X$, their defining ideals satisfy the relation $\mathfrak{h} := (h_1, \dots, h_{n-1}) \subset \mathfrak{g}$. This implies that X is not convenient or *commode* in French. Also the zero level hypersurface defined by f contains L, so $V := X \cap f^{-1}(0)$ is not convenient either. If X also contains another axis of the local coordinate system, then there is a point Q in the dual Newton diagram $\Gamma^*(h_1, \dots, h_{n-1}, f)$ of V such that Q is not strictly positive, not on the axes and the minimal value $d(Q; \mathfrak{h})$ of the linear function determined by the covector Q on the Newton polyhedron $\Gamma_+(h_1, \dots, h_{n-1})$ is positive, but $d(Q; f) = 0$ on $\Gamma_+(h_1, \dots, h_{n-1}, f)$. This means that the assumptions, called $\sharp$ and $\sharp'$ conditions in the literature (see for example [12, P.128, P.205]), are not satisfied. These "sharp" requirements seem essential in order to get the good resolution and zeta function along the last " principal direction" of the non-degenerate complete intersection $(h_1, \dots, h_{n-1}, f)$.

Nevertheless, in case X is a surface with isolated singularity, we really can replace these "sharp" conditions by a weaker one and obtain a good resolution of f on X. By A'Campo's theorem, we are able to compute the zeta function of the algebraic monodromy of F_f and the Euler-Poincaré characteristic of F_f.

In case f is a generic function contained in $\mathfrak{g}$, our work also supplies some information on the hypersurface intersection of X along the line contained therein.

If $f \in \mathfrak{g}^2$ and the transversal singularity type of f along L is A_1, practically to get the Euler-Poincaré characteristic of F_f we do not need to resolve the function f (which might be very general) since the theory developed in [6]. For example we can consider the Milnor fibre F_q of a generic quadric form q in the variables $y_1, \dots, y_n$. The Euler-Poincaré characteristic of F_f can be expressed by: the Euler-Poincaré characteristic of F_q, the number of Morse points outside L, and the number of D_∞ points on $L \setminus 0$ of the Morsification $f + q$. If, moreover, F_q is connected, F_f is also connected, hence a bouquet of one circles.

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As applications, we prove that F_f is homotopically a bouquet of one cycles if $f \in \mathfrak{g}^2$ has transversal A_1 singularity along L and X is an $A_k - D_k - E_6 - E_7$ type surface singularity. We also prove that F_q is in general not connected when X is a Brieskorn-Pham surface.

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1. Preliminaries

1.1. Let $\mathcal{O}_{\mathbb{C}^m}$ be the structure sheaf of $\mathbb{C}^m$. The stalk $\mathcal{O}_{\mathbb{C}^m,0}$ of $\mathcal{O}_{\mathbb{C}^m}$ at 0 is denoted by $\mathcal{O}$. Let $(X,0)$ be a reduced analytic space germ in $(\mathbb{C}^m,0)$ defined by the radical ideal $\mathfrak{h}$ of $\mathcal{O}$. Let $(L,0)$ be the germ of a subspace of X defined by the radical ideal $\mathfrak{g}$ of $\mathcal{O}$. Denote $\mathcal{O}_X := \mathcal{O}/\mathfrak{h}$, $\mathcal{O}_L := \mathcal{O}/\mathfrak{g}$.

Let $\mathrm{Der}(\mathcal{O})$ denote the $\mathcal{O}$-module of germs of analytic vector fields on $\mathbb{C}^m$ at 0. Define $D_X := \mathrm{Der}_{\mathfrak{h}}(\mathcal{O}) = \{\xi \in \mathrm{Der}(\mathcal{O}) \mid \xi(\mathfrak{h}) \subset \mathfrak{h}\}$, which is the $\mathcal{O}$-module of *logarithmic vector fields* along $(X,0)$ (cf. [1]). Geometrically, D_X consists of all germs of vector fields that are tangent to the smooth part of X. Equiped with X there is a so called *logarithmic stratification* induced by logarithmic vecotr fields [1]. Especially, when X is purely dimensional and has isolated singularity in 0, then $\{0\}$ and the connected components of $X\setminus\{0\}$ form a holonomic logarithmic stratification of X. And this stratification is a Whitney stratification.

Let $\mathcal{S} = \{S_\alpha\}$ be an analytic stratification of X, $f : (X,0) \longrightarrow (\mathbb{C},0)$ an analytic function germ. The *critical locus* $L_f^{\mathcal{S}}$ of f relative to the stratification $\mathcal{S}$ is the union of the closure of the critical loci of f restricted to each of the strata S_α, namely, $L_f^{\mathcal{S}} = \bigcup_\alpha \overline{L_{f|S_\alpha}}$. If $\dim L_f^{\mathcal{S}} = 0$, we say f is (or defines) an *isolated singularity* on $(X,0)$. Otherwise, f is (or defines) a *non-isolated singularity* on $(X,0)$. If $L_f^{\mathcal{S}}$ is a smooth curve, we say f is (or defines) a *line singularity* on $(X,0)$. In this article, we always use the logarithmic stratification to define singularities of functions.

All the functions whose critical loci contain L form an ideal of $\mathcal{O}$

$$\int_X \mathfrak{g} := \{f \in \mathcal{O} \mid (f) + J_X(f) \subset \mathfrak{g}\},$$

called the *primitive ideal* of $\mathfrak{g}$ (cf. [15, 14, 6]). This ideal collects all the functions whose zero level surfaces are tangent to X_{reg} along L. Obviously $\mathfrak{g}^2 + \mathfrak{h} \subset \int_X \mathfrak{g} \subset \mathfrak{g}$.

1.2. For $f \in \int_X \mathfrak{g}$ we define an ideal $J_X(f) := \{\xi(f) \mid \xi \in D_X\}$, called *Jacobian ideal* of f. Call $\mathfrak{g}/(J_X(f)+\mathfrak{h})$ the *Jacobian module* of f on X, and its dimension over $\mathbb{C}$ is called the *Jacobian number* of f on X and is denoted by $j(f) := \dim_{\mathbb{C}} \mathfrak{g}/(J_X(f)+\mathfrak{h})$. If $X_{\mathrm{sing}} \subset \{0\}$ and $\dim L = 1$, it is known [6] that $j(f) < \infty$ if and only if the transversal singularity type of f along $L \setminus \{0\}$ is A_1.

The $\mathcal{O}_L$-module $M := \bar{\mathfrak{g}}/\bar{\mathfrak{g}}^2 \cong \mathfrak{g}/(\mathfrak{g}^2+\mathfrak{h})$ is called the *conormal module* of $\bar{\mathfrak{g}}$ (as an ideal of $\mathcal{O}_X$). Denote $T := \int_X \mathfrak{g}/(\mathfrak{g}^2+\mathfrak{h})$, $N := \mathfrak{g}/\int_X \mathfrak{g}$. We have the exact sequence of $\mathcal{O}_L$-modules

$$0 \longrightarrow T(M) \longrightarrow M \longrightarrow N \longrightarrow 0.$$

If L does not contain any irreducible components of X_{sing}, $T(M)$ is the torsion submodule of M. In case $\dim L = 1$, $T(M)$ has finite length, called *torsion number* of (L,X), denoted by $\lambda(L,X)$. See [9] for generalizations of primitive ideals and torsion numbers.

1.3. Let L be a line (i.e. smooth curve). We choose L to be the x-axis of the local coordinate system in $(\mathbb{C}^{n+1},0)$. Then L is defined by ideal $\mathfrak{g} = (y_1,\dots,y_n)$. For a function $f \in \mathfrak{g}^2$, we have $f = \sum_{k,l=1}^{n} h_{kl}y_ky_l$ with $h_{kl} = h_{lk}$. Let $U = \{u := (u_{kl}) \in \mathbb{C}^{n^2} \mid u_{kl} = u_{lk}\}$, and $V = \mathbb{C}^{mn}$

with coordinates $v = (v_{jk})_{1 \le j \le m, 1 \le k \le n}$. Let $s = (u, v)$ be the coordinates of $S = U \times V$. Define $f_s(z) := f + q(s, z)$ with

$$q(s,z) := \sum_{k,l=1}^{n} \left(u_{kl} + \sum_{j=1}^{m} z_j v_{jk} \delta_{kl} \right) y_k y_l,$$

where δ_{kl} is Kronecker's delta, $z_0 := x, z_j := y_j (1 \le j \le n)$ are the local coordinates of $(\mathbb{C}^{n+1}, 0)$.
The following proposition is a generalization of a result due to Siersma-Pellikaan [17, 16], and the proof is similar to [16](cf. [6]).

Proposition 1. *Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be an icis of pure dimension $n - p + 1$, and $(L, 0) \subset (\mathbb{C}^{n+1}, 0)$ be a line. If $j(f) < \infty$. Then there exists a Zariski open dense subset $S' \subset U \times V$ such that*

(1) *For any $s \in S'$, f_s has only isolated Morse points on $X \setminus L$, and only A_∞ and D_∞ type singularities on $L \setminus \{0\}$;*
(2) *The module N is free, and if the images of $y_{p+1}, \dots, y_n$ form the basis of N,*

$$\delta := \sharp D_\infty = \dim_{\mathbb{C}} \frac{\mathcal{O}_L}{(\det(h_{kl})_{p+1 \le k,l \le n})}. \qquad \square$$

We say that f_s is *a good deformation* of f.

1.4. Let B_ϵ denote an open ball of radius ϵ centered at 0, Δ_η denote an open disk in $\mathbb{C}$ with center 0 and radius η. Let ϵ and η be admissible for the Milnor fibration of f. Namely, there exists the following local trivial topological fibration, the *Milnor fibration*

$$f : \bar{B}_\epsilon(0) \cap X \cap f^{-1}(\bar{\Delta}_\eta^*) \longrightarrow \bar{\Delta}_\eta^*,$$

where $\bar{\Delta}_\eta^* = \bar{\Delta}_\eta \setminus \{0\}$. The fibre F of this fibration is called *the Milnor fibre* of f. The Milnor fibre F^c of f_s is called the *central type* of the Milnor fibre F of f. The following proposition is a generalization of a result of Siersma [17, 18]. The proof of it can be found in [6]

Proposition 2. *Let L and X be the same as in Proposition 1 , and f_s be a good deformation of f. Let ϵ and η be admissible for the Milnor fibration of f. Then for $s \in S'$ with $|\, s \,|, \eta$ and ϵ sufficiently small, the map*

$$f_s : \bar{B}_\epsilon \cap f_s^{-1}(\bar{\Delta}_\eta) \longrightarrow \bar{\Delta}_\eta$$

has the following properties:

(1) *For all $t \in \bar{\Delta}_\eta$, $f_s^{-1}(t) \pitchfork \partial \bar{B}_\epsilon$ (as stratified spaces);*
(2) *For every $t \in \partial \bar{\Delta}_\eta$, and hence for every $t \in \bar{\Delta}_\eta \setminus \{$critical values of $f_s\}$, there is a homeomorphism: $F := f^{-1}(t) \cong \hat{F} := f_s^{-1}(t)$;*
(3) *There is a homeomorphism: $f^{-1}(\bar{\Delta}_\eta) \cong f_s^{-1}(\bar{\Delta}_\eta)$;*
(4) *Let F^0 be the intersection of $\hat{F}$ with a sufficiently small tubular neighborhood T of L such that inside T there is no Morse type points. Then F^0 can be obtained from F^c by attaching n-cells along a transversal vanishing cycle of F^c, the number of the n-cells is $2\,\sharp D_\infty$;*
(5) *If $\dim X = n - p + 1 > 3$ and F^c is simply connected $\hat{F} \simeq F^0 \vee S^{n-p} \vee \cdots \vee S^{n-p}$, the number of S^{n-p} is the number of Morse point on $X \setminus L$: $\sharp A_1$;*
(6) *If $\dim X = 2$ and F^c is connected $\hat{F} \simeq F^c \vee S^1 \vee \cdots \vee S^1$, the number of S^1 is $\sharp A_1 + 2\,\sharp D_\infty - 1$.* $\square$

In this article we study mainly the central type F^c of functions defining isolated line singularities on a two dimensional non-degenerate *icis* .

1.5. Let $g : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ be an analytic function germ. Let $g := \sum_\nu a_\nu z^\nu$ be the Taylor expansion of a representative of g. The *Newton polyhedron* $\Gamma_+(g)$ (with respect to the local coordinate z) is by definition the convex hull of $\bigcup_{\{\nu | a_\nu \neq 0\}} \{\nu + \mathbb{R}^{n+1}\}$. The *Newton boundary* $\Gamma(g)$ (with respect to the local coordinate z) is by definition the collection of all the compact facets of $\Gamma_+(g)$.

Let $P \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{n+1}, \mathbb{Z})$ with non-negative integral coordinates $p_0, \ldots, p_n$, and be denoted by $P = {}^{\mathrm{T}}(p_0, \ldots, p_n) \geq 0$, called *positive covector.* As an $\mathbb{R}$-linear function on $\mathbb{R}^{n+1}$, the restriction of P to $\Gamma_+(g)$ has a minimal value, denoted by $d(P; g)$. Denote also $\Delta(P; g) = \{z \in \Gamma_+(g) \mid P(z) = d(P; g)\}$. The *face function* of g with respect to P is by definition $g_P(z) = g_{\Delta(P;g)} := \sum_{\nu \in \Delta(P;g)} a_\nu z^\nu$.

Let X be a complete intersection defined by $\mathcal{O}$-regular sequence $h_1, \ldots, h_p$. The Newton polyhedron $\Gamma_+(h_1, \ldots, h_p)$ of X is by definition the mixed sum of $\Gamma_+(h_j)$, and the Newton boundary $\Gamma(h_1, \ldots, h_p)$ of X is the mixed sum of $\Gamma(h_j)$.

Two positive covector P, Q are equivalent if and only if $\Delta(P; h_j) = \Delta(Q; h_j)$ for $j = 1, \ldots, p$. The *dual Newton diagram* $\Gamma^*(h_1, \ldots, h_p)$ of X is a collection of all the equivalent classes of positive covectors under the aforementioned equivalence.

X is called *a non-degenerate complete intersection* (with respect to the local coordinate z) if $X \cap \mathbb{C}^{*n+1}$ is a reduced non-singular complete intersection in the complete torus $\mathbb{C}^{*n+1}$.

For more systematical introduction to toric modifications of non-degenerate complete intersections, we refer the reader to [12], where the notions and notations used in this article without explanations can be found.

2. Lines on singular spaces

2.1. Let $(X, 0)$ be a reduced analytic space germ in $(\mathbb{C}^{n+1}, 0)$. A smooth curve germ $(L, 0)$ in $(\mathbb{C}^{n+1}, 0)$ is call a line. If $L \setminus \{0\} \subset X_{\mathrm{reg}}$, we say that X *contains (or has) a line* passing through O.

On a singular space X in $\mathbb{C}^{n+1}$ one can not always find a line passing through (not contained in) the singular locus of X. Gonzalez-Sprinberg and Lejeune-Jalabert [4, 5] proved a criterion for the existence of smooth curve on any (two dimensional) surface.

The existence and number of lines on surfaces with isolated simple singularities and on Brieskorn-Pham surfaces have been studied in [8, 7].

2.2. Let $\mathcal{R}$ be the group of all the local automorphisms of $(\mathbb{C}^{n+1}, 0)$. $\mathcal{R}_L := \{\phi \in \mathcal{R} | \phi(L) = L\}$ is a subgroup of $\mathcal{R}$. Define ${}_L\mathcal{K} := \mathcal{R}_L \rtimes \mathcal{C}$, the semi-product of $\mathcal{R}_L$ with the contact group $\mathcal{C}$ [11]. This group has an action on the space $\mathfrak{m}\mathfrak{g}\mathcal{O}^p$ consisting of mapping germs $h : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}^p, 0)$ with components $h_j \in \mathfrak{m}\mathfrak{g}$. For $h = (h_1, \ldots, h_p) \in \mathfrak{m}\mathfrak{g}\mathcal{O}^p$, we define an analytic space $X = \mathcal{V}(\mathfrak{h})$, where $\mathfrak{h}$ is the ideal generated by $h_1, \ldots, h_p$. The image of the morphism:

$$\mathcal{O}^{n+1} \xrightarrow{dh^*} \mathcal{O}^p$$

is denoted by $\mathrm{th}(h)$, where dh is the differential of h. Define

$$\tilde{\lambda} := \tilde{\lambda}(L, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}^p}{\mathrm{th}(h) + \mathfrak{g}\mathcal{O}^p}.$$

Remark that $\tilde{\lambda}$ is ${}_L\mathcal{K}$-invariant.

Let $x, y_1, \ldots, y_n$ be the local coordinates of $(\mathbb{C}^{n+1}, 0)$. Let L be the x-axis defined by $\mathfrak{g} = (y_1, \ldots, y_n)$. Then L can be defined by $\mathfrak{g} = (y_1, \ldots, y_n)$.

2.3. **Theorem.** *Let L be a line on an icis X defined by $\mathfrak{g}$ and $\mathfrak{h}$ as above. Then h is ${}_L\mathcal{K}$-equivalent to a mapping germ with components*

$$\tilde{h}_j = b_j y_j \mod \mathfrak{g}^2, \quad j = 1, \dots, p \tag{2.3.1}$$

where $b_j \notin \mathfrak{g}$. Moreover

$$\lambda(L, X) = \tilde{\lambda}(L, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_L}{(\bar{b})} = \sum_{j=1}^{p} \lambda_j. \tag{2.3.2}$$

where $\bar{b}$ is the image of $b := b_1 \cdots b_p$ in $\mathcal{O}_L$, and λ_j is the order of $\bar{b}_j$ in $\mathcal{O}_L$.

Proof. Since $L \subset X$, $\mathfrak{h} \subset \mathfrak{g}$. Then for a given generator set $\{h_1, \dots, h_p\}$ of $\mathfrak{h}$, we have

$$h_k \equiv \sum \bar{b}_{kj} y_j \mod \mathfrak{g}^2, \quad k = 1, \dots, p.$$

where $\bar{b}_{kj} \in \mathcal{O}_L$, and for fixed k, $\bar{b}_{kj}$'s are not all zero since $X_{\text{sing}} = \{0\} \subsetneqq L$. Since $\mathcal{O}_L$ is a principal ideal domain, by changing the indices, we can assume $\bar{b}_{11} \mid \bar{b}_{kj}$. Let

$$y_1' = y_1 + \sum_{j=2}^{n} \frac{\bar{b}_{1j}}{\bar{b}_{11}} y_j.$$

Then

$$h_1 \equiv \bar{b}_{11} y_1' \quad (\mod \mathfrak{g}^2).$$

Let

$$h_k' = h_k - \frac{\bar{b}_{k1}}{\bar{b}_{11}} h_1, \quad k = 2, \dots, p.$$

Repeat the above argument will prove the first part of the theorem.

Consider the exact sequence

$$\mathcal{O}^{n+1} \xrightarrow{dh^*} \mathcal{O}^p \longrightarrow \operatorname{coker}(dh^*) \longrightarrow 0.$$

By tensoring with $\mathcal{O}_L$, we have exact sequence

$$\mathcal{O}_L^{n+1} \xrightarrow{d\bar{h}^*} \mathcal{O}_L^p \longrightarrow \operatorname{coker}(d\bar{h}^*) \longrightarrow 0.$$

However by the expression of $\tilde{h}_k$'s above, this is just

$$\mathcal{O}_L^p \xrightarrow{d\bar{h}^*} \mathcal{O}_L^p \longrightarrow \frac{\mathcal{O}^p}{th(h) + \mathfrak{g}\mathcal{O}^p} \longrightarrow 0.$$

Since $\bar{b} \neq 0$, by [3, A.2.6], we have the formula for λ. □

3. Toric modifications of line singularities on surfaces

3.1. In this section we study the toric modifications of functions with lines singularities on surfaces. Let $z_0 := x, z_1 := y_1, \dots, z_n := y_n$ be the local coordinates of $(\mathbb{C}^{n+1}, 0)$. Let $L = \{y_1 = \cdots = y_n = 0\}$ be contained in $X = \{z \in \mathbb{C}^{n+1} \mid h_1(z) = \cdots = h_{n-1}(z) = 0\}$, the germ at 0 of a two dimensional irreducible non-degenerate *icis*. Assume that h_i takes the form in (2.3.1). Consider a function germ $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ such that $V = X \cap f^{-1}(0)$ is a one dimensional non-degenerate complete intersection. Let $\hat{\pi} : \mathcal{X} \longrightarrow \mathbb{C}^{n+1}$ be the admissible toric modification for V associated with a small admissible regular simplicial cone subdivision Σ^*. Denote by $\tilde{X}$ the strict transform of X by $\hat{\pi}$. We denote by E_j the unit vector along the j-th axis of $\mathbb{R}^{n+1}$. For $P \in \Sigma^*$, denote by $\hat{E}(P)$ the exceptional divisor of $\hat{\pi}$, and $D(P) := \hat{E}(P) \cap \tilde{X}$. For a vector $Q = (q_0, q_1, \dots, q_n)$, define $I(Q) := \{j \mid q_j = 0\}$. Let $|A|$ denote the cardinality of a finite set A.

The following theorem generalizes [12, III(6.2)].

3.2. **Theorem.** *Let X be a 2-dimensional non-degenerate icis defined by $\mathfrak{h} = (h_1, \dots, h_{n-1})$ with the form of (2.3.1).*

(1) *There exists at least one primitive integral covector $Q = (0, p_1, \dots, p_n)$ in $\Gamma^*(\mathfrak{h})$, the dual Newton diagram of $\mathfrak{h}$, such that $\dim(\Delta(Q; \mathfrak{h}) \cap \Gamma(\mathfrak{h})) = n-1$;*

(2) *Assume that X is $(n-1)$-convenient. If on each $\mathrm{Cone}(E_0, \dots, E_{i-1}, \check{E}_i, E_{i+1}, \dots, E_n)$, there exist at most one point Q which belongs to $\Gamma^*(\mathfrak{h})$, such that $\dim(\Delta(Q; \mathfrak{h}) \cap \Gamma(\mathfrak{h})) \geq n-1$, then the small toric modification $\pi : \tilde{X} \longrightarrow X$ for $\mathfrak{h}$ is a good resolution of X.*

Proof. The first statement follows straightaway from Theorem 2.3.

The proof of (2) is similar to that of [12, III(6.2)].

Suppose X is not convenient, for each vertex $Q \in \mathrm{vertex}(\Sigma^*)$ with $|I(Q)| = 1$, $\hat{\pi} : \hat{E}(Q) \longrightarrow \mathbb{C}^{I(Q)} := \{z \in \mathbb{C}^{n+1} \mid z_j = 0 \text{ if } j \notin I(Q)\}$ is a surjective morphism with fibre $\mathbb{P}^1$. Since X is $(n-1)$-convenient, π is biholomorphic over $X \cap \left(\mathbb{C}^{n+1} \setminus \bigcup_{|I|=1} \mathbb{C}^I\right)$. Take such a point Q on, for instance, $\mathrm{Cone}(E_1, \dots, E_n)$ with $\dim(\Delta(Q; \mathfrak{h}) \cap \Gamma(\mathfrak{h})) \geq n-1$ by assumption. Hence the exceptional divisor $D(Q)$ is the only non-empty divisor which is surjectively mapped onto $L = \{y_1 = \dots = y_n = 0\}$. As $\dim D(Q) = 1$, the fibre of π on L consists of finite points. Indeed it contains exactly one point by [12, III(6.2.1)]. By using Riemann's removable singularity theorem, $\pi : \tilde{X} \setminus \pi^{-1}(0) \longrightarrow X \setminus \{0\}$ is a biholomorphism. □

The following theorem is a slight generalization of [12, III(3.4.11)].

3.3. **Theorem.** *Let $X, V, \hat{\pi}$ be as above. Suppose that X is $(n-1)$-convenient, and on each $\mathrm{Cone}(E_0, \dots, E_{i-1}, \check{E}_i, E_{i+1}, \dots, E_n)$, there exist at most one point which belongs to $\Gamma^*(\mathfrak{h})$, such that $\dim(\Delta(Q; \mathfrak{h}) \cap \Gamma(\mathfrak{h})) \geq n-1$. Assume that*

($\sharp''$) $\quad Q \in \mathrm{vertex}(\Sigma^*), 1 < |I(Q)| < n, \hat{E}(Q) \cap \tilde{X} \neq \emptyset \Longrightarrow d(Q; f) > 0.$

Then the restriction $\pi : \tilde{X} \longrightarrow X$ of $\hat{\pi}$ is a good resolution of f.

Proof. Since the dual Newton diagram $\Gamma^*(h_1, \dots, h_{n-1}, f)$ is finer than $\Gamma^*(h_1, \dots, h_{n-1})$, the smoothness of $\tilde{X}$ is obvious by [12, III(3.4)] as Σ^* is admissible for $\Gamma^*(h_1, \dots, h_{n-1})$. And the map

$$\pi : \tilde{X} \cap \left(\mathcal{X} \setminus \bigcup_{|T(\sigma)|<n} \hat{E}(\sigma)\right) \longrightarrow X \cap \left(\mathbb{C}^{n+1} \setminus \bigcup_{|I|<n} \mathbb{C}^I\right)$$

is biholomorphic.

However, for $Q \in \mathrm{vertex}(\Sigma^*) \setminus \{E_0, \dots, E_n\}$, if $1 < |I(Q)| < n$, then $\hat{E}(Q) \cap \tilde{X}$ is included in the zero locus of $f \circ \pi$. If $|I(Q)| = 1$, then Q is on a $\mathrm{Cone}(E_0, \dots, E_{i-1}, \check{E}_i, E_{i+1}, \dots, E_n)$. Hence even though $\hat{E}(Q) \cap \tilde{X}$ might not be included in the zero locus of $f \circ \pi$, π is still bijective on $\hat{E}(Q) \cap \tilde{X}$ by Theorem 3.2. Hence $\pi : \tilde{X} \longrightarrow X$ is a good resolution of f. □

3.4. **Remark.** Note that in case X is a surface in $\mathbb{C}^3$, the ($\sharp''$) condition is empty.

3.5. **The zeta function.** Let F_f be the Milnor fibre of f. We are interested in the zeta function $\zeta_f(t)$ of the Milnor fibration of f.

Let f, X, V be the same as before. Assume ($\sharp''$). Let Σ^* be the small regular simplicial subdivision of $\Gamma^*(h, f)$, where $h = (h_1, \dots, h_{n-1})$. Let $\hat{\pi} : \mathcal{X} \longrightarrow \mathbb{C}^{n+1}$ be the associated toric modification map. By Theorem 3.3, the restriction $\pi : \tilde{X} \longrightarrow X$ of $\hat{\pi}$ to the strict transform $\tilde{X}$ of X is a good resolution of f.

For $P \in \mathrm{vertex}(\Sigma^*)$, denote by

$$D(P) := \hat{E}(P) \cap \tilde{X}, \qquad E(P) := \hat{E}(P) \cap \tilde{V},$$

$$D(P)^* := D(P) \setminus \left(\bigcup_{P' \neq P} D(P') \right), \qquad E(P)^* := E(P) \setminus \left(\bigcup_{P' \neq P} E(P') \right),$$

$$\mathcal{V}^+(f) := \{P \in \text{vertex}(\Sigma^*) \mid d(P;f) > 0\}.$$

The total transform is

$$\tilde{V}^{\text{tot}} = \tilde{V} + \sum_{P \in \mathcal{V}^+(f)} d(P;f) D(P).$$

Note that the multiplicity of $\pi^* f$ along $D(P)$ is $d(P;f)$. Let

$$\check{D}(P) = \left(D(P) \setminus \left(E(P) \cup \bigcup_{P' \in \mathcal{V}^+(f) \setminus \{P\}} D(P') \right) \right) \cap \pi^{-1}(0).$$

By A'Campo formula we have the zeta function and Lefschetz number

$$\zeta_f(t) = \prod_{P \in \mathcal{V}^+(f)} \left(1 - t^{d(P;f)}\right)^{-\chi(\check{D}(P))}, \quad \Lambda_f^k = \sum_{d(P;f)|k} d(P;f) \chi(\check{D}(P)) \quad (k \geq 1).$$

3.6. Since $\tilde{X}$ is a surface, $D(P)$ is a smooth curve. Hence $D(P) \cap D(Q)$ and $E(P)$ are at most zero dimensional for all $P, Q \in \mathcal{V}^+(f)$. Define $e(P) := |E(P)|$, the cardinality of the set $E(P)$. $\tilde{e}(P,Q) := |\tilde{X} \cap \hat{E}(P) \cap \hat{E}(Q)|$. Then

$$\chi(\check{D}(P)) = \chi(D(P)) - e(P) - \sum_{Q \in \mathcal{V}^+(f)} \tilde{e}(P,Q).$$

Let $\{S_j\}_{j=1}^k$ be a set of simplexes in $\mathbb{R}^n$. We say that $\{S_j\}_{j=1}^k$ satisfies the (A_0) condition if for any $I \subset \{1, \dots, k\}$, the dimension of the mixed sum $\dim(\sum_{j \in I} S_j) \geq |I|$.

Let $P \in \mathcal{V}^+(f)$ be strictly positive. By [12, IV(6.2)], we know that

1) $e(P) > 0$ if and only if $\{\mathbf{\Delta}(P;h_1), \dots, \mathbf{\Delta}(P;h_{n-1}), \mathbf{\Delta}(P;f)\}$ satisfies the (A_0) condition;
2) $\tilde{e}(P,Q) > 0$ if and only if both $\{\mathbf{\Delta}(P;\mathfrak{h})\}$ and $\{\mathbf{\Delta}(Q;\mathfrak{h})\}$ satisfies the (A_0) condition, $\text{Cone}(P,Q) \subset \Sigma^*$ and $\dim \mathbf{\Delta}(P;\mathfrak{h}) \cap \mathbf{\Delta}(Q;\mathfrak{h}) \geq n-2$.

Hence we have (see [12, IV§7])

$$e(P) = \chi(E(P)) = \chi(E^*(P)) = n! V_n(\mathbf{\Delta}(P;h_1), \dots, \mathbf{\Delta}(P;h_{n-1}), \mathbf{\Delta}(P;f)),$$

where $V_n(\cdots)$ is the Minkowski's mixed volume.

Let $\sigma := \text{Cone}(P, Q, P_2, \dots, P_n) \in \Sigma^*$ be a regular simplex. By [12, III(3.4.10)], in the coordinate chart $\mathbb{C}_\sigma^{n+1}$

$$\begin{aligned} \hat{E}(P) \cap \hat{E}(Q) \cap \tilde{X} &= \{(0,0,y'_\sigma) \mid \tilde{h}_{1,\bar{P},\sigma}(y'_\sigma) = \cdots = \tilde{h}_{n-1,\bar{P},\sigma}(y'_\sigma) = 0\} \\ &= \{y'_\sigma \in \mathbb{C}^{*n-1}_\sigma \mid \tilde{h}_{1,\bar{P},\sigma}(y'_\sigma) = \cdots = \tilde{h}_{n-1,\bar{P},\sigma}(y'_\sigma) = 0\}, \end{aligned}$$

where $\bar{P} = P + Q$, and $\tilde{h}_{\alpha,\bar{P},\sigma}(y'_\sigma) := h_{\alpha,\bar{P},}(\hat{\pi}_\sigma(y_\sigma)) / \prod_{j=0}^n y_{\sigma,j}^{d(P_j;h_\alpha)}$. Hence

$$\tilde{e}(P,Q) = \chi(\hat{E}(P) \cap \hat{E}(Q) \cap \tilde{X}) = (n-1)! V_{n-1}(\mathbf{\Delta}(\bar{P};\mathfrak{h})).$$

If $P \in \mathcal{V}^+(f)$ is not strictly positive. By [12, IV(6.5)],

3) $e(P) = 0$ since $E(P)$ is empty;
4) $\tilde{e}(P,Q) > 0$ if and only if both $\{\mathbf{\Delta}(P';h_{1,P}), \dots, \mathbf{\Delta}(P';h_{n-1,P})\}$ and $\{\mathbf{\Delta}(Q;\mathfrak{h})\}$ satisfies the (A_0) condition, $\text{Cone}(P,Q) \subset \Sigma^*$ and $\dim \mathbf{\Delta}(P;\mathfrak{h}) \cap \mathbf{\Delta}(Q;\mathfrak{h}) \geq n-2$.

4. Lines singularities on certain surfaces

4.1. **Lemma.** *Let $(X,0)$ be a 2-dimensional icis containing the line L. Let $f \in \mathfrak{g}$ be a function with $j(f) < \infty$ such that $h_1, \dots, h_{n-1}, f$ define a complete intersection. Then for generic $s \in S$, the Milnor fibre F^c of f_s is homotopoy equivalent to the Milnor fibre of $q(s,z)$, if the Milnor fibre of q is connected.* **Proof.** We give the outline of the proof. Note that in this case $N = M/T(M)$ is free $\mathcal{O}_L$-module, and q is defined by

$$q(s,z) := \sum_{k,l=1}^{n} \left(u_{kl} + \sum_{j=0}^{m} z_j v_{jk} \delta_{kl} \right) y_k y_l,$$

where $z_0 = x$, and $z_j = y_j$ for $j > 0$. Hence for generic parameter value f_s is a good deformation of f. Fix such an s, define $\tilde{f}_t := t \cdot f + q(s,z)$. Then one proves that for $t \in \mathbb{C} \setminus \{$ finite points $\neq 0, 1\}$, $\tilde{f}_t$ has no critical points outside L and has only A_∞ type singularity on $L \setminus \{0\}$ in a small neighborhood of 0. By using a generalized version of *additivity of vanishing homology* (see *e.g.* [18, 6]), one proves that F^c and the Milnor fibre of q have the same homology, which implies that they also have the same homotopy type since we assume the connectedness of the Milnor fibre of q □

4.2. Denote by

$$q_1(u,z) := \sum_{k,l=1}^{n} u_{kl} y_k y_l.$$

Note that all the terms in $q - q_1$ are "above" the Newton boundary $\Gamma(q_1)$ of q_1 The following lemma is a corollary of Damon [2, Corollary 1].

4.3. **Lemma.** *If, for a fixed u, $h_1, \dots, h_{n-1}, q_1$ define a non-degenerate complete intersection, then the Milnor fibres of q and q_1 are homeomorphic.* □

4.4. In the remainder of this section we study certain functions whose zero level surfaces have higher order contact with a surface along a line contained therein. Let L be a line in $\mathbb{C}^3$ defined by $\mathfrak{g} = (y,z)$, and contained in a surface X defined by $\mathfrak{h} = (h) \subset \mathfrak{g}$. Assume that $X_{\mathrm{sing}} = \{0\}$. Define $f^{(\varsigma)} = \sum_{i=0}^{\varsigma} a_i y^{\varsigma - i} z^i \in \mathfrak{g}^\varsigma$, where $(a_0, \dots, a_\varsigma) \in \mathbb{C}^{\varsigma+1}$ are generic.

Let $\boldsymbol{\Sigma}^*$ be a regular simplicial cone subdivision of $\Gamma^*(h)$, the dual Newton diagram of h, such that the restriction of $\boldsymbol{\Sigma}^*$ to each two dimensional cone $\mathrm{Cone}(P,Q)$ is obtained by the canonical way as described in [12, II§2]. Associated with this $\boldsymbol{\Sigma}^*$ there is a toric modification $\hat{\pi} : \mathcal{X} \longrightarrow \mathbb{C}^3$, called *canonical toric modification.* The restriction π of $\hat{\pi}$ to the strict transform $\tilde{X}$ of X under $\hat{\pi}$ is called the canonical toric modification of X. Denote by $\Gamma^*(h)_2^+$ the union of two dimensional cones $\sigma_2 = \mathrm{Cone}(P,Q)$ of $\Gamma^*(h)$ such that for any $P_i \in \sigma_2 \cap \boldsymbol{\Sigma}^* \setminus \{P, Q\}$, $P_i >> 0$ and $\dim(\boldsymbol{\Delta}(P_i; h)) \geq 1$. Let $\mathcal{G}'_X$ be the graph of $\boldsymbol{\Sigma} | \Gamma^*(h)_2^+$. The dual resolution graph $\mathcal{G}_X$ of X can be obtained from $\mathcal{G}'_X$ in the way described by [12, III(6.3)].

Now we study $\Gamma_+(h)$ more carefully. In $\Gamma_+(h)$ we have a non-compact face $Q : qy + z = q$ by [12, III(6.1)] with vertices $A(a,1,0)$ and $C(c,0,q)$ (see the proof of loc. cit.). Let $P : \alpha x + \beta y + \gamma z = \delta$ be the face in $\Gamma_+(h)$ which intersects with Q along AC. Assume that $\gcd(\alpha, \beta, \gamma) = 1$. Hence in the dual Newton diagram $\Gamma^*(h)$ we have the point $Q = {}^{\mathrm{T}}(0, q, 1)$ on the edge $E_2 E_3$. And the $\mathrm{Cone}(P,Q)$ also belongs to $\Gamma^*(h)$. One sees that $P = {}^{\mathrm{T}}(\alpha, \delta - a\alpha, \frac{\delta - c\alpha}{q})$ and $\det PQ = \alpha$.

4.5. **Lemma.** *The divisor $E(Q)$ is a reduced smooth curve on $\tilde{X}$ intersecting the exceptional divisor $E(Q_1)$ transversally, and is biholomorphic to L under π. And $d(Q; f^{(\varsigma)}) = \varsigma$.*

Proof. Let $Q_1 = \frac{1}{\alpha}(P + k_1 Q) = {}^{\mathrm{T}}(1, q_1, q_2)$ be the first point ("near" Q) in the canonical subdivision of PQ. One sees that

$$q_1 = \frac{\delta - \alpha a + k_1 q}{\alpha}, \quad q_2 = \frac{\delta - \alpha c + k_1 q}{q\alpha},$$

where k_1 is the smallest integer such that $0 < k_1 < \alpha$, and both q_1 and q_2 are integers. Then the simplex σ determined by QQ_1E_2 is regular for h. The restriction of $\tilde{\pi}$ to this chart is

$$\tilde{\pi}_\sigma : x = v, y = u^q v^{q_1} w, z = uv^{q_2},$$

then

$$h \circ \tilde{\pi}_\sigma = uv^{a+q_1}(1 + w + \cdots).$$

One sees that $u = 0, v = t$ defines $\tilde{L}$, which is mapped on to L biholomorphically. □

4.6. As the Newton polyhedron $\Gamma_+(f^{(\varsigma)})$ consists of one non-compact face: $U : y + z = \varsigma$, we assume from now on that $f^{(\varsigma)} = a_0 y^\varsigma + a_\varsigma z^\varsigma$. The Newton polyhedron $\Gamma_+(h, f^{(\varsigma)})$ consists of two kind of faces: 1) certain faces coming from the parallel transformations of the faces of $\Gamma_+(h) \cup \Gamma_+(f^{(\varsigma)})$; 2) the faces spanned by the parallel transformations in y-direction and z-direction of the edges of $\Gamma(h)$. A calculation shows that each face from class 2) has equation of the form $P' : \alpha' x + \beta' y + \beta' z = \gamma'$. Hence the dual Newton diagram $\Gamma^*(h, f^{(\varsigma)})$ is a subdivision of $\Gamma^*(h)$ by adding the point $U = {}^{\mathrm{T}}(0, 1, 1)$ to E_2E_3 and certain points of the form $P' = {}^{\mathrm{T}}(\alpha', \beta', \beta')$ to some two dimensional cone of $\Gamma^*(h)$.

Note that if all the points of form P' which are qualified to be added to $\Gamma^*(h)$ are equal to some points in $\Gamma^*(h)$, then the canonical toric modification of X is also a good resolution of $f^{(\varsigma)}$. And V and X have the same resolution graph (including the self intersection numbers of the exceptional divisors). Although in general this is not the case, the dual resolution graph $\mathcal{G}_V$ and the total dual resolution graph $\mathcal{G}_V^{\mathrm{tot}}$ of $f^{(\varsigma)}$ can be obtained from $\mathcal{G}_X$ by adding some vertices. To do this one only needs to identify the faces of the form P'. In the remainder of this section we will do this for certain classes of surfaces.

4.7. **Theorem.** *If X is a surface with isolated simple singularity and contains a line, the toric modification of X is already a good resolution of $f^{(\varsigma)}$ and the Milnor fibre of $f^{(\varsigma)}$ is a bouquet of 1-cycles for any integer $t > 0$. In particular, the Milnor fibre of any function f with $j(f) < \infty$ is a bouquet of 1-cycles. The zeta function $\zeta_{f^{(\varsigma)}}(t)$ and Milnor number $\mu(f^{(\varsigma)})$ are listed in table 1.*

Table 1

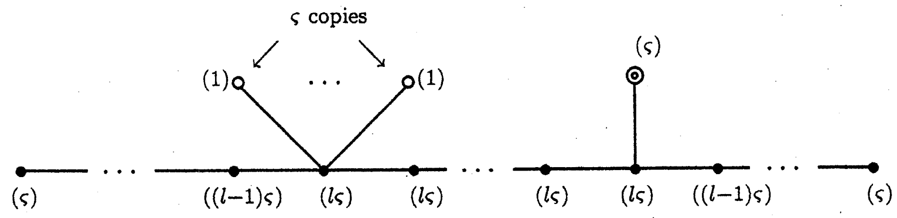
Type of X	Equations	λ	$\zeta_{f^{(\varsigma)}}(t)$	$\mu(f^{(\varsigma)})$
$A_{k,l}$	$x^l y + x^s z^2 + yz = 0$ $(k = 2l + s - 1, l \geq 1, s \geq 0)$	l	$\frac{(1-t^{l\varsigma})^{\varsigma+1}}{(1-t^\varsigma)^2}$	$l\varsigma^2+(l-2)\varsigma+1$
$D_{k,2}$	$x^2 y + y^{k-1} + z^2 = 0$ $(k \geq 4)$	2	$\frac{(1-t^{2\varsigma})^{\varsigma+1}}{(1-t^\varsigma)^2}$	$2\varsigma^2+1$
$D^*_{5,2}$	$x^2 y + xz^2 + y^2 = 0$	2	$\frac{(1-t^{3\varsigma})(1-t^{2\varsigma})^{\varsigma-1}}{(1-t^\varsigma)}$	$2\varsigma^2+1$
$D_{2l,l}$	$x^l y + xy^2 + z^2 = 0$ $(l \geq 3)$	l	$\frac{(1-t^{2\varsigma(l-1)})(1-t^{l\varsigma})^\varsigma}{(1-t^\varsigma)(1-t^{(l-1)\varsigma})}$	$l\varsigma(\varsigma+1)-2\varsigma+1$
$D_{2l+1,l}$	$x^l y + xz^2 + y^2 = 0$ $(l \geq 3)$	l	$\frac{(1-t^{(2l-1)\varsigma})(1-t^{l\varsigma})^{\varsigma-1}}{(1-t^\varsigma)}$	$l\varsigma(\varsigma+1)-2\varsigma+1$
$E_{6,2}$	$x^2 z + y^3 + z^2 = 0$	2	$(1-t^{4\varsigma})(1-t^{2\varsigma})^{\varsigma-2}$	$2\varsigma^2+1$
$E_{7,3}$	$x^3 y + y^3 + z^2 = 0$	3	$\frac{(1-t^{6\varsigma})(1-t^{3\varsigma})^{\varsigma-1}}{(1-t^{2\varsigma})}$	$3\varsigma^2+\varsigma+1$

Proof. By studying $\Gamma(h, f^{(\varsigma)})$ case by case, one sees that the resolution of X is already a good resolution of $f^{(\varsigma)} : X \longrightarrow \mathbb{C}$. One only need to resolve X. By toric modification (cf. [12]), we obtain a "canonical" resolution of X. The dual resolution graph $\mathcal{G}$ can be obtained by the way described in §4.4.

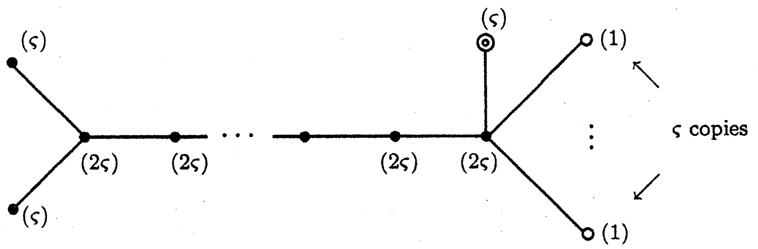
Note that the strict transform of $f^{(\varsigma)}$ only intersects with the reduced components of $\mathcal{Z}_X$. The weight of each component $E(T)$ of $\mathcal{Z}_X$ can be computed on the line $x + y = \varsigma$, the only compact 1-facet of $\Gamma^*(f^{(\varsigma)})$.

We include the total resolution graph of $f^{(\varsigma)}$. In the graphs, a bullet • denotes an (compact) exceptional divisor of the resolution of X. A small circle ∘ denotes a branch of the strict transform of V. A circled circle ⊚ denotes the lifting of L, the divisor corresponding to the point Q in §4.4. Each number in the parentheses denotes the multiplicity of $f^{(\varsigma)} \circ \pi$ along the divisor to which the number attached.

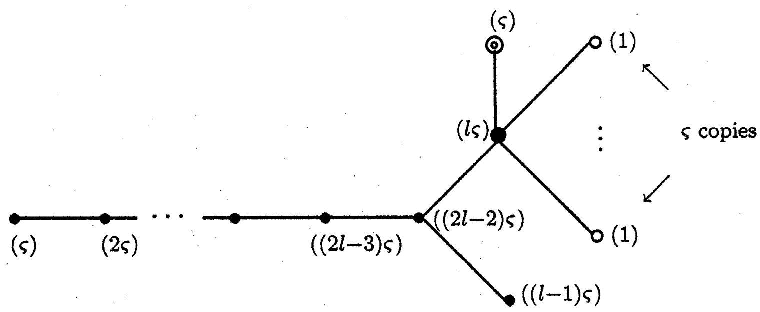
$\mathcal{G}^{\mathrm{tot}}_{A_{k,l}}$:



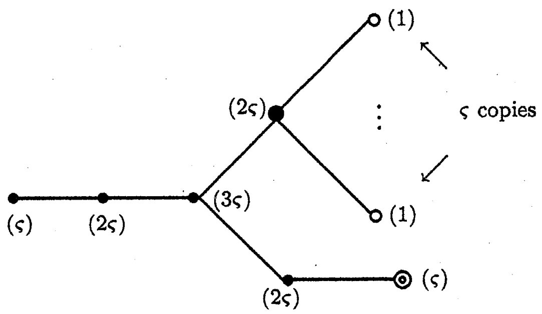
$\mathcal{G}^{\mathrm{tot}}_{D_{k,2}}$:



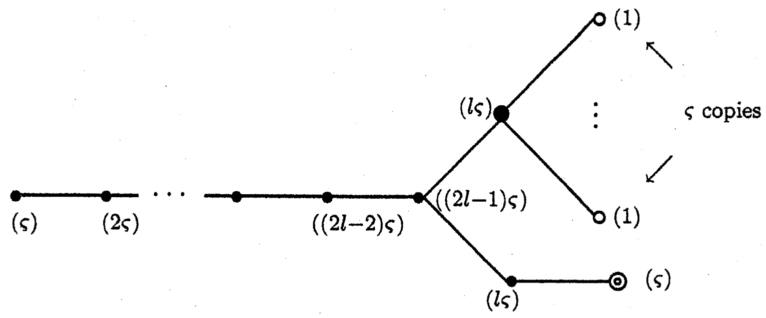
$\mathcal{G}^{\mathrm{tot}}_{D_{2l,l}}$:



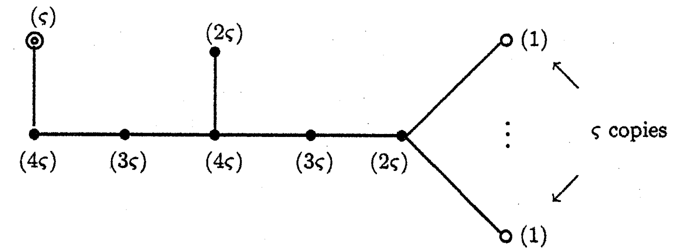
$\mathcal{G}^{\mathrm{tot}}_{D^*_{5,2}}$:



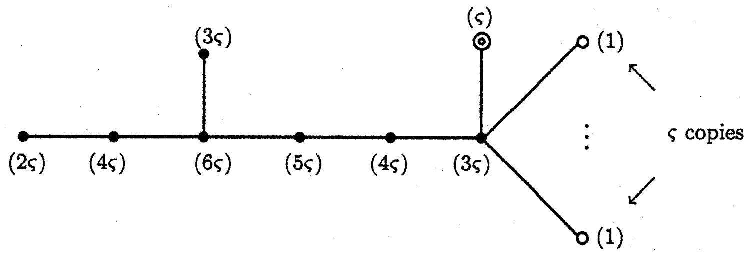
$\mathcal{G}^{\mathrm{tot}}_{D_{2l+1,l}}$:
$(l \geq 3)$



$\mathcal{G}^{\mathrm{tot}}_{E_{6,2}}$:



$\mathcal{G}^{\mathrm{tot}}_{E_{7,3}}$:



From the total resolution graphs we see immediately the zeta functions and the Euler-Poincaré characteristics. □

4.8. **Remark.** Among simple surface singularities only $A_k - D_k - E_6 - E_7$ type surfaces have lines and their definition equations are given in the table 1 (cf. [8]). If $\varsigma = 1$, the above theorem gives information about the hyperplane intersections of X by a generic plane passing through the line. If $\varsigma = 2$, the zeta functions and Milnor numbers are those of the central type of a function with line singularity and $j(f) < \infty$. One sees clearly how the torsion number ($\lambda = l$) enters the resolution data. The theorem also provides information about the topology of generic functions coming from $\bar{\mathfrak{g}}^{\varsigma}/\bar{\mathfrak{g}}^{\varsigma+1}$.

4.9. Let X be a Brieskorn-Pham surface $G(p,q,r) : h = x^p + y^q + z^r = 0$. Assume that $1 < p < q < r$ and $\gcd(p,q) = 1$. By [7], if $r > pq$ and $p \nmid r, q \nmid r$, there exists $\left[\frac{r}{pq}\right]$ different families of lines on $G(p,q,r)$. Let $\mathcal{L}_{T_{k+1}}$ be the family of lines with $\lambda = \lambda_{k+1} := (k+1)(p-1)q$ ($k = 0, 1, \ldots, \left[\frac{r}{pq}\right] - 1$). We first choose a line in $\mathcal{L}_{T_{k+1}}$ on $G(p,q,r)$ to be the last axis in a local coordinate system x', y', z' of $\mathbb{C}^3$. Then the line is defined by $\mathfrak{g} = (x', y')$. Define function $f^{(\varsigma)}_{k+1} := ax'^{\varsigma} + ay'^{\varsigma}$, where $\varsigma > 0$ is an integer as before, and a, b are generic constants. Then we consider the transformed function of $f^{(\varsigma)}_{k+1}$ under the inverse coordinate transformation. We still denote this function by $f^{(\varsigma)}_{k+1}$.

4.10. **Theorem.** *The Milnor fibre of* $f^{(1)}_{k+1}$ *is a bouquet of 1-cycles. The Milnor fibre of* $f^{(\varsigma)}_{k+1}$ *is not connected and consists of* ς *disjoint pieces. The zeta function is*

$$\zeta_{f^{(\varsigma)}_{k+1}}(t) = \frac{(1-t^{(k+1)p\varsigma})^p(1-t^{(k+1)p^2q\varsigma})}{(1-t^{p\varsigma})(1-t^{(k+1)p^2\varsigma})(1-t^{(k+1)pq\varsigma})},$$

and the Euler-Poincaré characeristic of the Milnor fibre is $\chi(f^{(\varsigma)}_{k+1}) = -\varsigma p(\lambda_{k+1}+k)$.

Proof. Note that $\Gamma^*(h)^+_2$ consists three arms: PE_1, PE_2 and PE_3. Let R_i, S_j and T_k denote the points added to these arms in order to get the canonical subdivision of the respective 2-simplex. One sees that (cf. [7]) the exceptional divisor corresponding to $T_{k+1} = {}^T((k+1)q, (k+1)p, 1)$, $(k = 0, \dots, \left[\frac{r}{pq}\right] - 1)$ are reduced. And they are the only reduced ones in $\mathcal{Z}_X$. The lines in $\mathcal{L}_{T_{k+1}}$ can be parameterized as

$$x = c^{kq}u_1^{\frac{1+(kp+\alpha)q}{p}}t^{(k+1)q}, \quad y = c^{kp}u_1^{kp+\alpha}t^{(k+1)p}, \quad z = cu_1t,$$

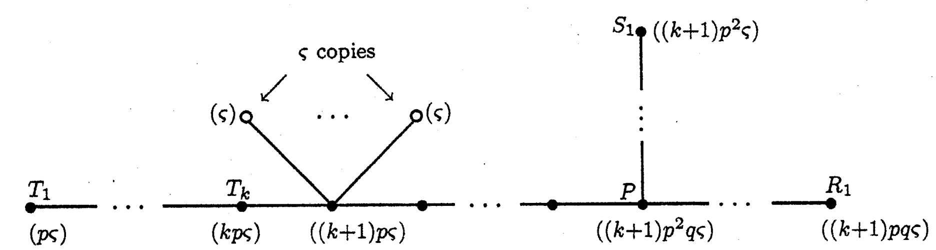
where u_1 is a unit satisfying $1 + u_1 + c^{r-kpq}u_1^{r-(kp+\alpha)q}t^{r-(k+1)pq} = 0$, and $0 \le \alpha < p$ such that $\frac{1+(kp+\alpha)q}{p}$ is an integer. The torsion number of the lines in $\mathcal{L}_{T_{k+1}}$ are: $\lambda_{k+1} := (k+1)(p-1)q$. Then

$$f^{(\varsigma)}_{k+1} = a(x - \bar{u}_1z^{(k+1)q})^\varsigma + b(y - \bar{u}_2z^{(k+1)p})^\varsigma,$$

where $\bar{u}_1$ and $\bar{u}_2$ are unit functions of z.

From the Newton boundary $\Gamma^*(h, f^{(\varsigma)}_{k+1})$, one sees that the canonical toric modification of X is a good resolution of $f^{(\varsigma)}_{k+1}$. The following is the total resolution graph.

$\mathcal{G}^{tot}_{G(p,q,r)}$:



From the total resolution graph one sees immediately the zeta function. The Milnor fibre $F^{(1)}_{k+1}$ of $f^{(1)}_{k+1}$ is connected since there are reduced components in $\mathcal{G}^{tot}_{G(p,q,r)}$. In case $\varsigma > 1$, all the multipilcities of the divisors in $\mathcal{G}^{tot}_{G(p,q,r)}$ have common divisor ς. Hence the Milnor fibre $F^{(\varsigma)}_{k+1}$ of $f^{(\varsigma)}_{k+1}$ is a disjoint union of $F^{(1)}_{k+1}$. □

4.11. **Remark.** The reason for the Milnor fibre $F^{(\varsigma)}_{k+1}$ $(\varsigma > 1)$ being not connected is that the function $f^{(\varsigma)}_{k+1}$ does not have D_∞ in its deformation. In the following example, the function considered has a D_∞ point in its good deformation, and its Milnor fibre is a bouquet of one cycles. This is similar to the case in which X is smooth [17, 18].

4.12. **Example.** Let X be defined by $h = x^2 + y^3 + z^7$. There is a line L on X parameterized by (see [7])

$$x = -c^{21}(1+t)^{11}t^3, y == -c^{14}(1+t)^7t^2, z == -c^6(1+t)^3t.$$

Let $\alpha := \alpha(z), \beta := \beta(z)$ be analytic functions such that $\alpha(0)\beta(0) \neq 0$ and $x - \alpha z^3 = 0, y - \beta z^2 = 0$ define L. Consider the function $f = (x - \alpha z^3)^2 + z(y - \beta z^2)^2$. The Newton polyhedron $\Gamma_+(h, f)$

is as Figure 1. The equations of the faces other than the coordinate planes in $\Gamma_+(h,f)$ are as follows.

$$\begin{aligned} FHZ &: & 21x+14y+6z=72 &\leadsto P\in\Gamma^*(h,f)\\ CDFH &: & 3x+2y+z=11 &\leadsto P_1\in\Gamma^*(h,f)\\ ABCD &: & 3x+2y+2z=12 &\leadsto P_2\in\Gamma^*(h,f)\\ ADF &: & 5x+4y+2z=20 &\leadsto R\in\Gamma^*(h,f)\\ BC\infty &: & x+2z=2 &\leadsto Q\in\Gamma^*(h,f)\end{aligned}$$

Part of the minimal regular subdivision Σ^* of the dual Newton diagram $\Gamma^*(h,f)$ of $V := X\cap f^{-1}(0)$ is as Figure 2, where $R_1 = {}^T(11,7,3)$, $S_1 = {}^T(7,5,2)$, $S_2 = {}^T(13,9,4)$, $Q_1 = {}^T(2,1,2)$, $Q_2 = {}^T(4,3,2)$. From the total resolution graph Figure 3 we see a reduced branch. This implies the Milnor fibre F of f is connected and is a bouquet of $\mu = 16$ copies of S^1.

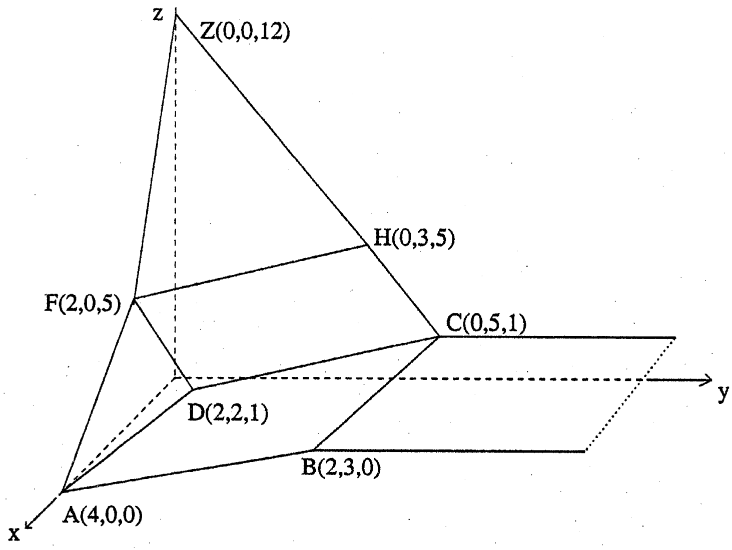


FIGURE 1. The Newton polyhedron $\Gamma_+(h,f)$

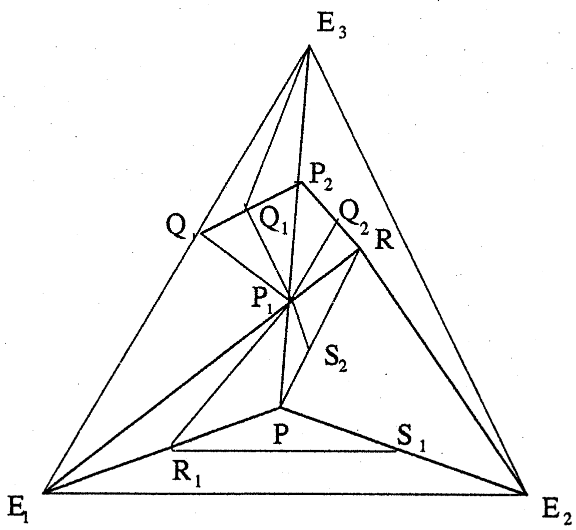


FIGURE 2. The dual Newton diagram $\Gamma^*(h,f)$ and part of Σ^*

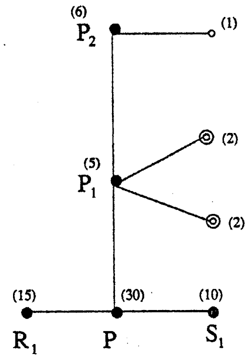


FIGURE 3. The Total resolution graph of V

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DEPARTMENT OF MATHEMATICS, JINZHOU NORMAL UNIVERSITY, JINZHOU, LIAONING 121000, P. R. CHINA
E-mail address: `jzjgf@mil.jzptt.ln.cn`

DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, MINAMI-OHSAWA 1-1, HACHIOJI-SHI, TOKYO 1920397, JAPAN
E-mail address: `oka@comp.metro-u.ac.jp`